Symbolic Dynamics of the Fixed-Position, Non-Equilibrium Farmer-Joshi Market Model Or How To Evenly Color Your Bread Dough And Get Rich

D. J. Albers*†

in collaboration with
C. R. Shalizi‡

Chaos and Complexity
Seminar
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*Department of Physics, University of Wisconsin, Madison. E-mail: albers@cow.physics.wisc.edu
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‡Department of Physics, University of Wisconsin, Madison, and the Santa Fe Institute. E-mail: shalizi@santafe.edu

Outline

- Motivation: Why Do We Care About Chaos
- Chaos: How We Define and Argue for the Existance of Chaos
- The General Farmer-Joshi Market Model
- Discretizing the Farmer-Joshi Model
- Analysis of Stability
- Extending The Market: De-Discretzing the Market Model
- Conclusions and Recap

Chaos; Why Do I Care?

- It Is, In Many Systems, The Most Common Behavior
- It Affects The Stability, Or Is The Degree Of Instability In The System
- Has Profound Affects On What Can Be Said About The System; I.E. Predictability, Geometry, Structure Of The Dynamics

What Makes Chaos Chaos?

- Qualitative Characteristics of Chaos:
  - Instability
  - No Predictability
  - Strange Attractors
  - Sensitive Dependence on Initial Conditions
  - Exponential Spreading of Near By Trajectories
  - Boundedness
  - Folding: Dye On Dough

- Six Formal Definitions, Or Characterizations For Chaos:
  - Devaney's Definition Of Chaos; Analytical
  - Strange Attractor Theorem: Topological Chaos; Analytical
  - Eigenvalues; Analytical/Computational
  - Lyapunov Exponents; Computational
  - Entropy; Analytical/Computational
  - Engineering Chaos; Computational

- Devaney's Definition of Chaos: Let f be a map on a compact set Λ. We say f is chaotic if and only if:
  - the periodic orbits of f are dense (potential for exploration) in Λ, and
  - f is topologically transitive (mechanism for exploration) on Λ.
- Used in a Mathematical Proofs of Chaos
- Extremely Difficult to Use in a Rigorous Fashion
- Reveals Mechanisms of Instability from a Orbit Viewpoint; i.e. states what has to happen to the dye on the dough, the end results shows the existence.

- Topological Chaos: The Existence of Smale Horseshoes And Strange Attractors
  - Used for Abstract Mathematical Analytical/Geometric Arguments, Directly Shows the Mechanism, The Existence of the Smale Horseshoe
  - Ingredients:
  - Topological Conjugacy to the Full Shift
  - Existence of Homoclinic Bifurcation: Extremely Geometric Viewpoint, Consists of Transverse Intersections of Stable and Unstable Manifolds
  - It is the Mechanism with which the Dye Uses to Get Everywhere in the Dough (i.e. to satisfy the Devaney Conditions), the existence and occurrence of the mechanisms guarantees the existence of chaos

- Local Eigenvalue Analysis: Absolute Value of the Modulus of the Largest Eigenvalue is Greater Than One.
  - Analytical and Numeric Calculation; Involves Calculation of the Jacobian (Generalized Derivative)
  - Measure of the Degree of Exponential Spreading of Nearby Trajectories.

- Largest Lyapunov Exponent: Largest Lyapunov Exponent is Greater than Zero; Note: $\lambda_{Lyap} = log(|\lambda_{eigen}|)$.
  - Numerical Calculation of the Tangent Space (Different Method of Calculation of the Derivatives)
  - Measure (a more "Experimental" Measure than the Eigenvalues) of the Degree of Exponential Spreading of Nearby Trajectories.

- Entropy: Count the Number of States, and Take the Limit As $t \to \infty$; if It is Positive, The System Is Chaotic
  - Used In Symbolic Dynamics
  - Can Be Used On Empirical Data
  - Calculates The Places You Can Go In The Space

- Engineering Chaos: Filter Noise From The Time-Series; Test For Randomness and Patterns Using Every Available Tools, If You See No Patterns, Call It Chaos
  - Used On Empirical Data
  - Used When All Other Methods Fail, or Cannot be Applied.

The General Farmer-Joshi Market Model

- Agents And Their Beliefs
- Market Maker
- Price Formation Rule

$$p_{t+1} = \frac{1}{\lambda} \sum_{i=1}^{N} (\tilde{x}_{t+1}^{i} - x_t^i) + p_t \tag{1}$$

Characters:

- p_t price at time t (in the FBM case, there will exist 3^N possible prices at each time step, as each agent of the N can have three possible positions.)
- p_0 the initial price.
- c_t^i agent i's capital. This consists of the cash left after purchases plus the injected capital (and hence the money for the next time step).
- c_0^i agent i's initial capital (for the FBM, $c_0^i = 0$, but each agent has a fixed amount of capital in which he/she can bet with at each time step, $\frac{c_0}{p_0}$, a fixed number, hence the fixed bet market.)
- m_t^i agent i's money at time t. The money is the cash an agent has available to spend at any given time. More formally, $m_t = m_{t-1} - p_t(x_t - x_{t-1})$; i.e. the money the agent had minus the money spent on stocks. (This is not a relevant property for the FBM, the agent always has a fixed amount of cash to use, $\frac{c_0}{p_0}$.)
- w_t^i agent i's wealth at time t. Wealth is the sum of the agent's money, and the stock holdings. More formally, $w_t = p_t x_t + m_t$. (This is also an somewhat irrelevant parameter for the FBM, it only factors in to the picture in relation to the gains of each of the strategies.)
- x_t^i agent i's position at time t. (in the FBM there exist three possible positions per agent: $\pm\frac{c_0}{p_0}$, and 0.)

- a is the re-investment rate.
- λ is the liquidity. It works against the re-investment rate. It is a very relevant parameter in the RMM, but not so much in the FBM.
- $\tilde{x}^i_{t+1}$ agent i's order for time $t+1$. What the order is will depend on the given set-up. For the specific case we are considering in this brief write-up, $\tilde{x}^i_{t+1} = a\frac{c_0}{p_0}$. This is what we mean when we say fixed bet size.
- x^i_{t+1} this is the order that gets filled by the market maker for each agent. In general, $x^i_{t+1} = a\frac{m^i_t}{p_{t+1}}$; but in our fixed bet case, $x^i_{t+1} = a\frac{c_0}{p_0}$.
- N this is the number of agents.
- n this is the number of past time steps the agents consider.
- s this is the number of strategies that the agents have to consult.
- $g^i_{j,t}$ is the gain for each strategy, j, for each agent, i, at each time, t. This can become a very complex calculation upon including re-investment, thus we will only state it for the FBM in general, $g^i_{j,t} = (p_t - p_{t-1})x_t$

The Binary Strategies:

- Strategies are $n \times n$ Matrices of 1's and 0's, Based On The Previous n Price Movements
- 2^{2^n} Strategies
- 1 Means Go Long If Short Or Even, Stay Long If Long
- 0 Means Go Short If Short or Even, Stay Short If Short

Iterating the Market:

- Agents Pick Active Strategy Based On Previous Gains
- Agents Use Binary Strategies to Predict Price Movements, and Place Orders
- Market Maker Absorbs the Orders, and Sets the Price Via the Price Formation Gain.

$$p_{t+1} = \frac{1}{\lambda} \sum_{i=1}^{N} (\tilde{x}^i_{t+1} - x^i_t) + p_t \tag{2}$$

- Agents Calculate Gains Base Upon the Price Movement and Their Order.

$$g^i_{j,t} = x^i_{t-1}(p_t - p_{t-1}) \tag{3}$$

Discretizing the Farmer-Joshi Model

- Fix the Bet Size Per Agent: $x^i_{t+1} = \frac{c_0}{p_0}$
- Fixes the Positions At: $x^i_{t+1} = \frac{c_0}{p_0}$, 0
- Fixes the Number of Possible Prices At: $3N$
- Fixes The Possible Gains if Considered Over A Finite Past Time, (Finite Combinations of Finite Symbols)
- Given this Alphabet, We Can Write The Fixed Position Market as a Shift
- Discretizing the Market Allows for Upper Bounds on the Period;
  - $\tau = 1$ or $\tau \leq n$
  $$k_n = (3N)^n \tag{4}$$
  - $\tau > n$
  $$k_\tau = (3N)^\tau \tag{5}$$
  - Fixed Position Market With 1 Strategy Will Have Period No Higher Than k_n
  - $n \geq \tau$: The Fixed Position Market s Strategies Will Have Period No Higher Than:
  $$s^N k_n = s^N (3N)^n \tag{6}$$
  - $\tau > n$: The Fixed Position Market s Strategies Will Have Period No Higher Than:
  $$s^N k_\tau = s^N (3N)^\tau \tag{7}$$

Analysis Of Stability

- Symbolic Dynamics
  - Construct An Alphabet (Can be Finite, Or Countable, But Not Uncountable)
  - Create Mappings From Letters To Letters, Or Words To Words
  - The Mapping Is Called A Shift Map
  - Example: $\{0, 1\}$ Shift

- Using Shift Maps To Show Chaos
  - Chaos In THE Shift Map (Shift Automorphism On 2 Symbols)
    - Contains Countable Periodic Orbits
    - Contains Uncountable Non-Periodic Orbits
    - Existence Of A Cantor Set (And Thus The Existence Of Strange Attractors (Roughly))
    - A Dense Orbit
    - Smale Horseshoes
    - Sensitive Dependence On Initial Conditions In The Invariant Cantor Set
  - Methods Of Showing Chaos
    - Conjugacy (I.E. Equivalence) With THE Shift Map
    - Symbolic Dynamics Gives Us Tools To Show Topological Conjugacy And Density of Orbits

- Chaos And Instability In The Fixed Position Market
  - Skip The Specifics And Identify What Must Happen In Shift Maps For Transitivity To Occur
  - Two Examples: The Non-Transitive and Transitive-Case
    * Non-Transitive:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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    * Transitive:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

- Our Market Is Many To One
- Our Market Is Not Topologically Transitive; And Consists of Many To One Mappings; I.E. No One Can Get To Everyone Else
- The Fixed Position Market Is Not Chaotic

Extending The Market: De-Discretizing the Market Model

- Make The Alphabet A Uncountable Set: Allow For Re-Investment
  - Making The Bet Size Not Discrete Makes Both The Price And Positions Real Numbers
  - Gains (And Hence Price Histories) No Longer Necessarily Discrete.
  - Introduces Wealth In A Meaningful Way; $w_t^i = m_t^i + p_t x_t^i$; The Wealth Is The Money In The Market Plus The Cash (Before It Was Not Relevant, Because It Didn't Influence Their Purchases)
  - Agents Invest Wealth, Which Is Now Not Discrete
  - The Agents Money For Investment Is Based Upon How Well They Do; Their Money Is Thus No Longer Discrete

- What Does This Due To Our Results?
  - Lose Symbolic Dynamics Framework Due To Uncountability Of State
  - Lose Finite Sequences In The Bounded Gains; Thus It Appears That It Can Fold
  - Still Have Many To One Relationships, But Everything Is Uncountable
  - Dense Orbits Appear
  - Topological Transitivity Hard To Show
  - Still Working On This

Conclusions and Recap

- Methods Of Showing Chaos
  - Devaney
  - Strange Attractor Theorem: Topological Chaos
  - Eigenvalues
  - Lyapunov Exponents
  - Entropy
  - Engineering Chaos
- Shift Market Model Results
  - Upper Period Bounded; Depends More Upon Number Of Strategies Than Number Of Agents
  - No Topological Transitivity (Orbits Can't Mix)
  - No Chaos
- Re-Investment Market
  - Uncountable States
  - Dense Orbits
  - Still Many To One; But Doesn't Have To Repeat
  - Possibly Chaotic