What the heck is a PughShub?

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Agenda:

- Why do you care?
- What are the proper ingredients?
- How do we use geometry to describe dynamics?
- Examples: the pendulum, motion on the torus.
- What the heck does ergodic mean?
- THE PughShub: the theorem.
- Unpacking the properties
- Remarks

Why do you care about Pugh-Shub?

- We model phenomena with mathematics, we need to understand the dynamical properties of our tool.

- Having a intuitive geometric understanding for how geometry drives dynamics is useful.

- The stability conjecture addresses many dynamic behaviors in a negative light.

- It is beautiful (to me at least).

What are the proper ingredients of equivalence of dynamics under perturbation?

- A good notion for dynamic stability:
  - Applies to the wide range of dynamics - especially "chaotic" dynamics
  - Is specialized enough so that we can make SPECIFIC (mathematical) statements with respect to geometry.

- Geometric properties might we wish to include; what properties might be required to make statements with respect to the long time average of the dynamics.
  - Notion of recurrence

– Ability to “distinguish” geometric objects

How do we use geometry to describe dynamics?

- Define the mappings with which we are concerned.

- These mappings allow us to separate the dynamics into three categories - fast expansion, fast contraction, and neither expansion or contraction

- Represent these categories as geometric objects in the state space.
  - The dynamics are governed by these objects.

- Analyze how these objects interact.

Examples:

- Pendulum - motion on a cylinder
  - Plain, vanilla pendulum
  - Damped pendulum
- Anosov diffeomorphism - Motion on a torus - or the unit square
- A “stack” of Anosov diffeomorphisms

General facts to know about manifolds:

- Basically Euclidean objects.
  - Euclidean - real line $= R$, plane $= R \times R = R^2$, 3$-$space $= R \times R \times R = R^3$

- Orbits can be manifolds, the state space can be made up of manifolds, and spatial or temporal variations of the orbits can be manifolds.

Stable, unstable and center manifolds:

- Unstable manifold for a point: moving forward in time, the trajectory expands exponentially away from the point - fast expansion

- Stable manifold for a point: moving forward time, the trajectory contracts exponentially toward the point - fast contraction

- Center manifold for a point:
  - less than exponential expansion or contraction, i.e. slow.
  - no contraction or expansion

Relationship between derivatives and stable, unstable, and center manifolds:

- Derivatives measure rates of expansion and contraction along the orbit

- The stable manifold of the orbit is the direction tangent to the exponential contraction - and has an associated rate

- The unstable manifold of the orbit is the direction tangent to the exponential expansion - and has an associated rate

- Identify "time scales"

What the heck is ergodic?:

- Measure - just a way of specifying a volume
- Transformation - maps a set to into another set (it can be the same set)
- A transformation is ergodic if (and only if) the orbit of almost every point visits each set of positive measure
- An ergodic transformation takes almost all sets all over the space. The only sets it doesn't move are some sets of measure zero and the entire space.

Ergodic examples:

- Flow (continuous time) on a torus with an irrational rotation is ergodic

- The hyperbolic Anosov diffeomorphism is ergodic

- Periodic orbits are NOT ergodic

- Irrational rotations of a discrete time mapping of the torus are NOT ergodic

- Many "chaotic" dynamical systems (specifying which chaotic dynamical systems are is the point of the Pugh-Shub theorem)

The Pugh-Shub theorem:

- f is a continuous, two times differentiable function on a bounded set (with a smooth measure) IF:

- f is partially hyperbolic

- f is center bunched

- f is dynamically coherent

- f is stably essentially accessible

- THEN: f is ergodic

Partially hyperbolic:

- The rates of expansion, contraction and neutrality can be fully expressed via combinations of stable, unstable and central manifolds

- The stable, unstable, and neutral directions can be defined and can be identified with their respective manifolds

- Does not mention how distinct these manifolds are, or speak to the make-up of the geometric structure.

- Point: includes dynamical systems with notions of exponential growth and decay AND non-exponential growth and decay and zero growth (central direction)

$$TM = E^u \oplus E^c \oplus E^s \tag{1}$$

Center bunched:

- Somewhat technical (based upon Holder continuity) relating "distinctness" of the "splitting" of the tangent vectors.

- Recall the eigenvalues of a map - bunching estimates regarding how close eigenvalues are to each other

- Separation of time scales

Dynamically coherent:

- The unstable, stable, and central orbit classes fit together "nicely"

- Center manifold is integrable - the integral is continuous and differentiable one time

Accessibility property:

- Generalized notion of recurrence
- Points are connected by a us-path
- There exist dynamical systems that are accessible but not transitive

Essential accessibility property:

- Each measurable set is that is the union of accessible sets has EITHER measure zero or one

- Example: “stacked” Anosov$\times id$

Final remarks:

- This encompasses MANY types of complicated dynamical behaviors
  - Anosov diffeomorphisms
  - Any hyperbolic - chaotic dynamical systems (Such as Smale horseshoes)
  - Many of the structurally unstable dynamical systems - that fail to be structurally stable due to the existence of a central manifold

Final remarks con't:

- The desired result is a removal of dynamical coherence and center bunched from the required characteristics of ergodicity.

- Desired conjecture: If f maps a bounded set to itself, is partially hyperbolic and essentially accessible, then f is ergodic.

- Desired conjecture: Accessibility is an open and dense property of partially hyperbolic dynamical systems

- Center bunched partially hyperbolic diffeomorphisms (maps) which are stably dynamically coherent and stably accessible are stably ergodic.