Structural Stability and Partial Hyperbolicity in Large Dynamical Systems

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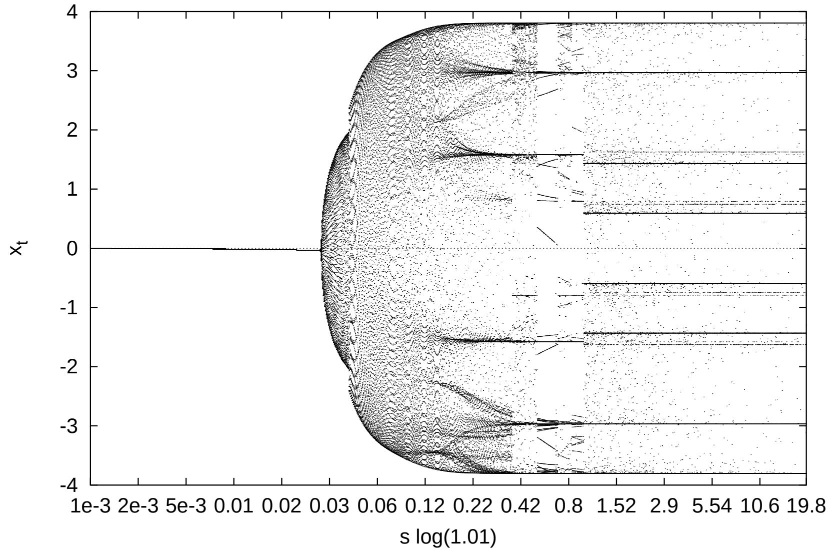
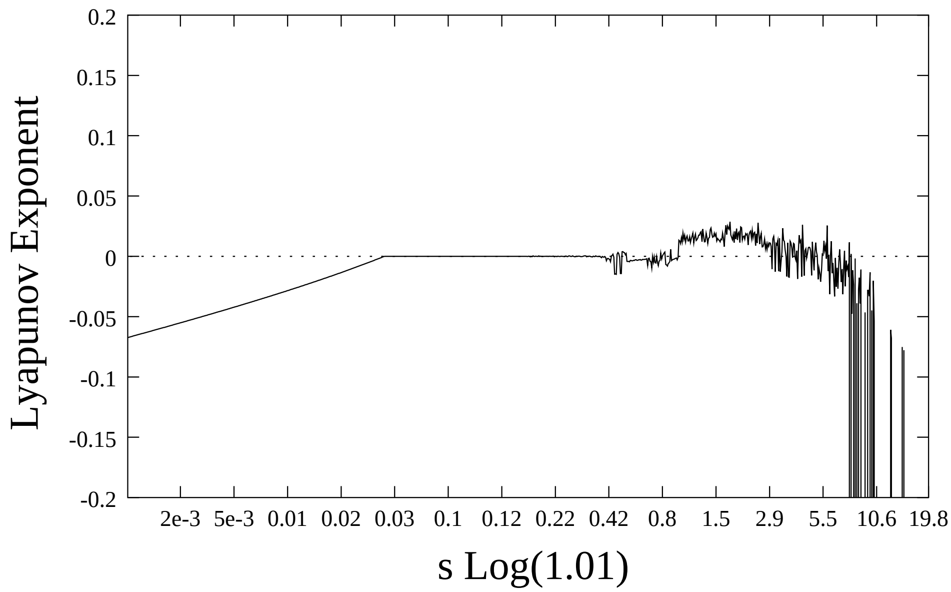
Outline

- Initial motivation
- Preliminary notions and definitions
- Background and explanation of the problem
- Outline of arguments
- Arguments
- Other directions of work and summary

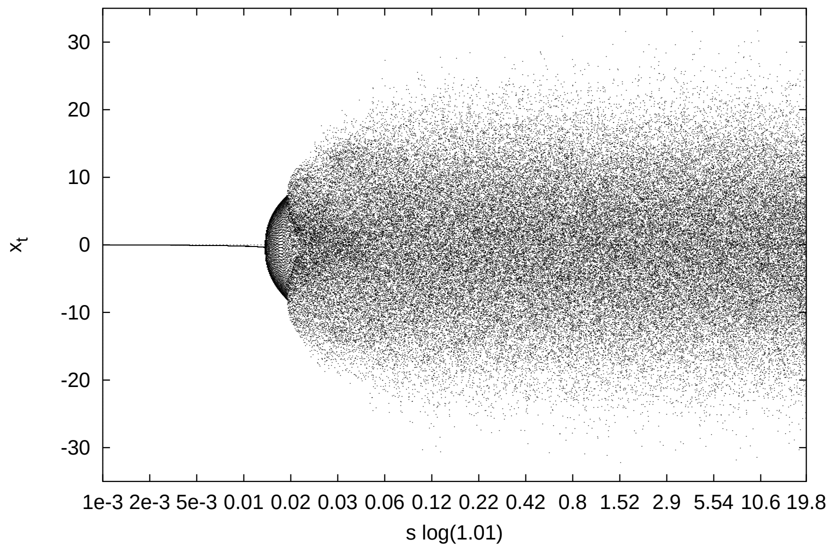
Source of motivation

- Bifurcation diagram and largest Lyapunov exponent for $n = 4$ and $d = 64$

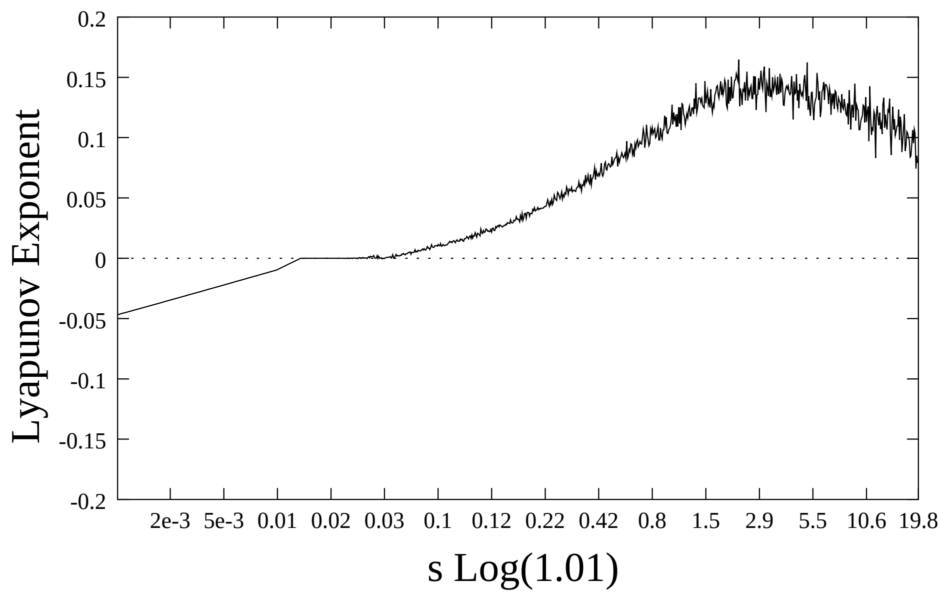




- Bifurcation Diagram for $n = 64$ and $d = 64$



- Lyapunov exponent for $n = 64$ and $d = 64$



Preliminary notions

- Stable (E^s), unstable (E^u), and center manifolds (E^c): defined with respect to fixed points, define boundaries between different orbit types, define the geometry of orbits.
- Pictures

- Topological conjugacy:

  Definition 1 (Topological conjugacy) *Consider two C^r diffeomorphisms $f : R^n \rightarrow R^n$ and $g : R^n \rightarrow R^n$. f and g are said to be C^k ($k \geq r$) conjugate if there exists a C^k diffeomorphism such that $g = h \circ g \circ h^{-1}$. If $k = 0$, f and g are said to be topologically conjugate.*

- Structural stability:

  Definition 2 (Structural Stability) *A C^r discrete time map, f, is structurally stable if there is a C^r neighborhood, V, such that any $g \in V$ is topologically conjugate to f, i.e. there exists a homeomorphism h such that $f = h^{-1} \circ g \circ h$.*

- Pictures

- Perturbation: three types - functional form, parameter variation, initial condition variation

- Hyperbolicity:

  Definition 3 (Hyperbolic linear map) *A linear map of R^n is called hyperbolic if all of it's eigenvalues are different from one.*

  Definition 4 (Hyperbolic periodic point) *p is a hyperbolic periodic point for f if $(Df^n)_p : T_pM \rightarrow T_pM$ is a hyperbolic linear map. It's orbit will be called a hyperbolic periodic point.*

  Definition 5 (Hyperbolic map) *A discrete time map f is said to be hyperbolic on a compact invariant set Λ if there exists a continuous splitting of the tangent bundle, $TM|_\Lambda = E^s \oplus E^u$, and there are constants $C > 0$, $0 < \lambda < 0$, such that $||Df^n|_{E^s_x}|| < C\lambda^n$ and $||Df^{-n}|_{E^u_x}|| < C\lambda^n$ for any $n > 0$ and $x \in \Lambda$.*

  where the stable manifold E^s [respectively unstable E^u] of $x \in \Lambda$ is the set of points $p \in M$ such that $|f^k(x) - f^k(p)| \rightarrow 0$ as $k \rightarrow \infty$.

- Lyapunov exponents: correspond to stable, unstable, and center manifolds OF AN ORBIT; measure rates of expansion and contraction

Definition 6 (Lyapunov Exponents) *Define the discrete dynamical system $f : R^n \to R^n$ and a point in the domain, $x \in R^n$. Suppose there are subspaces $V_i^{(1)} V_i^{(2)} \cdots V_i^{(n)}$ in the tangent space of $f^i(x)$ and scalars $\chi_1 \leq \chi_2 \leq \cdots \leq \chi_n$ with the properties:*
1. $Df(V_i^{(j)} = V_{i+1}^{(j)}$
2. $dim V_i^{(j)} = n + 1 - j$
3. $\lim_{N\to\infty} \ln ||\sqrt{(Df^N)^T(Df^N)} \cdot v|| = \chi_j$ *for all* $v \in V_0^{(j)} - V_0^{(j+1)}$

The numbers χ_j are called the Lyapunov exponents of f at x.

$$\chi_j = \lim_{N\to\infty} \frac{1}{N} \Sigma_{k=1}^N \ln(< (Df_k \cdot \delta x_j)^T, (Df_k \cdot \delta x_j) >) \quad (1)$$

where $<, >$ is the standard inner product, δx_j is the j^{th} component of the x variation and Df_k is the "orthogonalized" Jacobian of f at the k^{th} iterate of $f(x)$.

- Non-wandering set:

  Definition 7 (Non-wandering set) *A point x_0 is called non-wandering if the following holds for any neighborhood U of x_0 for some $n \neq 0$:*

$$g^n(U) \cap U \neq 0 \qquad (2)$$

  The set of all such points is called the non-wandering set.

- Attractor (or orbit):

  Definition 8 (Attractor) *A closed invariant set $\Lambda \subset R^n$ is called an attracting set if there is some neighborhood U of Λ such that:*

$$g^n(x) \in U \text{ and } g^n(x) \to \Lambda \text{ as } n \to \infty \qquad (3)$$

- Dense periodic orbits: The given the invariant set (attractor) Λ has dense (maybe not stable) periodic orbits.

- Strong transversality: YIKES - $M_x = E^s_x + E^u_x$ for all $x \in M$ - i.e. a continuous "splitting" of the manifolds into E^s and E^u

- Axiom A:

  Definition 9 (Axiom A) *A C^r f satisfies axiom A if and only if Ω is hyperbolic and the periodic points of f are dense.*

- Structurally stable dynamical systems satisfy axiom A (most importantly are hyperbolic):

  Theorem 1 (Mane - theorem A) *Every C^1 structurally stable diffeomorphism of a closed manifold satisfies Axiom A.*

- Axiom A and strong transversality guarantees structural stability:

  Theorem 2 (Robbin - Structural Stability Theorem) *A C^2 diffeomorphism (on a compact, boundaryless manifold) which atisfies axiom A and the strong transversality conditions is structurally stable.*

- Partially hyperbolic dynamical systems

Definition 10 (Partial hyperbolicity) *The discrete time map f is said to be partially hyperbolic if the tangent bundle TM splits as a $Tf-$invariant sum:*

$$TM = E^U \oplus E^C \oplus E^S \tag{4}$$

where at least two of the sub bundles are non-trivial, and four constants, $a < b < 1 < c < d$, and a Finsler structure $||\cdot||$ on M such that for all $x \in M$, and all $v \in T_xM$:

$$v \in E^U(x) \Rightarrow d||v|| \leq ||T_xfv|| \tag{5}$$

$$v \in E^C(x) \Rightarrow b||v|| \leq ||T_xfv|| \leq c||v|| \tag{6}$$

$$v \in E^S(x) \Rightarrow ||T_xfv|| \leq a||v|| \tag{7}$$

Where E^U, E^S, E^C are the unstable, stable and center bundles for f, and a Finsler structure on the tangent bundle can be defined:

Definition 11 (Finsler structure) *A Finsler strucure on the tangent bundle of a Banach manifold M is a continuous function $||\cdot|| : TM \to [0, \infty)$ such that:*

(i) For every $x \in M$, the restriction $||\cdot||_x = ||\cdot|||_{T_xM}$ is and euivalent norm on the tangent space T_xM,

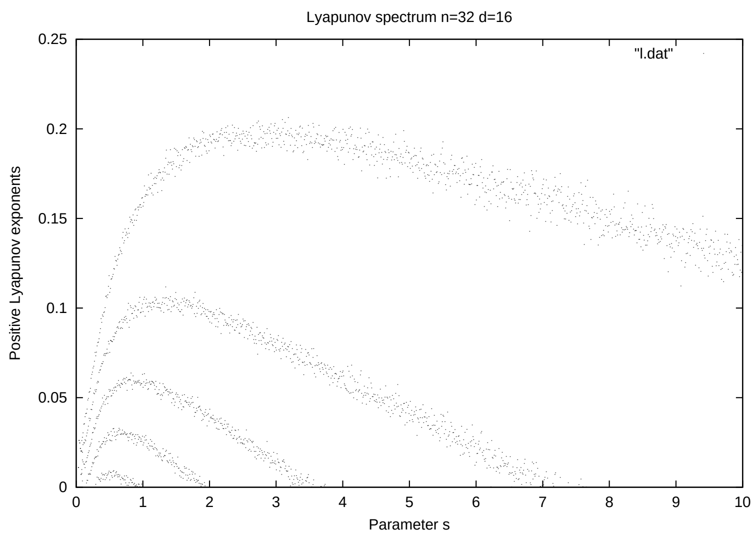
(ii) For every $x_0 \in M$*, and* $k > 1$*, thre is a trivializing neighborhood* U *of* x_0 *within which*

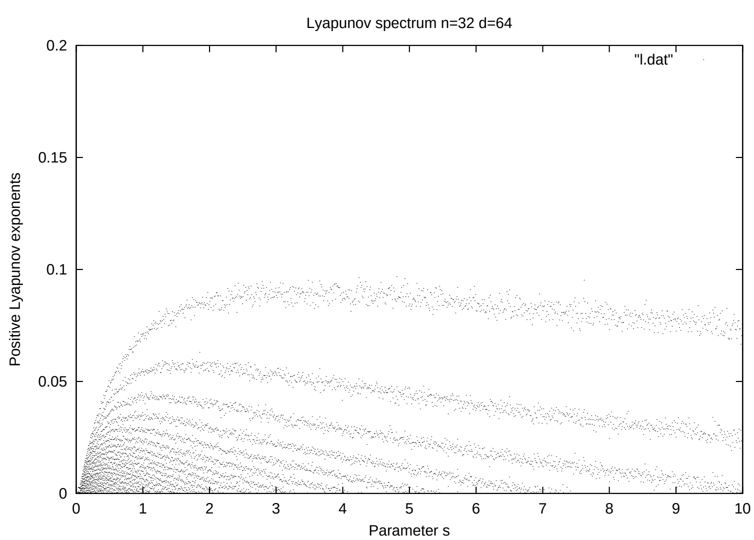
$$\frac{1}{k}||\cdot||_x \leq ||\cdot||_{x_0} \leq k||\cdot||_x \qquad (8)$$

A C^1 *Banach manifold* M *together with a Finsler structure on it's tangent bundle is said to be a Finsler manifold.*

Intution

- Assume C^r one parameter discrete time maps transforming bounded subsets of R^d to themselves



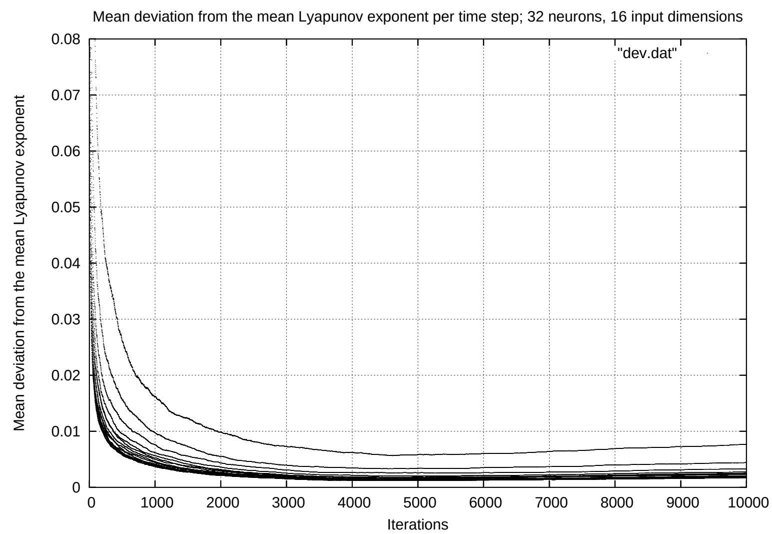


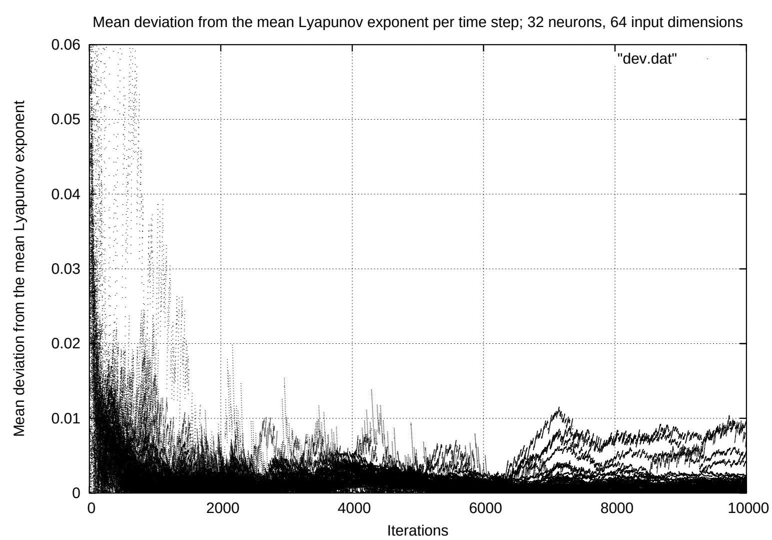
Outline of arguments: i.e. the ingredients

- As d increases we need:
  - increasing continuity of Lyapunov exponents
  - increase in the maximum number of Lyapunov exponents
  - decrease in the distance between exponent zero crossings
- With the above trends, given arbitrarily high d, we can find a subset in parameter space such that we can approximate violation of the stability conjecture
- Difference between strict mathematics and computational or experimental observation

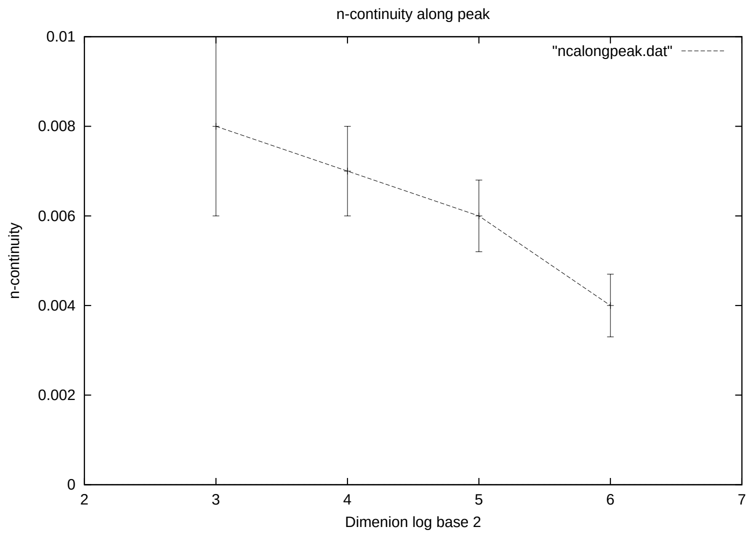
Numerical arguments: i.e. making sure we are seeing what we think we are seeing

- Error in Lyapunov exponent calculation:

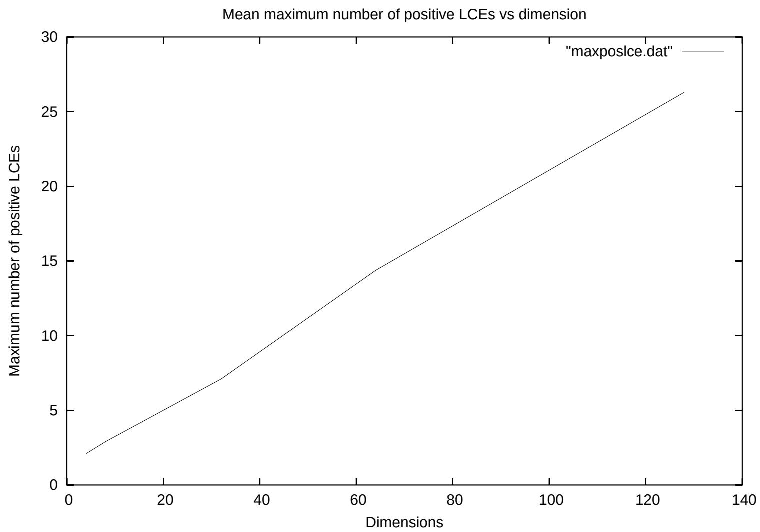




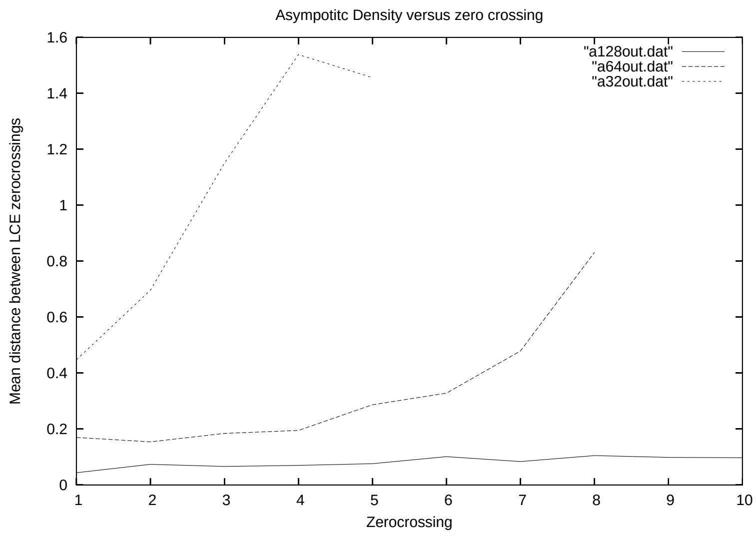
• Continuity of Lyapunov exponents



- Increase in the number of exponents



- Decrease in the distance between exponent zero crossings



- Upon varying our parameter we see:
  - Hyperbolicity violation on an “increasingly” dense - yet not open and Lebesque measure zero - set.
  - I.e. we can find a subset of parameter space such that as the dimension is increased, the “chance” of topological change versus “small” parameter variation becomes small (zero codimension bifurcation volume)
  - With increasing dimension, very low probability of periodic windows in certain subsets of parameter space

Future work

- General: understand the effects of perturbation of initial conditions and d parameter variation
- Achieve a better understanding of the seemingly robust nature of chaos and the non-existence of periodic windows - specifically Yorke's windows conjecture.
- Achieve a better understanding of the basins of attraction - existence of Milnor attractors, effects of SRB measures, and partial hyperbolicity.
- Achieve an understanding of the route out of chaos - the "high s" limit