Generic Dynamical Behaviors and Transitions along Curves in Parameter Space in a Various Sets of Mappings: A Potpourri of Confusing Results

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Abstract

Every researcher using a computer to model any sort of phenomena is selecting their model equations from some set of abstract mappings. Further, nearly all of these models contain parameters whose variation provides important and interesting insight. In fact, often the very point of setting up the model is to discover what can happen upon parameter variation. In this talk, I will discuss a potpourri of results from mathematicians and numerical scientists regarding common types of dynamics, common transitions between dynamics, apparent contradicting results, open questions, and possible solutions - all along curves and surfaces in parameter space. Basically, I will try to paint a picture of what can happen dynamically in many models and will, of course, make my best attempt at defusing the techincal lingo whenever possible in an attempt to remove the obstruction between mathematicians and applied folks. Thus, this talk should be accessible and useful to a wide audience.

The Problem:

We study nature with models.

Our models have parameters.

How does changing the parameters lead to different dynamics, and what are the specific mechanisms for those changes.

What might typical changes or paths through parameter space be like?

What type of framework do we need so that we can begin making statements analytically.

My World

I will only discuss discrete time maps and ODEs.

All maps and ODEs are C^r, $r > 0$ - i.e. everything is as smooth as we wish.

All mappings are from compact sets to themselves - think squares to squares, spheres to spheres, tori to tori, etc.

Everything is at least a Riemannian manifold (i.e. has a Riemannian structure)

$$f : V \times U \to U \tag{1}$$

$U \subset R^n$, $V \subset R^m$, U is the state space, V is the parameter space

Flows versus Maps

Flows: continuous time - parameterized in time by R^1 (not to be confused with the set of "parameters")

Maps: discrete time steps - parameterized in time by N^1

$n-$dimensional flows can be "cut" by a transverse section forming the "time one" map which is $n-1-$dimensional (called a Poincare section). These cross sections are NOT unique.

$n-$dimensional discrete time map can be "suspended" to form $n+1$ dimensional flows

"Every" map has a suspension to a flow - but every flow does not have a time one map (e.g. the Reebs foliation). This is important.

Perturbations

Colloquially: knobs on a radio or some other type of mechanical device

Two types: general (abstract) C^r perturbations, and (concrete) parameter perturbations

General C^r perturbations are an abstract notion that is very useful for topological and geometric arguments (draw picture - close in the Whitney topology)

General C^r perturbations are with respect to the mappings, and may have not practical or concrete mechanism (a radio with invisible knobs that exist in parallel dimension and can affect the radio)

Parameter perturbations may or may not be C^r perturbations - but parameter perturbations are with respect to "concrete" knobs (e.g. viscosity in a fluid)

Often all possible parameter perturbations do not correspond to all possible C^r perturbations as all C^r perturbations include perturbation of the "functional form"

I will be speaking about perturbation of parameters.

Invariant sets and attractors

Sets that are invariant under the parameterization by time - e.g. orbits on tori, circles, squares, etc.

Invariant sets can be attractive - i.e. invariant once you have waited long enough for the transients to disappear - think of moth flake water ball things

Invariant sets can be "neutral" (think Yuzuro and Jim's Hamiltonian matching pennies cycles) - the "basin of attraction" is the invariant set.

$$\Lambda = \cap_t f^t(\Lambda) \tag{2}$$

For all $t > T$ for some $T > 0$

Basins of attraction

Draw picture Davo.

Set of initial conditions corresponding to the given invariant sets.

The basins are subsets of the compact set U; they partition U.

Each basin can have a very different type of attractor.

Each basin has (hopefully) a measure (used to calculate ergodic properties, such as Lyapunov exponents) associated with it - an SRB measure.

The partitioning can be very complicated - to the point of being "riddled"

Basins and attractors can "interact," there exist attractors for with the intersection of the basin boundary and attractor is a measurable set - possibly with positive measure (Milnor attractors - draw picture).

Often the basin structure varies with parameters - i.e. there can exist basin bifurcations upon parameter variation (more in a minute).

Bifurcations

Bifurcations = qualitative changes in the dynamics upon parameter variation. This is vague - being specific is a bit nasty.

3 types - local, global, basin (such as collisions) - I will refrain from a discussion of global bifurcations.

Local: flip (period doubles), fold (saddle + node collide and disappear), Hopf (flows) or Naimark-Sacker (maps) (fixed points to invariant circles with periods greater than 3 or quasi-periodic orbits).

There are many open questions left about bifurcations of periodic orbits. These questions are hard.

Basin: Basin collisions, basins vanishing, appearance of "riddled" basin structure - there are a lot of open questions left - much of the work was done by physicists. - Milnor attractors are "bifurcation points" in this scheme, only they might persist under parameter perturbation.

Much of the math required to for a good analysis of the basin type bifurcations is still in the "developmental" stage - i.e. the SRB measure stuff.

TWO EXAMPLES:

THE ROUTE TO TURBULENCE

THE ROUTE TO SENSORY BASED LEARNING

Landau-Hopf route to turbulence in fluids

Bifurcation parameter - viscosity.

Equations: Flows.

Forget that you heard anything about basin style bifurcations - you only know about local bifurcations.

Cascade of tori - i.e. fixed point $\rightarrow$ invariant circle $\rightarrow$ T^2 (two torus) $\rightarrow$ T^3 $\rightarrow \ldots \rightarrow$ turbulence/chaos.

Or: μ_i is the bifurcation parameter and $\mu_i < \mu_{i+1}$, $x_{\mu_1}(t) = f(\omega_1, \omega_2)$, $x_{\mu_2}(t) = f(\omega_1, \omega_2, \omega_3)$, ..., $x_{\mu_{k-1}}(t) = f(\omega_1, \omega_2, \ldots, \omega_k)$ where the frequencies are not rationally related.

Yikes.

Enter the topologists: Peixoto

Flows on M^2 - the only generic (generic = common topologically speaking) (i.e. a property is generic if it exists on subset $E \subset B$, where E contains a countable intersection of open sets that are dense in the original set B) behaviors are fixed points and periodic orbits.

The manifolds must be orientation preserving - i.e. no Mobius strips or Klein bottles.

Think orbits on tori or spheres.

The same goes for discrete time maps of the circle, the only structurally stable (and generic) orbits are fixed points and periodic orbits. (in the C^1 norm - and again orientation preserving)

Ruelle - Takens - Newhouse

Landau was nutty, and his cascade of tori will never (in a topological sense of common) happen.

R and T Gave an independent proof of the Neimark-Sacker normal form theorem (Naimark was a Russian who they didn't know about and Sacker's proofs existed on as notes in the New York Library of Science (i.e. this thesis))

I.e. Ruelle and Takens proved a theorem speaking to the bifurcation of a MAP from a fixed point to a period orbit (of period greater than 5) or a quasi-periodic orbit.

A bifurcation of a map from a fixed point to a quasi-periodic orbit corresponds to the bifurcation of an invariant circle of a flow to T^2 - a two torus.

By Peixoto, this bifurcation should be to that of a periodic orbit.

Newhouse, Takens, and Ruelle (a generalization of a result of the T-R paper) showed that for quasi-periodic orbit of a FLOW on tori of dimension 3 or greater, there is an open set such that a C^2 perturbation will yield a strange attractor (i.e. a chaotic attractor).

We understand everything now yes?

For flows it goes: fixed point $\rightarrow$ periodic orbit $\rightarrow$ chaos (the quasi-periodic three tori are unstable - and can be perturbed away)

For maps it goes: fixed point to chaos

Right?

Right?

Of COURSE NOT.

Enter the Russians, a Frenchman, and a German (Measure Theorists)

Quasi-periodic orbit on tori play an important roll in the KAM celestial mechanics problems - hence K and A (M was German).

For maps of the circle, the measure one (measure = a different notion of common) dynamics are quasi-periodic orbits.

For FLOWS on a two torus, the orbit is periodic if and only if it's coordinates are RATIONAL - and the rational coordinates are a zero measure set. THIS IS IMPORTANT!

SHOOT. WHAT THE HECK PEIXOTO?

Moral: different notions of common give different answers to the question "what is common?"

The Russians say for FLOWS, the route should be something like: fixed point $\rightarrow$ quasi-periodic orbit on a two torus $\rightarrow$ who the heck knows, because bifurcations from quasi-periodic orbits are still hazy and there will not be a Newhouse-Takens-Ruelle style theorem for T^2.

Of course just because there exists some type of common behavior on T^2 doesn't imply that a map can bifurcation to that type of behavior.

Also, all the quasi-periodic orbits are structurally unstable, i.e. they can be perturbed to period orbits or different quasi-periodic orbits by a perturbation of any size.

Enter some French guys

Chenciner and Iooss proved that bifurcations of FLOWS from T^2 (with two frequencies) to T^3 (i.e. three frequencies) is highly non-generic (or very unlikely topologically speaking.) These bifurcations seem likely from a measure theoretic standpoint also.

Made some conjectures about routes to turbulence I will refrain from stating.

Of course I am skipping a lot of stuff.

Enter a CRAZY Russian, a Chinese man, and an American

Random Matrix Theory the Circular law

Given a $n \times n$ matrix whose elements are drawn from a distribution with a finite sixth moment (e.g. Gaussian normal distribution), the normalized spectra will tend toward a uniform distribution on unit disk in the complex plane as $n \to \infty$

Relevance: Naimark-Sacker bifurcations correspond to complex eigenvalues of the Jacobian of the fixed point hitting and transversally crossing the unit circle. If the angle is rational, the orbit will be periodic, if the angle is irrational, the orbit will be quasi-periodic (a la the Russians irrational/rational coordinates theorems).

Problem: the convergence in distribution is not absolutely continuous with respect to Lebesgue measure. Oh well.

Enter: Us (Clint, Dee, Me)

As the dimension of the dynamical system is increased, any time a Jacobian can be constructed, and the elements have a finite sixth moment, the eigenvalues should be nearly uniform on the unit disc, and thus it SEEMS probable that the most likely bifurcation will be a Neimark-Sacker to a quasi-periodic orbit.

For bifurcation from fixed points of maps, this doesn't seem so bad.

Then what? Suspensions of the map, vector field approximations - it is a mess, and we (read everybody) don't know what happens.

What can we manage to observe?

(This is not the original reason we began studying neural networks, just so you know.)

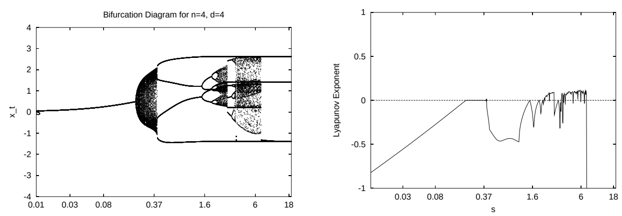
Figures...

Dynamical systems: Neural networks - very general discrete time MAPS.

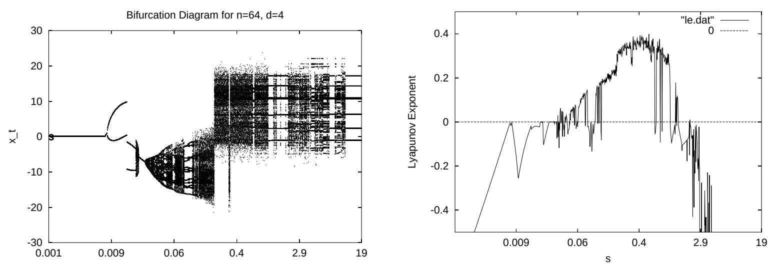
Examples of bifurcation diagrams and Lyapunov spectra, - these cases do not correspond to each other.

This is the real world - so there are multiple basins, multiple attractors, many SRB measures (corresponding to different Lyapunov spectra), etc.

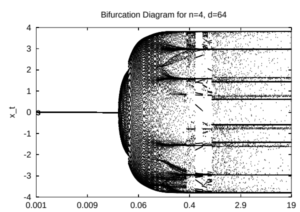
Some of this stuff is actually running at the moment.

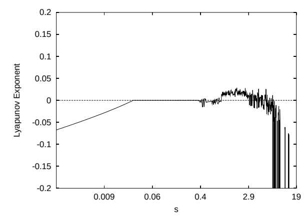


Bifurcation diagram with the largest Lyapunov exponent for $n = 4$ and $d = 4$.

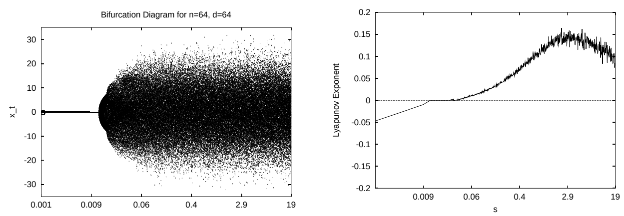


Bifurcation diagram with the largest Lyapunov exponent for $n = 64$ and $d = 4$.



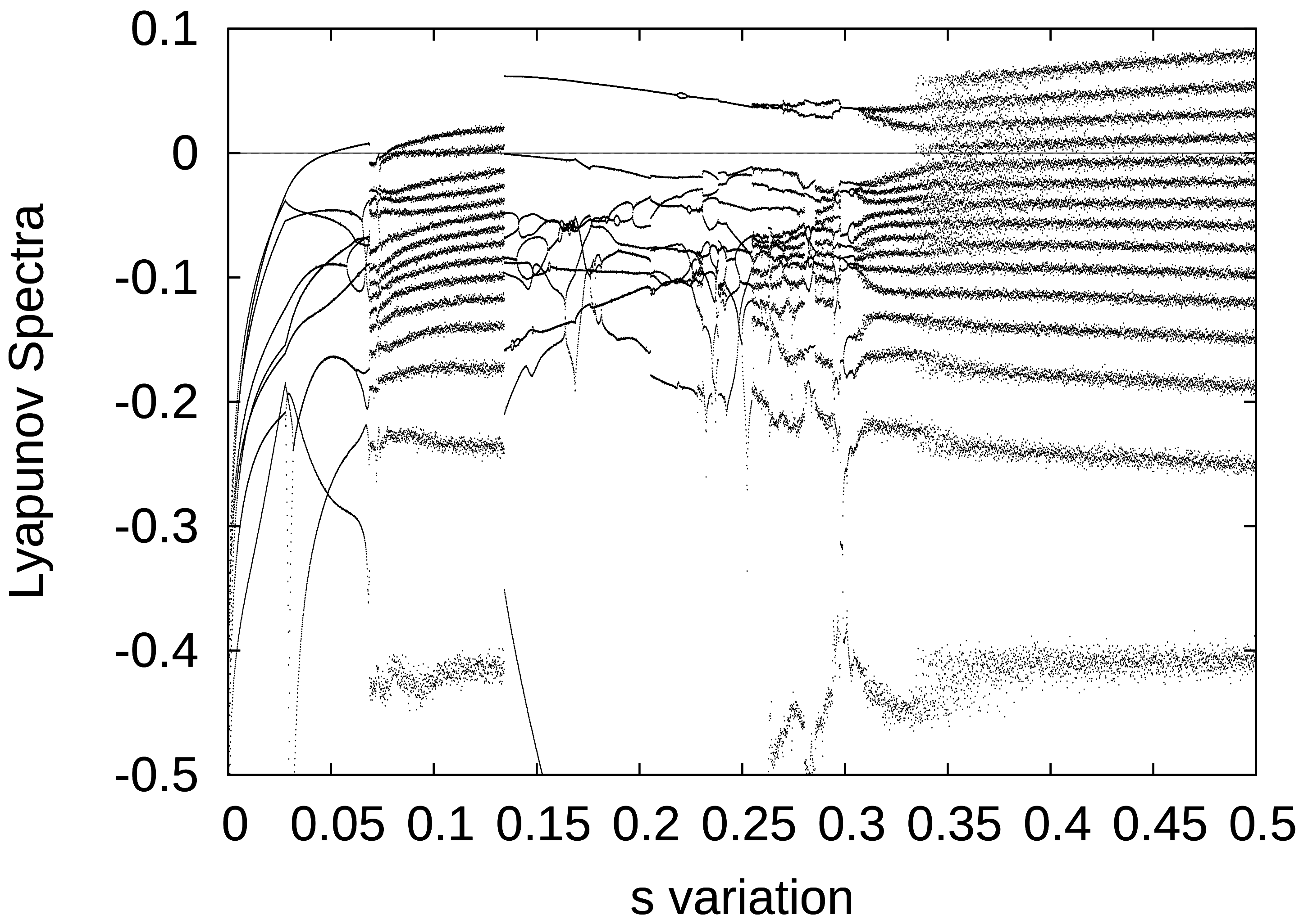


Bifurcation diagram with the largest Lyapunov exponent for $n = 4$ and $d = 64$.

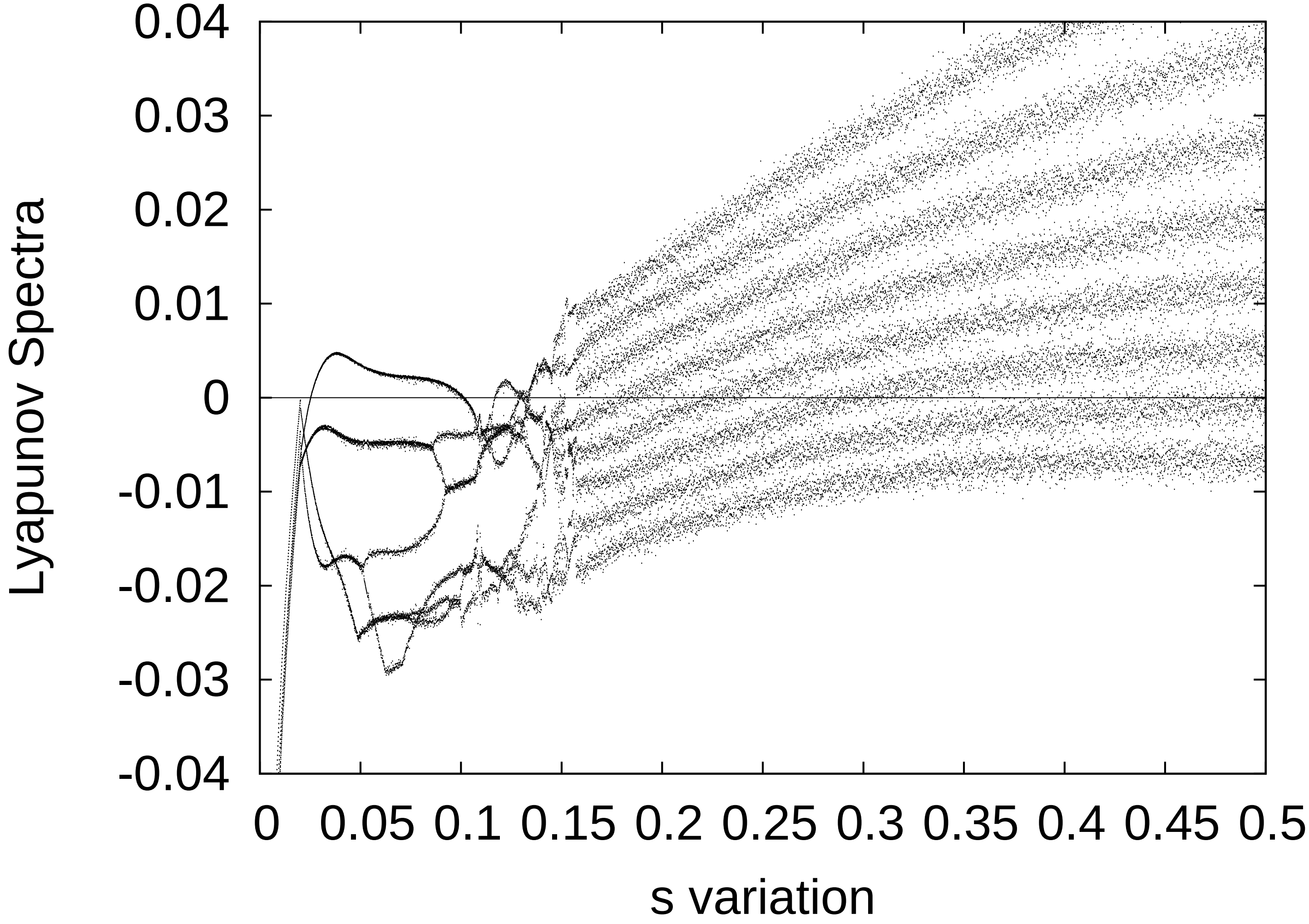


Bifurcation diagram with the largest Lyapunov exponent for $n = 64$ and $d = 64$.

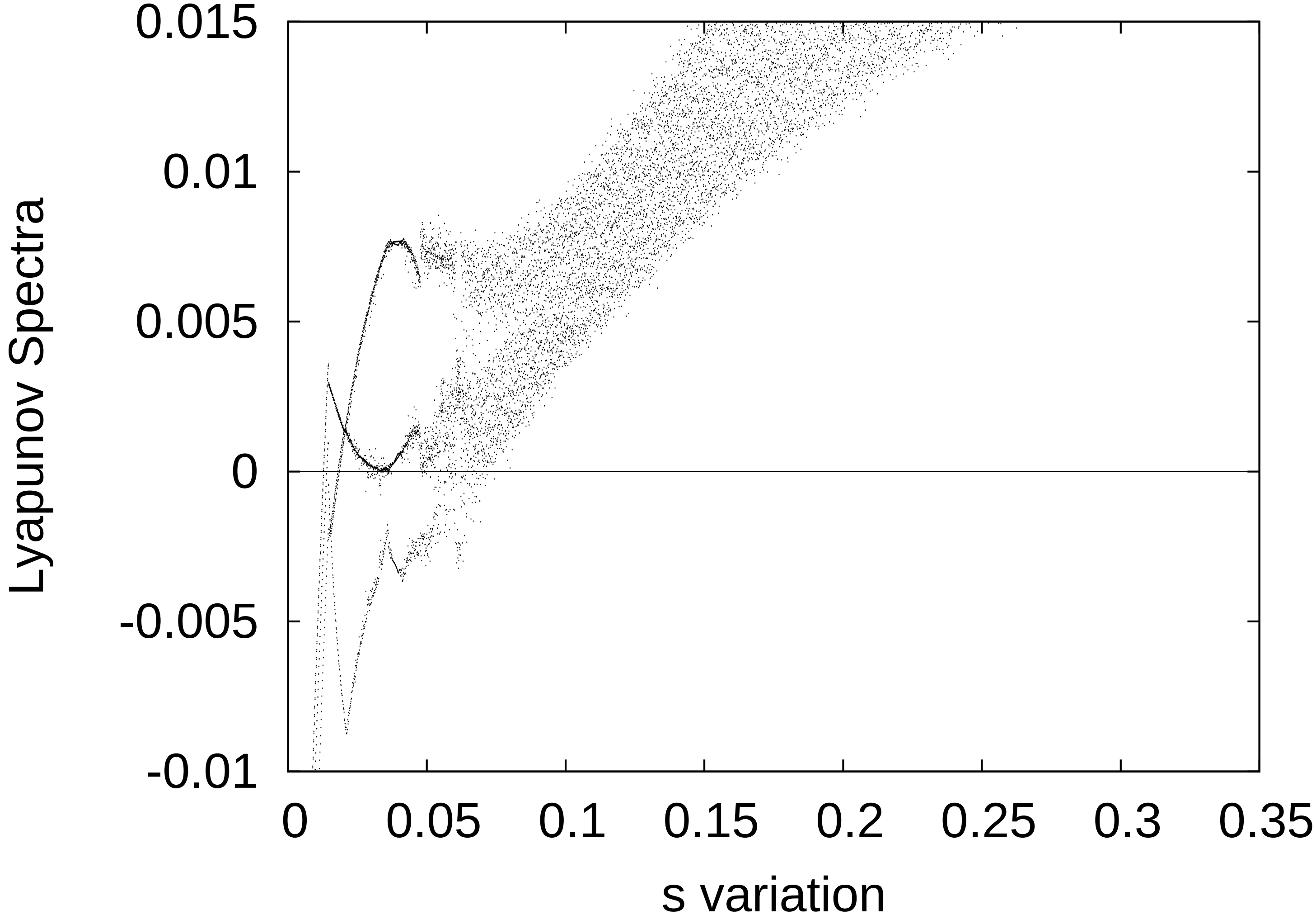
Bifurcation diagram for an typical network; $n = 32$, $d = 16$.



Bifurcation diagram for an atypical network; $n = 32$, $d = 32$.



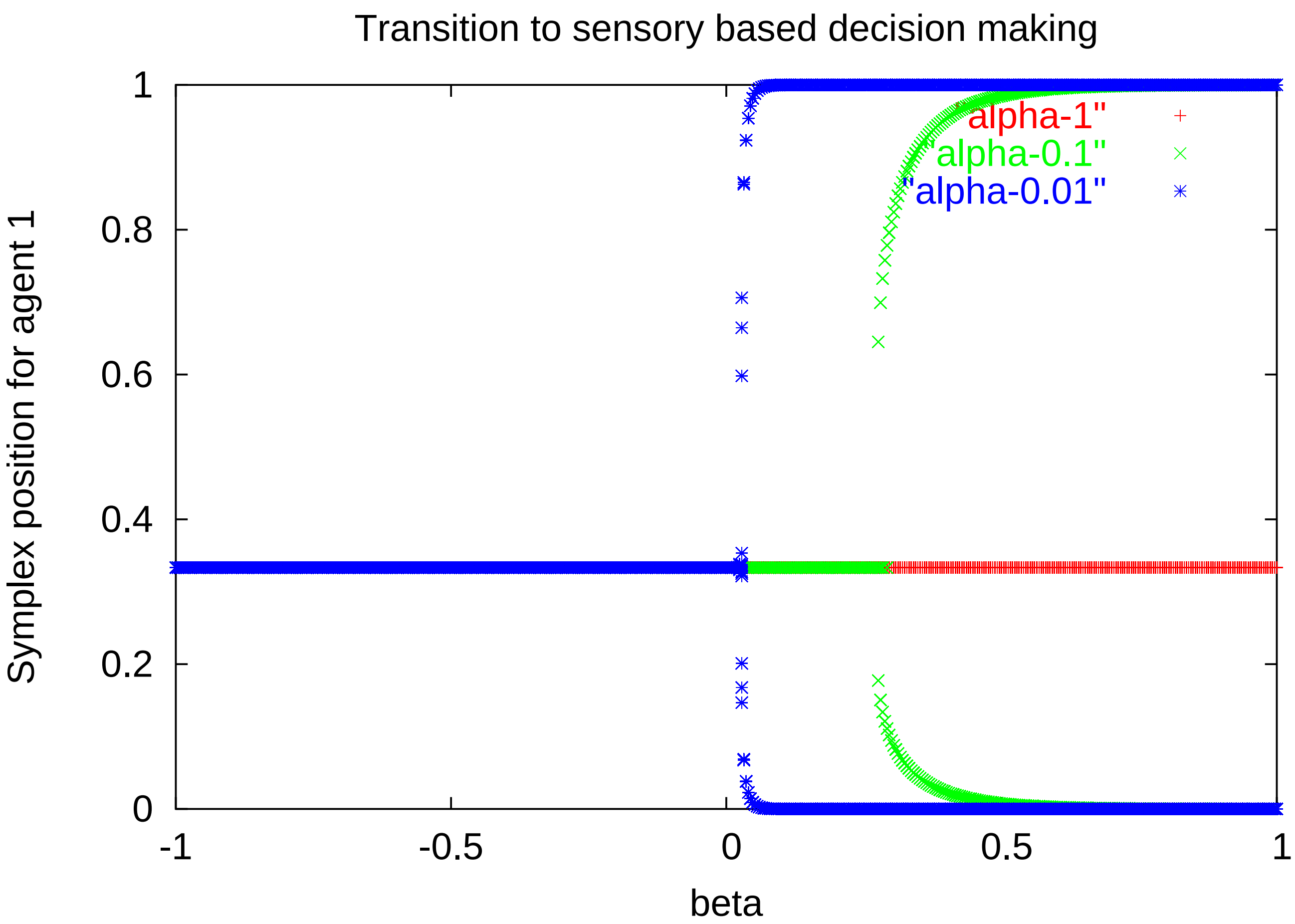
Bifurcation diagram for an typical network; $n = 32$, $d = 64$.



THE ROUTE TO SENSORY BASED LEARNING

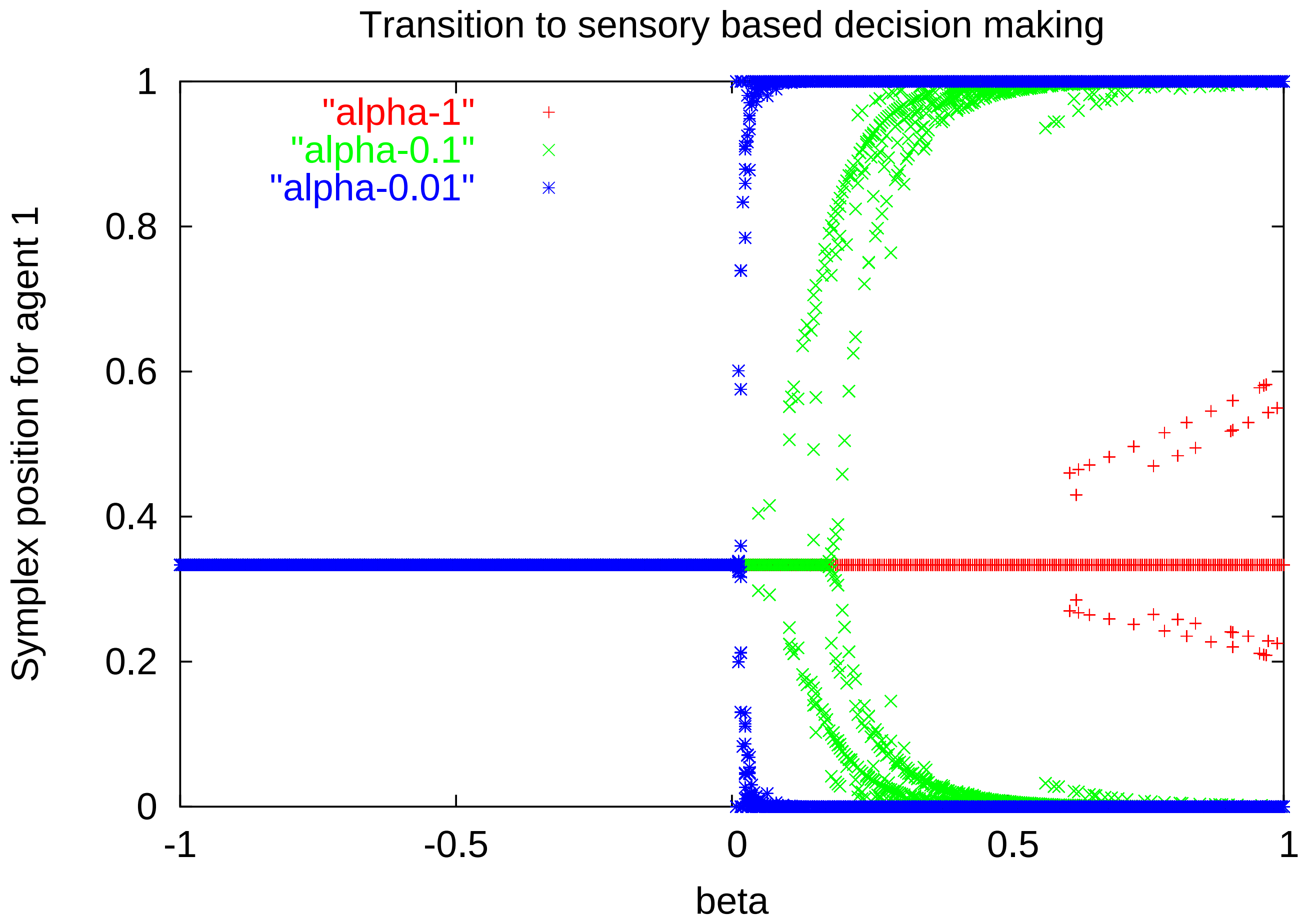
A problem with basins - no chaos - just qualitative change along a curve/surface in parameter space.

No dependence on other agents:



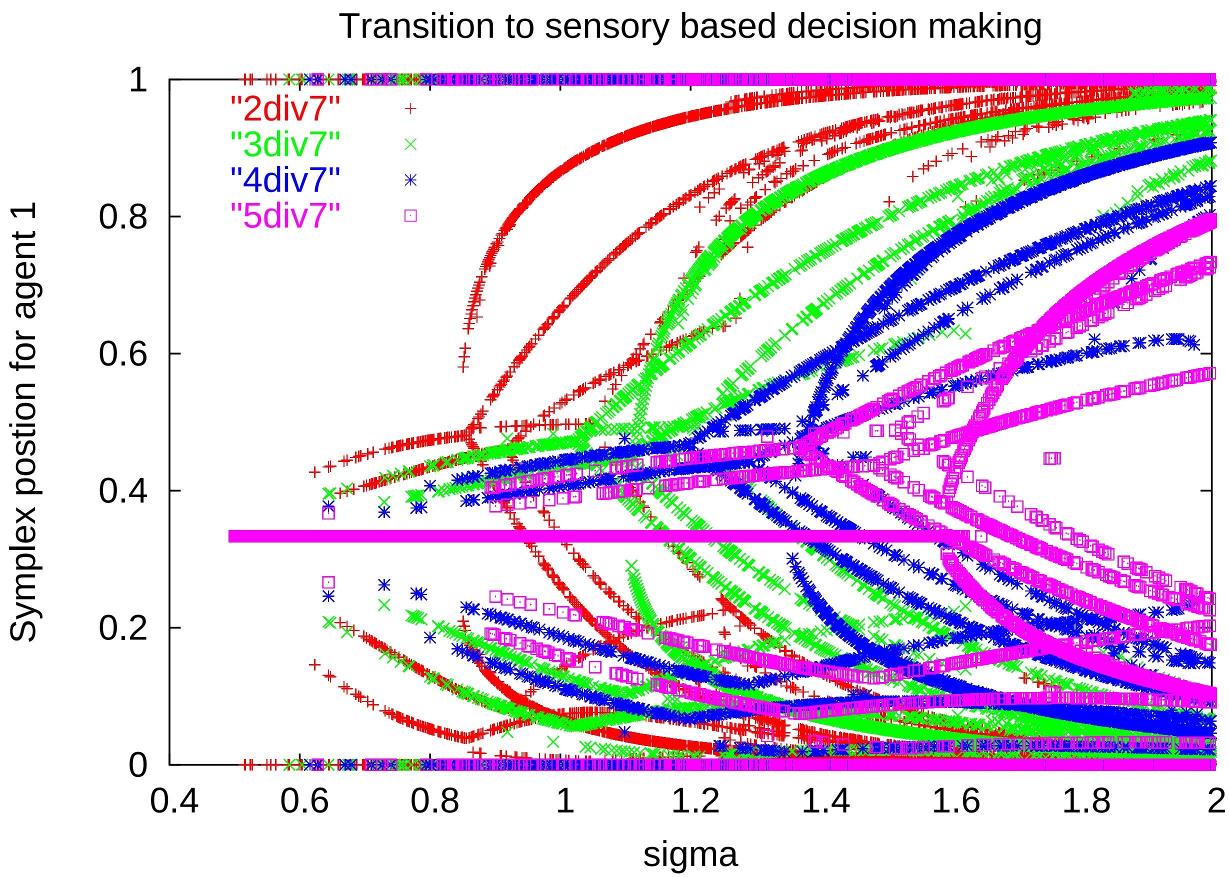
Add dependence of agents upon other agents:

Example: we have no idea what is happening yet - but we have just started:



Add variable dependence of agents upon other agents:

Examples: again, we have no idea what is happening yet - but we have just started:



Final Remarks

We are all still pretty confused about how this all goes

Lots of interesting open problems and work to be done...