Structural Stability and Robustness in Dynamical Systems

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Introduction

- Background: Initial Hopes and Dreams of Mathematicians and Other Scientists
- Formulation of solutions to dynamical systems
- Notions of stability and similarity
- Discussion of low dimensional mathematical models
- Discussion of high dimensional mathematical models
- Questions
- Conclusions

Initial hopes and dreams of mathematicians and other scientists

- Mathematical models will represent nature
- Science will be able, given enough time, to have an unlimited and complete understanding of nature through mathematics
- Eventually we will be able to predict all types of behaviors with our models

Formulation of solutions of dynamical systems

- Existence and uniqueness theorems
- Polynomial approximation
- Other examples

- Existence and uniqueness theorems

$$\frac{dx}{dt} = f(x,t) \tag{1}$$

  - f must be continuous; C^r on $U \subset R^n \times R^1$
  - Equation 1 has a solution at (x_0, t_0) and the solution for a particular f at(x_0, t_0) is unique; i.e. any solution of equation 1 at (x_0, t_0) will be the same on the common interval of existence; the solution, X is C^r
  - Any continuous function that evolves in time has a solution and that solution is unique at that time and position.
  - Fantastic: nature seems pretty continuous; if we can figure out the right assumptions make and measure the parameters correctly to model our system with a function of the from 1 it will have a solution.
  - Problem: solution might be hard to find

- Weierstrass polynomial approximation theorem
  - Polynomials are dense in the set of continuous functions
  - Great: we can use these functions as the solutions to the O.D.E.'s via the existence and uniqueness theorems; these are our solutions and they are "easy" to find
  - Problem: solution might not have a closed form, approximation might be necessary; in real life the solution is an infinite series

- Other approximation theorems
  - Neural Networks
  - Fourier Series
  - PDE's and Special Functions
  - Of course there are others for various situations
  - All follow the same formulation

Notions of stability and similarity

- Robustness
- Structural stability
- $\Omega-$stability
- Eigenvalue type equivalence

- Robustness
  - Notion: persistence of a property relative to changes in parameter space
  - Property x of object y is robust if x holds on an open set of y (or on an open set in y's parameter space)
  - Robust chaotic attractor: arbitrary change in parameters cannot destroy chaos; attractor is unique (???)

- Structural stability
  - f is topologically conjugate to g if $\exists$ homeomorphism h such that $g = h \circ f \circ h^{-1}$
  - f is structural stable if $\forall g \in N(f)$ in the C^1 topology, g is topologically conjugate to f
  - A C^2 diffeomorphism (on a compact manifold without boundary) which satisfies axiom A and the strong transversality condition

- $\Omega(f)$ - the non-wandering set: for any neighborhood U of x_0 $\exists\, n > 0$ such that $f^n(U) \cap U \neq 0$
- Axiom A: f is axiom A if and only if $\Omega(f)$ is hyperbolic and periodic points of f are dense in $\Omega(f)$
- Strong transversality: f satisfies the strong transverality if and only if $E_x^s + E_x^u = M_x$
- Remark: if f is axiom A, then f satisfies the strong transversality conditions if and only if every stable manifold intersects every unstable manifold transversality

- $\Omega-$stability
  - Like structural stability restricted to $\Omega(f)$
  - $\Omega-$conjugate: $\exists$ a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $gh = hf$
  - $\Omega-$stable: f is $\Omega-$stable if and only if $\exists$ $N(f)$ such that all $g \in N(f)$ are $\Omega-$conjugate to f

- Eigenvalue type equilvence
  - k-jets: equivalent spectrum of eigenvalues at a point
  - Equivalence in Lyapunov exponents
  - Note: eigenvalue equivalence notions do not imply structural stability and visa versa

Discussion of low dimensional mathematical models

- The real quadratic family
- The circle map
- Neural networks
- Other examples

- Quadratic family

$$f_a(x) = ax(1-x),\ 0 < a \leq 4 \qquad (2)$$

  - Open-dense set of attracting periodic orbits; i.e. arbitarly close to any periodic orbit, fixed point, or chaotic orbit is an attracting periodic orbit or fixed point
  - Measure of parameter values that give chaos is positive, i.e. the probability of finding chaos while sweeping the parameter space from 0 to 4 is greater than zero
  - Same can be said of periodic orbits and fixed points
  - Structure of the parameter space: pictures

- Circle map: rational versus irrational rotations

- Rational rotations repeat, irrational rotations do not repeat
- Rational rotations are dense, but measure zero
- Sweep continuously through rotations and you will observe rational rotations (periodic orbits)
- Sweep randomly through rotations and you will NEVER observe rational rotations (periodic orbits)

- Neural networks

$$f(y) = \beta_0 + \sum_{i=1}^{n} \beta_i \phi \left(s w_{i0} + s \sum_{j=1}^{d} w_{ij} y_j \right) \tag{3}$$

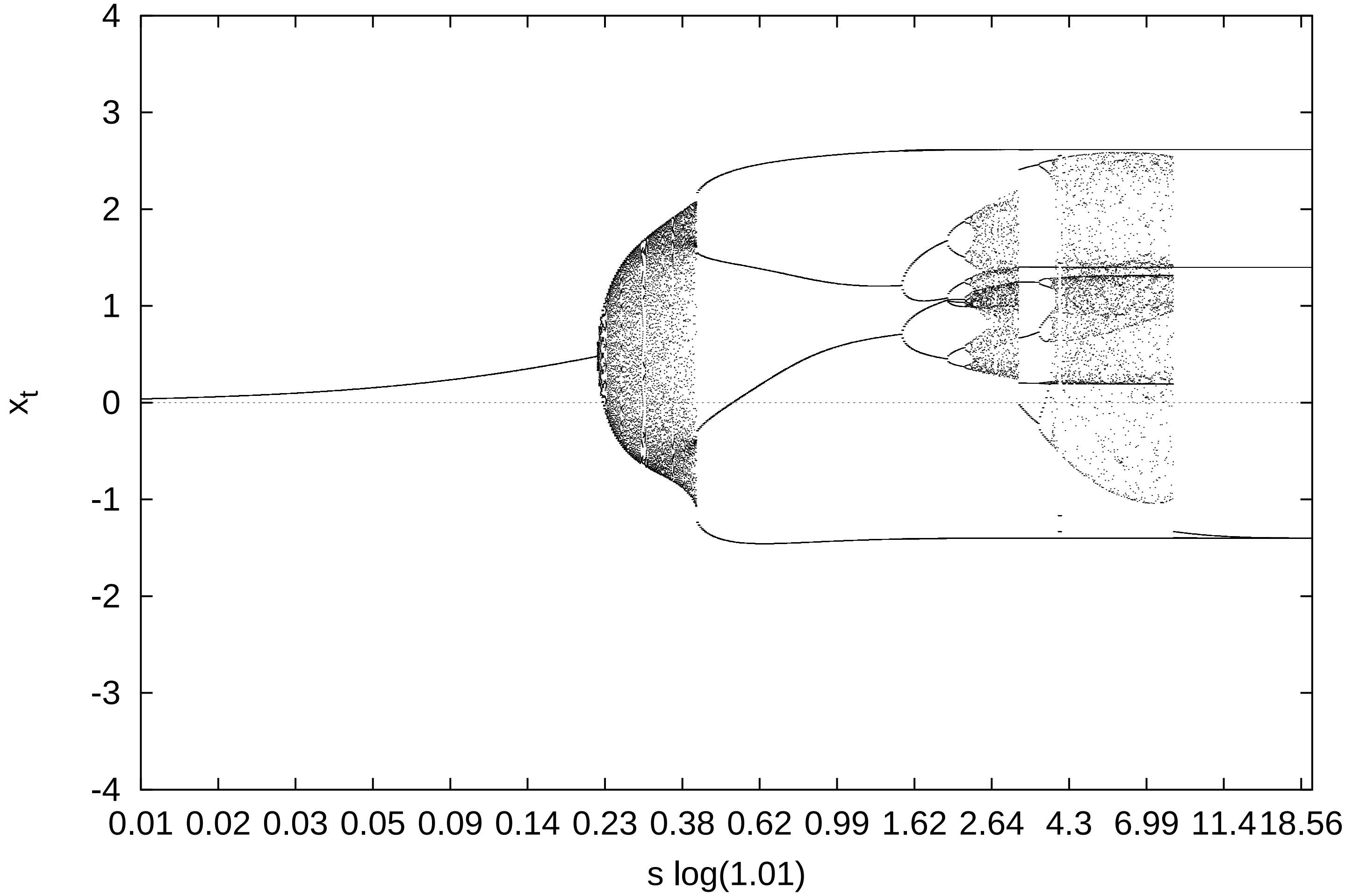
  - n, the Number of neurons
  - d, the "Dimension" of the network, or the number of inputs
  - s, the spread of the Gaussian of the w matrix of weights, used as the bifurcation parameter
  - Squashing function ϕ ($tanh(x)$)

- Observations
  - Many types of behavior
  - "Robust" chaos in low dimensional special altered networks

$$f(x_i) = |tanh(s(x_i + a\Sigma_{j=1}^{d} w_{ij} x_i - c)| \quad (4)$$

  - Pictures

- Other examples
  - Cat map
  - Smale's horseshoe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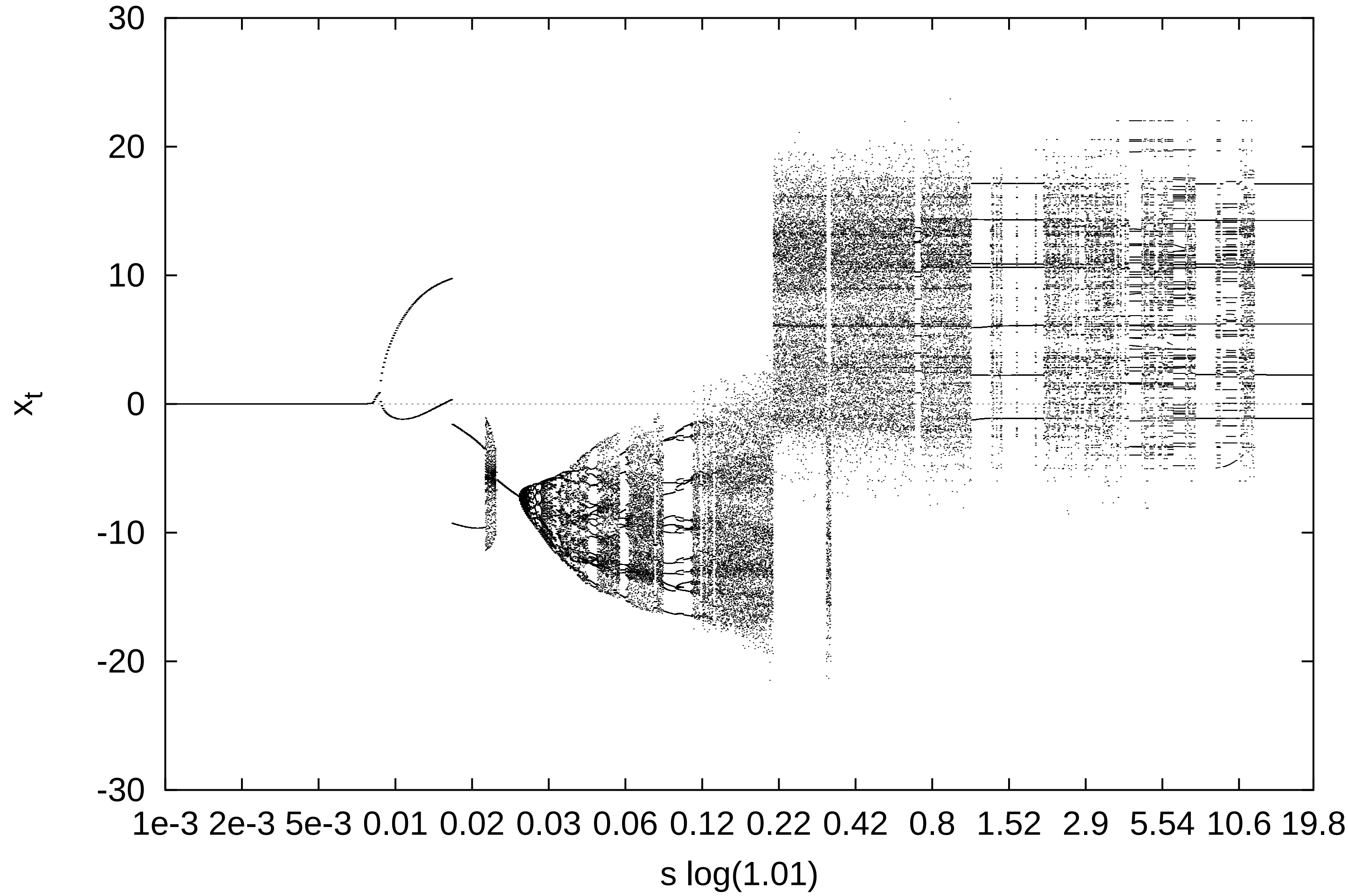

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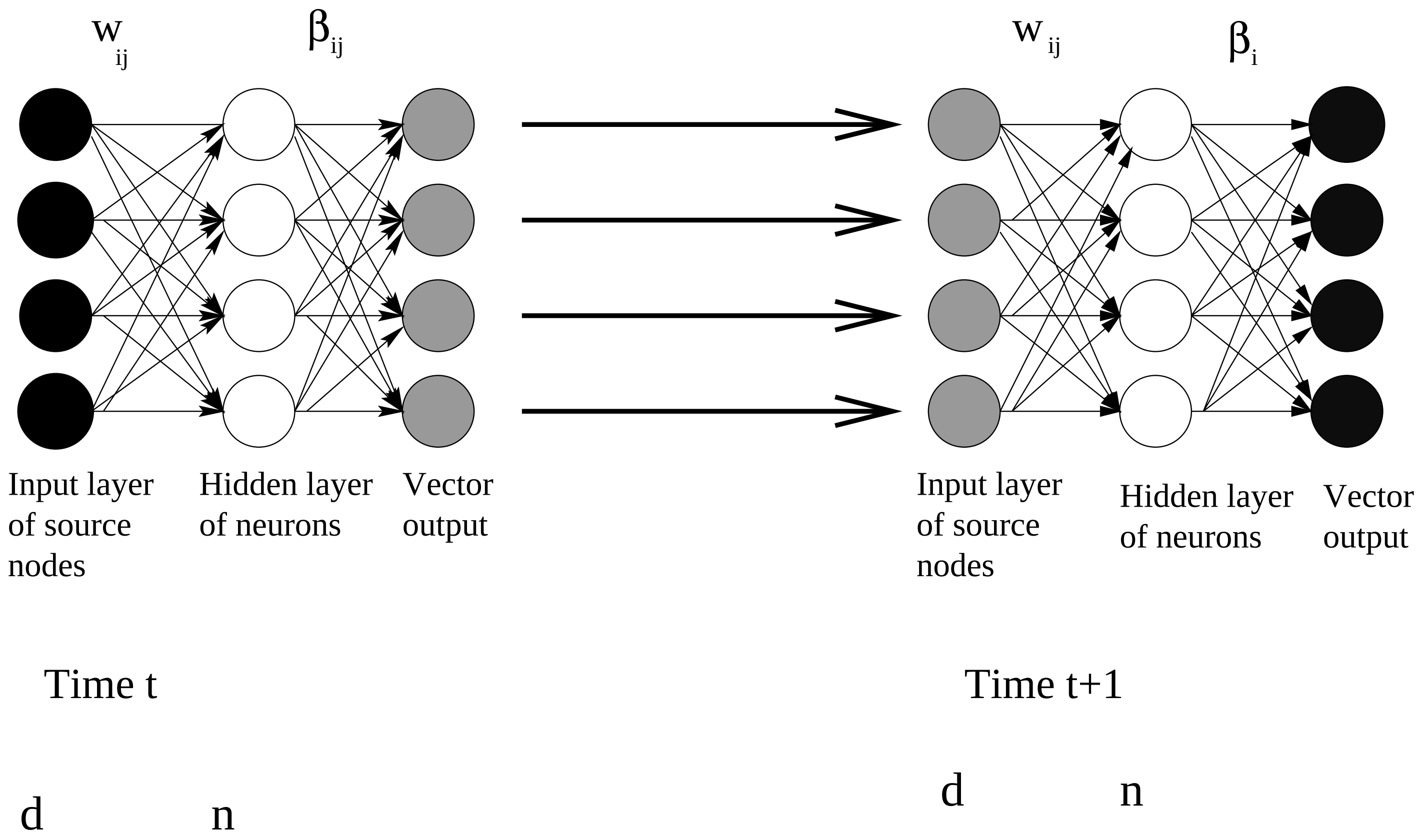
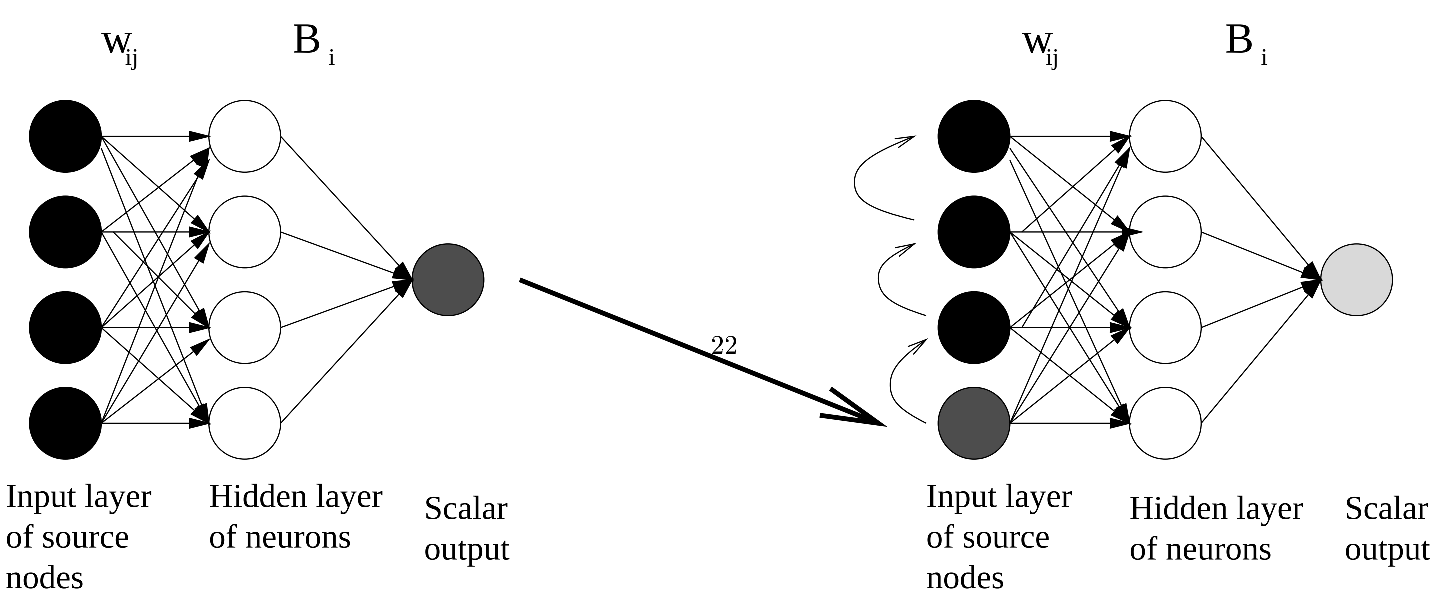
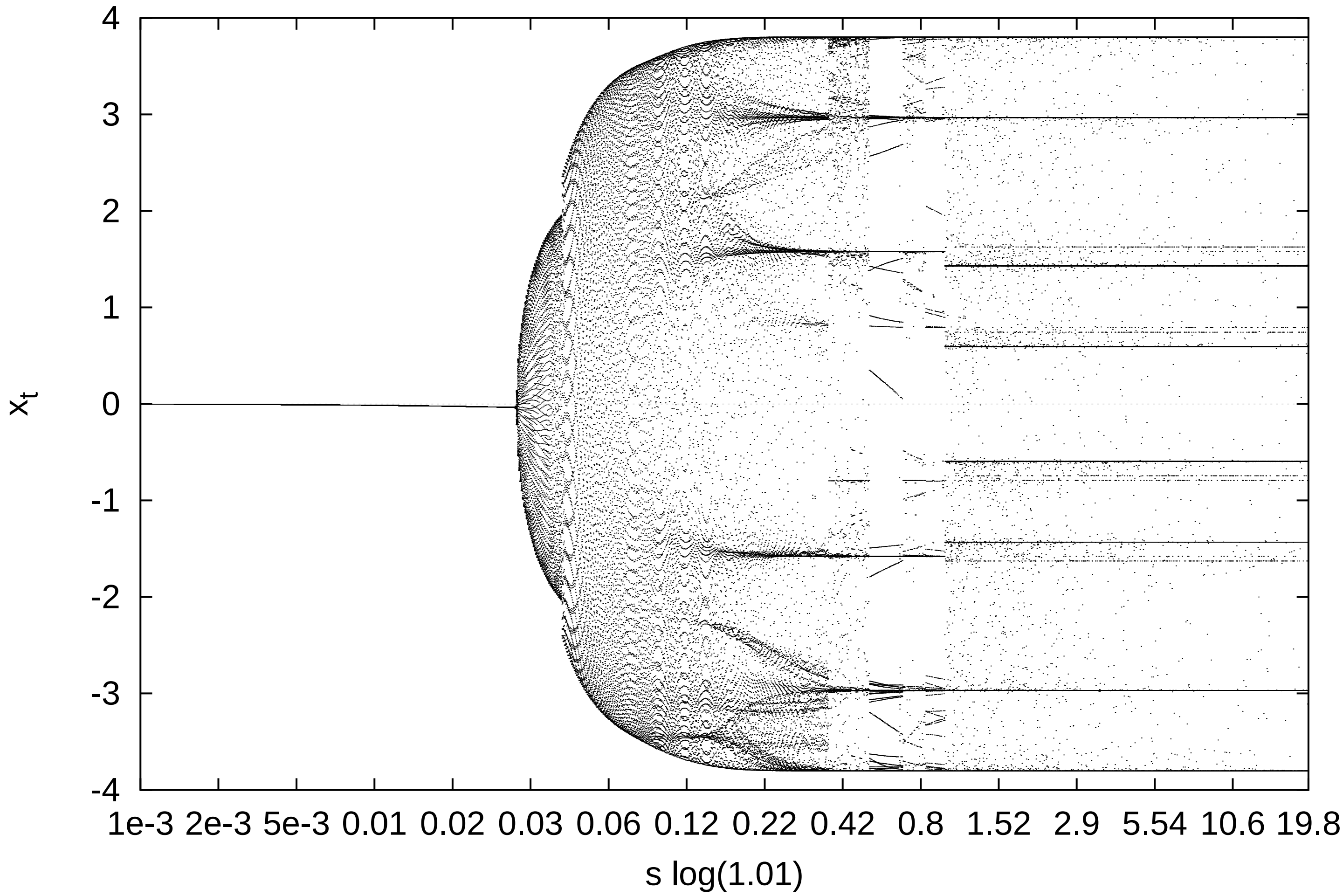
Diagrams of Scalar versus Vector Networks

Vector Networks



Scalar Networks





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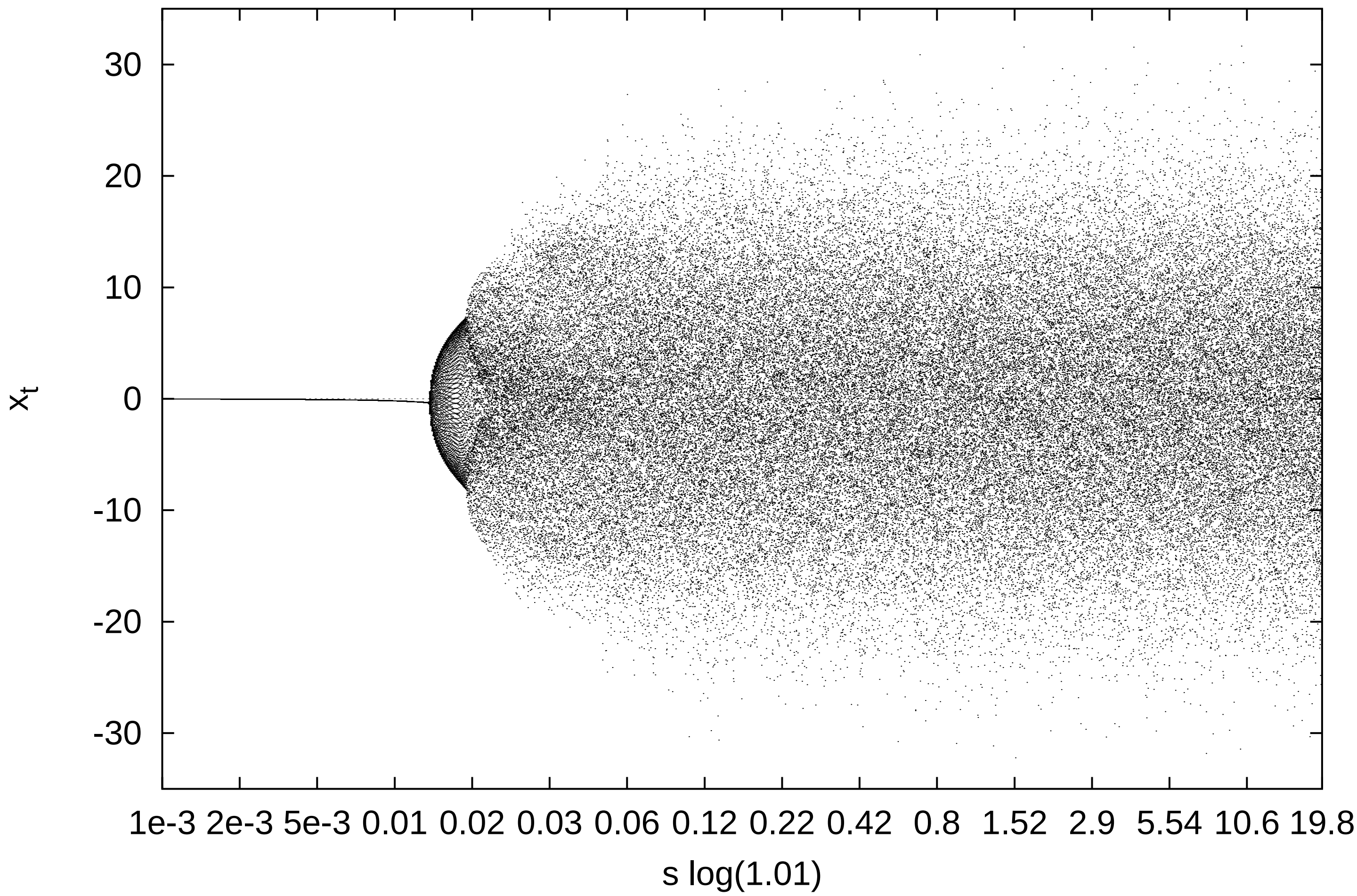

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Discussion of high dimensional mathematical models

- Non-genericity of stability and similarity
- Structure of parameter space
- Examples

- Non-genericity of structural stability (the simplest counter example was given in R^4)
  - Structural stability is not generic in R^4
  - Map: cat map crossed with the horseshoe ($S^2 \times T^2$),
  - Condition violated: axiom A, specifically hyperbolicity; violation occurs on at least a open set
  - Implications: In higher dimensional functions spaces (as low as R^4), "near-by" functions don't necessarily exhibit the same behaviors and can exhibit wildly differing behaviors (whether they will is still up of contention)

- Non-genericity of Ω-stability
  - Attempt to restrict f
  - Point, structural stability wasn't common, what about restricting functions orbits that stay close to themselves
  - Ω-stability not generic
  - Map same as the structural stability buster
  - Violation: Ω-conjugacy via approximations
  - Implications: Even after restricting functions to bounded orbits that stay near themselves, bumping that function can, in general give rise to different phenomenological behaviors

- Structure of parameter space
  - No one really knows yet besides the non-genericity of structural stability
  - Some possibilities
    * Persistent chaos
    * Persistent periodic orbits
    * Intermixed chaotic and period windows; can be bizarre, strange basin structure may exist for each set of parameter values; use your imagination to dream up pathologies, they will probability exist
  - Fragile conjecture

– Fragile

  * Λ is a chaotic attractor with k positive Lyapunov exponents
  * Λ is dispelled for g if ALMOST ALL points in a neighborhood of Λ belong to basins of attracting periodic orbits of g
  * If there exists $g \in N(f)$ such that Λ is dispelled for g then Λ is fragile
  * Given an n-parameter family of diffeomorphisms; the window set of f, W is the set of parameter values for which Λ is dispelled

- Windows conjecture: given $f : R^n \times R^m \to R^m$ with Λ having $k \geq 1$ positive Lyapunov exponents (invariant) that "exhibits" a fragile chaotic attractor
  * Given W is a typical window set for f_a $a \in R^n$
  * If $n < k$ then there exists $N(a)$ entirely outside of W
  * If $n = k$ W is dense in $N(a)$ but W is limited (limited means the "size" of the $w_i \in W$ shrinks as a is approached)
  * $n > k$ W is dense in $N(a)$ and W is extended (extended means not limited)
  * Number of positive Lyapunov exponents = the number of parameters needed to be perturbed to remove all the expanding directions
  * Number of parameters that determine the stability region determine the codimension in phase space needed to perturb stability away

- Problems
  - Defining "typical" f_a
  - "Size" if w_i
  - Defining parameters in an orthogonal way
  - Counterexamples: cat map, logistic map (?), etc...
  - Fixed point theorem
  - The words ALMOST ALL implies g has a chaotic attractor with k positive Lyapunov exponents implies that, with probability 1, every perturbation of $n > k$ parameters of g will result in a periodic orbit

- Examples:
  - Neural networks
    - Can approximate any Lebesgue integrable function (i.e. almost any function you can think of, including non-continuous functions) and it's derivatives
    - Always bounded
    - Pictures

* Observations from computer simulations
  - As the dimension is increased the number of chaotic cases increases
  - Many different attractors for the same set of parameters
  - Possible non-genericity of Ω and structural stability observed; i.e. very near a given function, there exist qualitatively different functions
  - Existence of very similar functions (in dimension, number of positive Lyapunov exponents, value of the largest Lyapunov exponent) near qualitatively different functions
  - Persistence of chaotic dynamics over a large portion of parameter space given high enough dimension
  - For high dimensions periodic windows are not observed for parameter perturbations of orders 10^2 to 10^{-8}

Conclusions: some problems with the current framework, and some re-assurances

- Problems:
  - When the models are perturbed qualitatively different behaviors can arise
  - "Structure of nature," impossible to pick out with complicated systems, i.e. high periodic, chaotic orbits and noise
  - High dimensional "fitting" of data being representative of the phenomena; with many parameters many models can be rationalized
  - In high dimensional models, connections with reality beyond stylized facts are much harder due to the diversity of possible behaviors
  - Many models begin to push the envelope of empiricism

- Reassurance
  - Qualitative effects can be captured
  - In neural networks with high dimension and number of neurons, while perturbations yielded qualitatively different behavior, the behaviors were not pathologically different
  - Parameters in a neighborhood that yield wildly different behaviors might be rare (robust chaos; extremely high periods vs. quasi-periodic or chaotic orbits)

Questions

- How do causal states and $\epsilon-$machines fit into this framework; are they fundamentally different and how might we show this?
- Approximation theorems for $\epsilon-$machines
- What are notions of equivalence between $\epsilon-$machines
- Are the result I presented for diffeomorphisms going to be fundamentally different for the space of $\epsilon-$machines