Modeling Phenomena with Mathematics

D. J. Albers*

10/14/02

*Department of Physics, University of Wisconsin, Madison. E-mail: albers@cow.physics.wisc.edu

Introduction

- Initial Hopes and Dreams of Mathematicians and Other Scientists
- Early success in physics and astronomy
  - Newton, Einstein and all those boys
- Mathematicians take the wheel
- Discussion of low dimensional mathematical models
- Discussion of high dimensional mathematical models
- What we can say at the end of the day
- Conclusions

Initial hopes and dreams of mathematicians and ether scientists

- Mathematical models will represent nature
- Science will be able, given enough time, to have an unlimited and complete understanding of nature through mathematics
- Eventually we will be able to predict all types of behaviors with our models

Early success in physics and astronomy

- Success from first principles
  - Newton and calculus
  - Einstein and general relativity
- Experimentally "fit" models
  - Electricity and magnetism
  - Quantum mechanics
- First principles but solution requires approximated even in simple cases
  - Navier-Stokes equations; fluid dynamics in general
- Note as we move down the list, the similarity between the models and the world becomes more difficult

Mathematicians take the wheel

- Existence and uniquness theorems
- Polynomial approximation
- Other examples

- Existance and uniqueness theorems

$$\frac{dx}{dt} = f(x,t) \tag{1}$$

  - f must be continuous; continuous = you can't lift your pen when graphing
  - Equation 1 has a solution at (x_0, t_0) and the solution for a particular f at(x_0, t_0) is unique.
  - Any continuous function that evolves in time has a solution and that solution is unique at that time and position.
  - Fantastic: nature seems pretty continuous; if we can figure out the right assumptions make and measure the parameters correctly to model our system with a function of the from 1 it will have a solution. We can solve nature.
  - Problem: solution might be hard to find

- Weierstrass polynomial approximation theorem
  - Polynomials are dense in the set of continuous functions
    - Dense = arbitary close to any continuous function is a polynomial that can approximate that function
  - Great: we can use these functions as the solutions to the O.D.E.'s via the existance and uniqueness theorems; these are our solutions to nature
  - Problem: solution might not have a closed form, approximation might be necessary (i.e. terms might need to be truncated)

- Other approximation theorems
  - Neural Networks
  - Fourier Series
  - PDE's and Special Functions
  - Of course there are others for various situations

Discussion of low dimensional mathematical models

- The quadratic family
- Structure of parameter space
- Neural networks

- Quadradic family

$$f_a(x) = ax(1-x),\ 0 < a \leq 4 \qquad (2)$$

  - Open-dense set of attracting periodic orbits; i.e. arbitarly close to any periodic orbit, fixed point, or chaotic orbit is an attracting periodic orbit or fixed point
  - Measure of parameter values that give chaos is positive, i.e. the probabilty of finding chaos while sweeping the parameter space from 0 to 4 is greater than zero
  - Same can be said of periodic orbits and fixed points
  - Structure of the parameter space: pictures

- Some other possible structures in parameter space
  - 1-Dimensional parameter

– Circle map: rational versus irrational rotations

– Rational rotations repeat, irrational rotations do not repeat

– Rational rotations are dense, but measure zero

– Sweep continuously through rotations and you will observe rational rotations (periodic orbits)

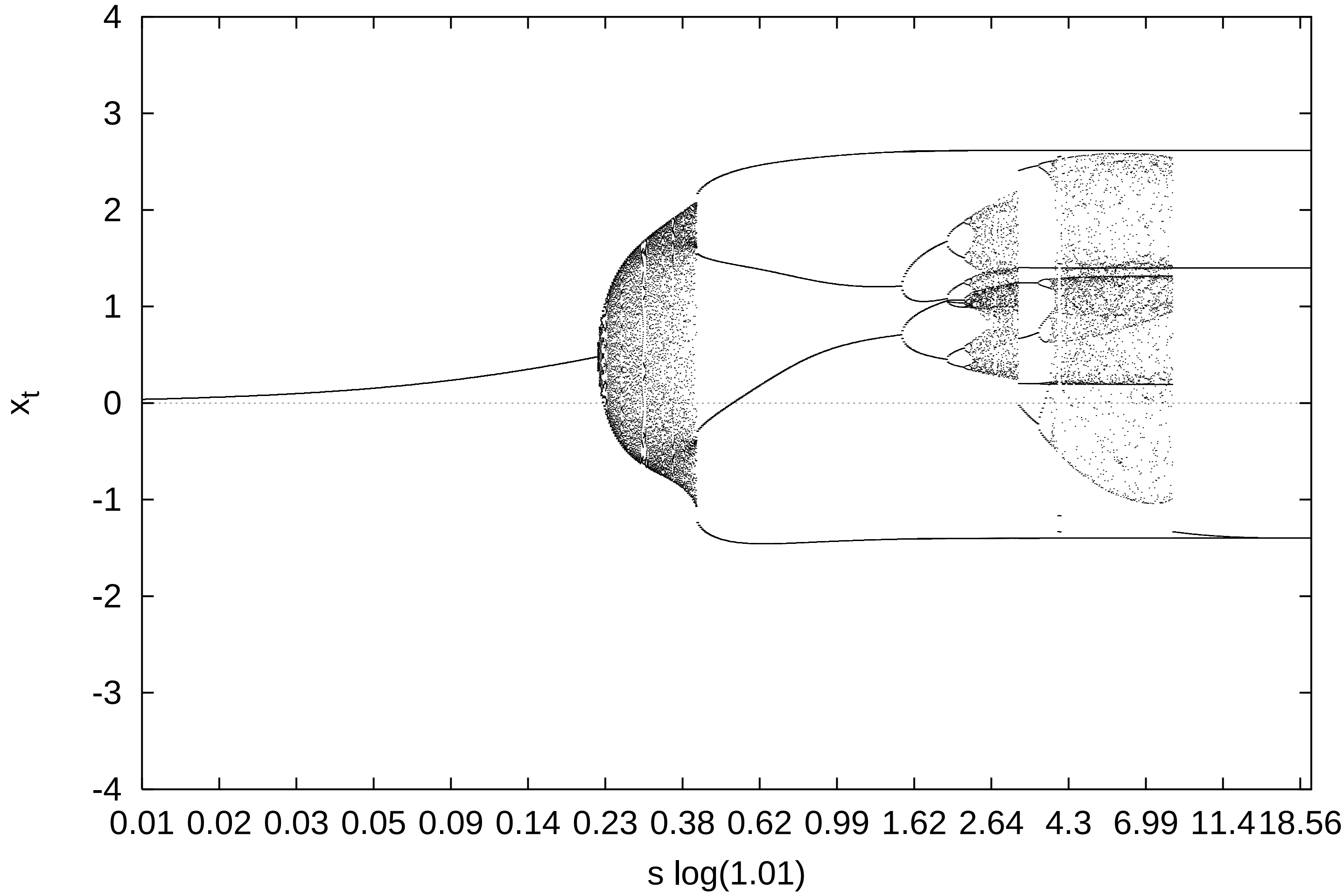
– Sweep randomly through rotations and you will NEVER observe rational rotations (periodic orbits)

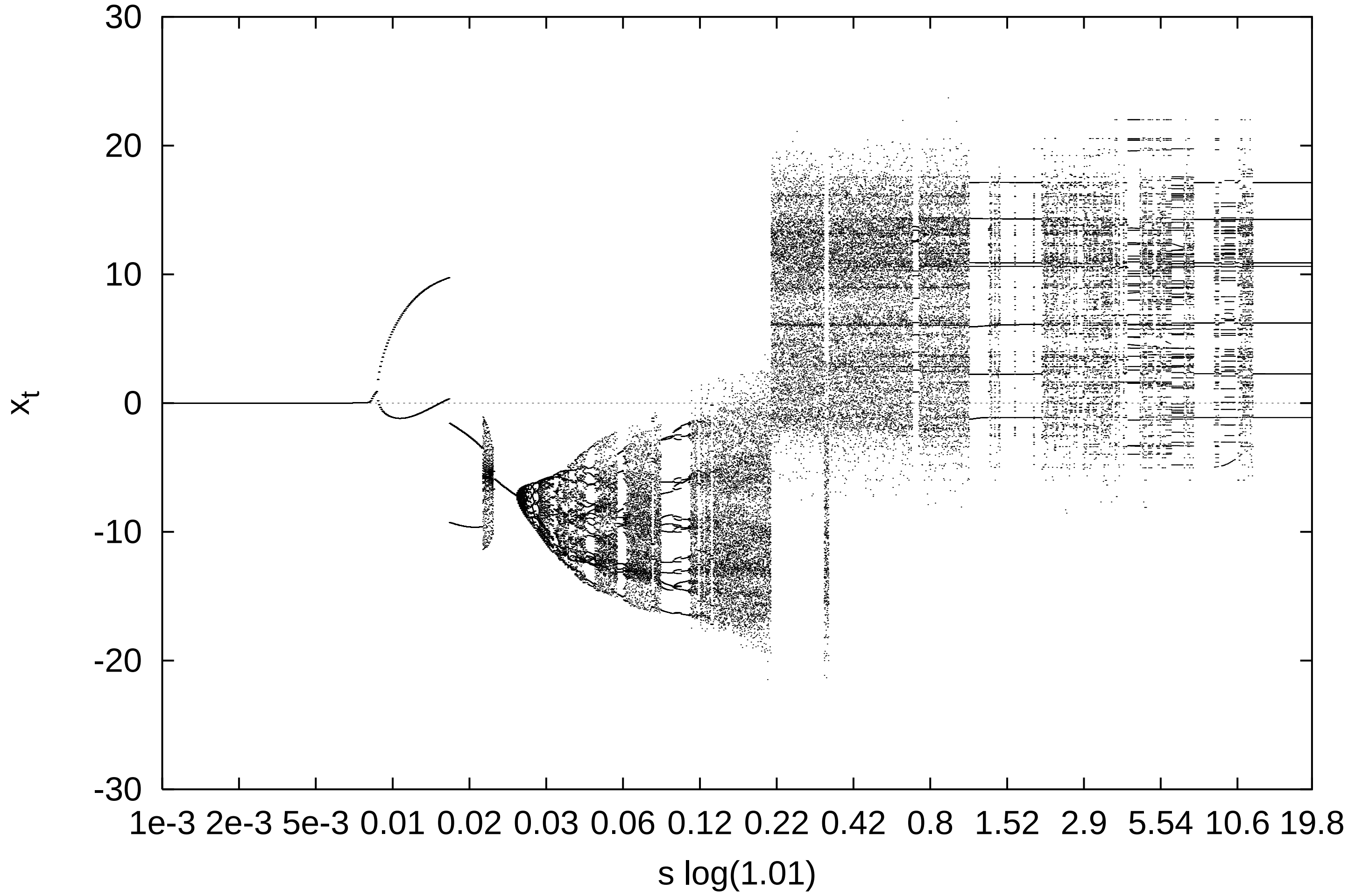
- Neural Networks

$$f(y) = \beta_0 + \sum_{i=1}^{n} \beta_i \phi \left(s w_{i0} + s \sum_{j=1}^{d} w_{ij} y_j \right) \tag{3}$$

  - n, the Number of Neurons
  - d, the "Dimension" of the Network, or the Number of Inputs
  - s, the Spread of the Gaussian of the w Matrix of Weights, Used as the Bifurcation Parameter
  - Squashing Function ϕ ($tanh(x)$)

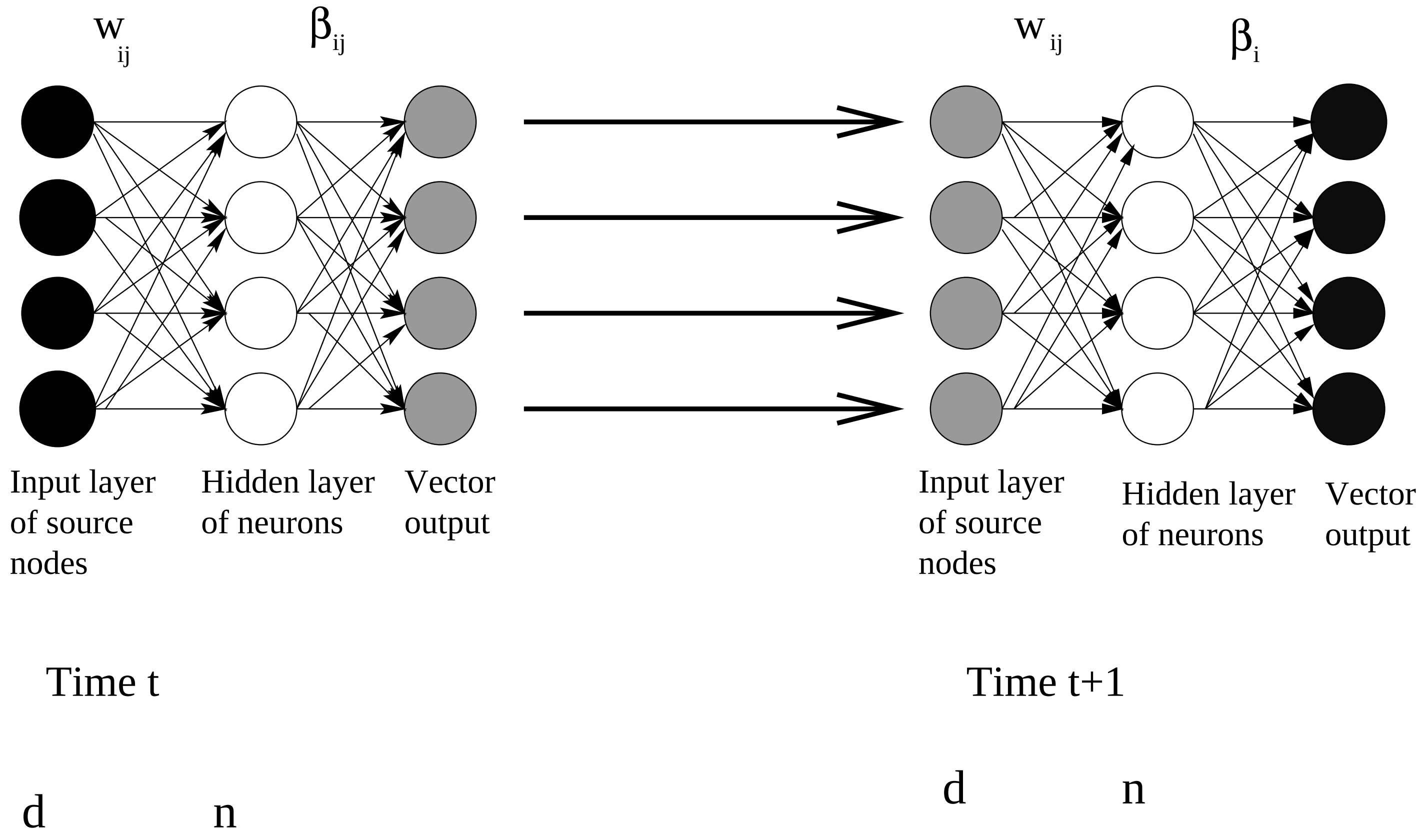
- Observations
  - Many types of behavior
  - “Robust” chaos in low dimensional special altered networks
  - Pictures


4
3
2
1
0
-1
-2
-3
-4
x_t
0.01 0.02 0.03 0.05 0.09 0.14 0.23 0.38 0.62 0.99 1.62 2.64 4.3 6.99 11.4 18.56
s log(1.01)

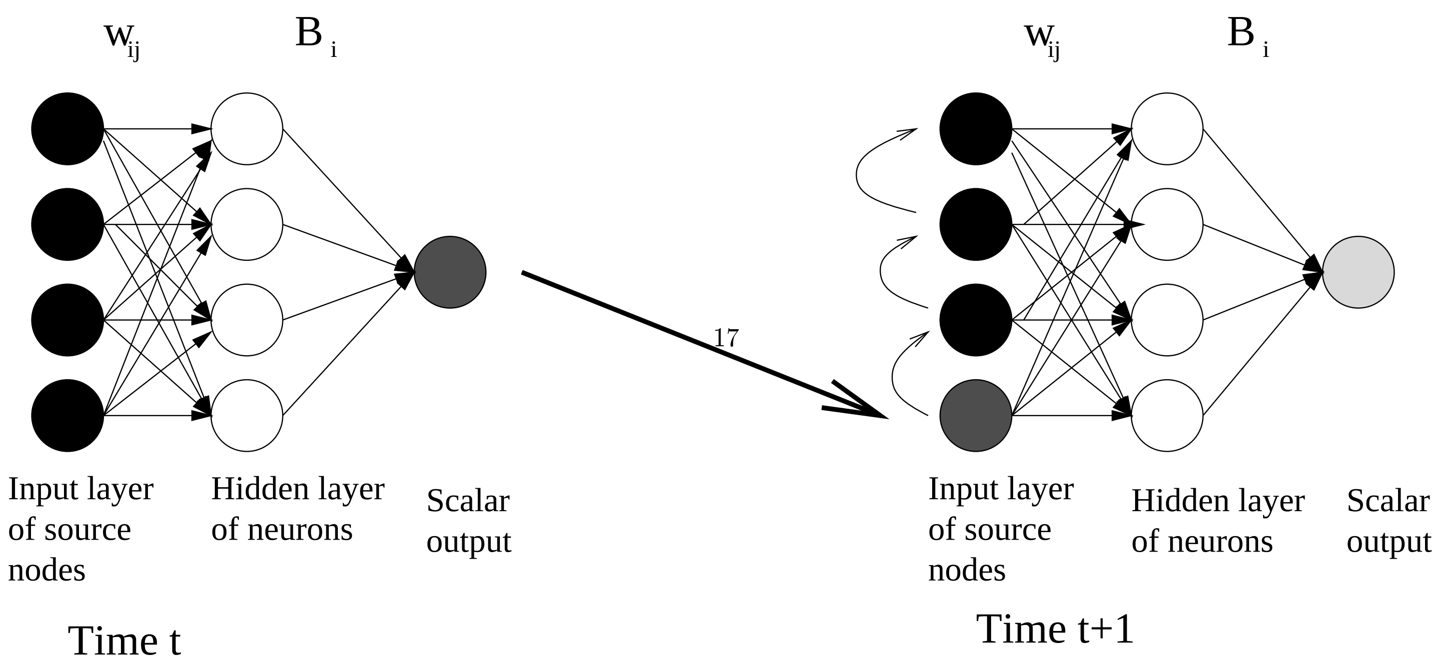

30
20
10
0
-10
-20
-30
x_t
1e-3 2e-3 5e-3 0.01 0.02 0.03 0.06 0.12 0.22 0.42 0.8 1.52 2.9 5.54 10.6 19.8
s log(1.01)

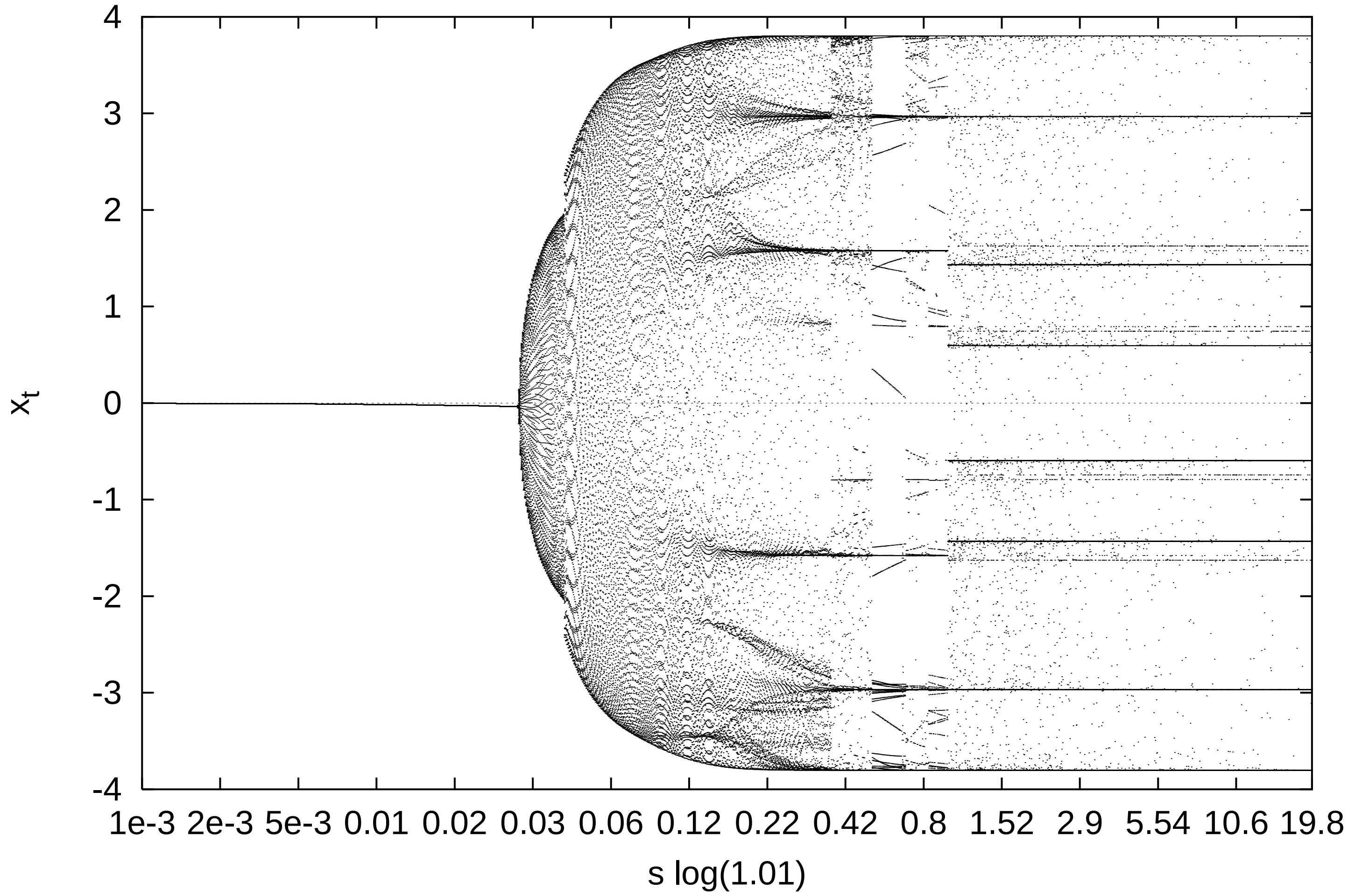
Diagrams of Scalar versus Vector Networks

Vector Networks

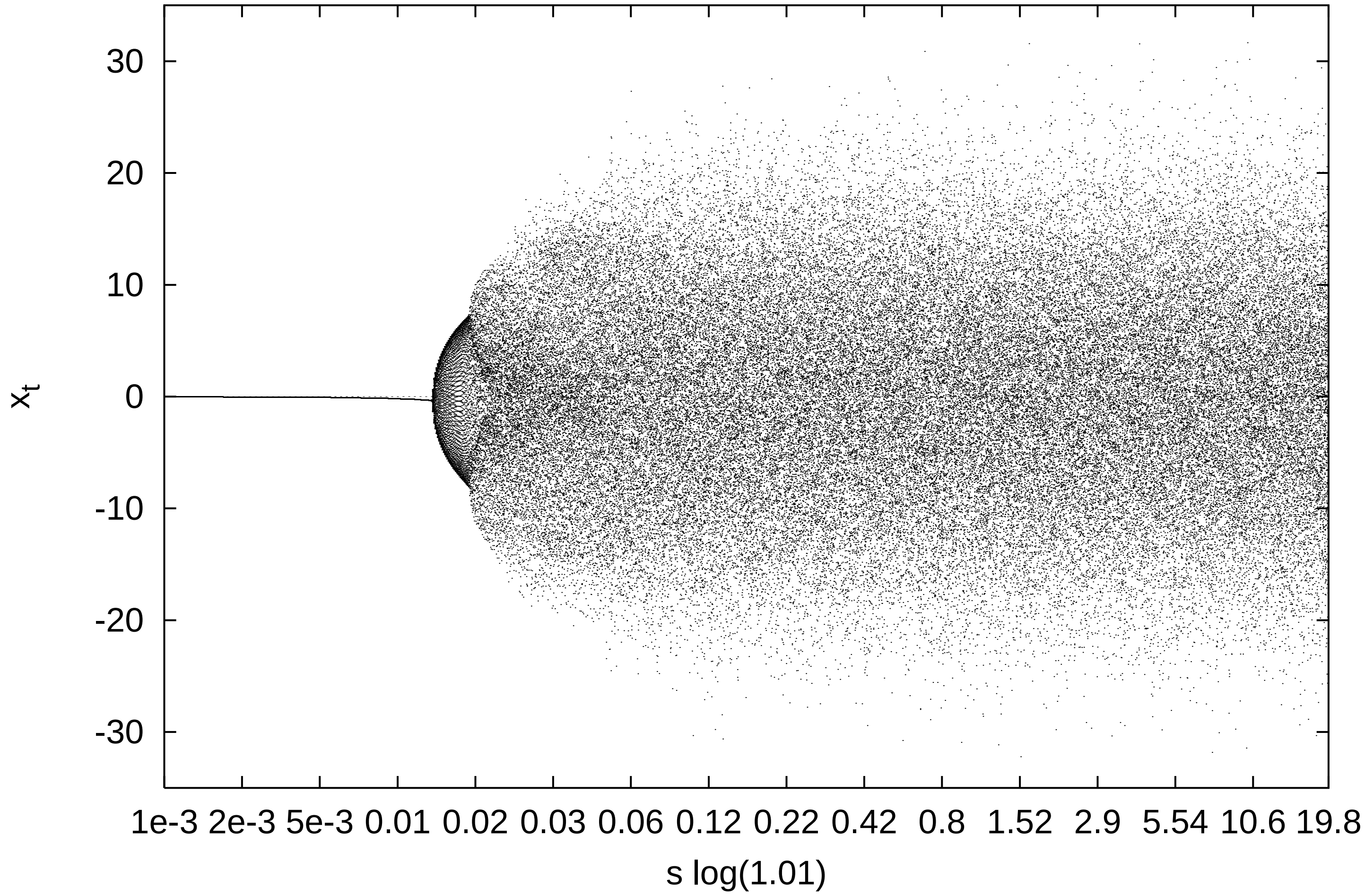


Scalar Networks





4
3
2
1
0
-1
-2
-3
-4
x_t
1e-3 2e-3 5e-3 0.01 0.02 0.03 0.06 0.12 0.22 0.42 0.8 1.52 2.9 5.54 10.6 19.8
s log(1.01)


30
20
10
0
-10
-20
-30
x_t
1e-3 2e-3 5e-3 0.01 0.02 0.03 0.06 0.12 0.22 0.42 0.8 1.52 2.9 5.54 10.6 19.8
s log(1.01)

Discussion of high dimensional mathematical models

- Notions of stability and similarity
- Structure of parameter space
- Examples

- Non-genericity of structural stability (the simplest counter example was given in R^4)
  - f is topologically-conjugate to g if there is a function (homeo) h that can "get" you from f to g
    - Example: all period 3 orbits are topologically conjugate, but period 3 and period 4 orbits are not topologically conjugate
  - f is structurally stable if all the functions "close" to f are conjugate
  - The point: you can perturb (or be wrong w.r.t. approximating) f a little bit and still retain the correct dynamics

- What functions are structurally stable: "hyperbolic," dense periodic points, no heteroclinic tangencies; i.e. technical conditions for being able to bump the system while retaining qualitatively similiar behavior
- Non-genericity = Not common; structural stability is non-generic
- Implications: In higher dimensional functions spaces (as low as R^4), "near-by" functions don't necessarly exhibit the same behaviors and can exhibit wildly differing behaviors

- Non-genericity of Ω-stability
  - Attempt to restrict f
  - $\Omega(f)$ is the set of points that f's orbit remains "near" to or exactly on (in the periodic orbit case)
  - Point, structural stability wasn't common, what about restricting functions orbits that stay close to themselves
  - Example of orbits that stay close to themselves: a broad periodic window; chaotic orbit
  - This was shown to also not be common
  - Implications: Even after restricting functions to bounded orbits that stay near themselves, bumping that function will in general give rise to different phenomenological behaviors

- Structure of parameter space
  - No one really knows yet besides the non-genericity of structural stability
  - Fragile conjecture
  - Some possibilities
    * Persistent chaos
    * Persistent periodic orbits
    * Intermixed chaotic and period windows; can be bizzare, strange basin stucture may exist for each set of parameter values; use your imagination to dream up pathologies, they will probability exist

- Examples:
  - Polynomials
    - Can be used to approximate any continuous function (Remember, the Weierstrass polynomial approximation theorem)
    - Can be unbounded
    - Observations from computer simulations
      - As the dimension is increased, the number of bounded cases decreases
      - As the dimension is increased, the number of chaotic cases decreases
      - Dimension scales like $\sqrt{d}$
      - Degree of polynomial makes little difference

- Neural networks
  - Can approximate any Lebesque integrable function (i.e. almost any function you can think of, including non-continuous functions) and it's derivatives
  - Always bounded
  - Pictures

* Observations from computer simulations
  - As the dimension is increased the number of chaotic cases increases
  - Many different attractors for the same set of parameters
  - Possible non-genericity of Ω and structural stability observed; i.e. very near a given function, there exist qualitatively different functions
  - Existance of very similar functions (in dimension, number of positive Lyapunov exponents, value of the largest Lyapunov exponent) near qualitatively different functions
  - Persistance of chaotic dynamics over a large portion of parameter space give high enough dimension
  - For high dimensions periodic windows are not observed for parameter perturbations of orders 10^2 to 10^{-8}

What We Can Say at the End of the Day

- Old physics style analogies in simple systems
- Stylized facts, qualatitive analysis
  - Economics, ecology, etc
- Difficulties
  - "Structure of nature," i.e. high periodic, chaotic orbits and noise
  - High dimensional "fitting" of data being representative of the phenomena; with many parameters many models can be rationalized
  - In high dimensional models, connections with reality beyond stylized facts are much harder due to the diversity of possible behaviors
  - "Rigorous" complicated predictions very unlikely
  - Many models begin to push the envelope of empiricism

Conclusions

- High dimensional models are terribly complicated
- When the models are perturbed qualitatively different behaviors can arise
- Knowing whether this reflects the phenomena under consideration can be very difficult
- High dimensional models still work well for understanding mechanisms that generate stylized phenomena
- It is very difficult to get good, empirically testable, sensible results from high dimensional models, especially truncated models
- No way can we solve the world with mathematical model, nor could we know if we had