Vol. 42, No. 3 DUKE MATHEMATICAL JOURNAL© September 1975

SOME INEQUALITIES FOR THE SPECTRAL RADIUS OF NON-NEGATIVE MATRICES AND APPLICATIONS

by

S. FRIEDLAND AND S. KARLIN

Contents

1. Introduction. The purpose of this work is to establish useful lower and upper estimates for the spectral radius of certain classes of positive matrices which apart from their independent interest are pertinent to the study of a number of mathematical models of population genetics and also apply to the solution of some cases of inverse eigenvalue problems.

In the stability analysis of certain equilibria states of physical and biological systems, it is relevant to determine useful conditions indicating when the largest eigenvalue ρ for a matrix of the type MD (composed from a general non-negative and positive diagonal matrix) exceeds or is smaller than 1. In the physical setting, $M = ||m_{ij}||$ is commonly an $n \times n$ matrix of non-negative elements corresponding to a Green's function for a vibrating coupled mechanical system of n mass points, while D is a diagonal matrix with positive diagonal entries $\{d_1, d_2, \cdots, d_n\}$ such that d_i, $i = 1, 2, \cdots, n$, relates to the mass at position i.

In the genetics context, a population is distributed in n demes (habitats, $\{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n\}$) subject to local natural selection forces and inter-deme migration pressures. The changes in the population composition of a trait expressed by two possible types (genes) labeled **A** and **a** are observed over

Received March 13, 1975. The first author supported in part by NSF-MPS72-05055-A02; second author supported in part by NSF-MPS71-02905-A03.

successive generations. The transformation of gene frequency accountable to the local selection forces in deme $\mathcal{P}_i$, is characterized by a non linear relation

$$\xi' = f_i(\xi)$$

such that if ξ is the A-frequency in $\mathcal{P}_i$ at the start of a generation then after the action of mating and natural selection the resulting A-frequency prior to migration is ξ'. Generally $f_i(\xi)$ defined for $0 \le \xi \le 1$ is continuously differentiable and monotone increasing obeying the boundary conditions

$$f_i(0) = 0, \qquad f_i(1) = 1 \tag{1.1}$$

signifying that selection operates to maintain a pure population composition. (Thus, in this formulation mutation events are ignored, i.e., new mutant forms arising in the time frame under consideration cannot be established.)

The dispersal (= migration) pattern is described by the matrix $M = \|m_{ij}\|$ where m_{ij} is the proportion of the population in $\mathcal{P}_i$ immigrating from deme $\mathcal{P}_j$. Thus, from the interpretation, M is a stochastic matrix. Let x_i denote the proportion of type A in deme i at the start of a generation and x_i' the frequency for the next generation. The global transformation equations connecting $\mathbf{x} = (x_1, \cdots, x_n)$ to $\mathbf{x}' = (x_1', \cdots, x_n')$ over two successive generations takes the form

$$x_i' = \sum_{k=1}^{n} m_{ik} f_k(x_k), \qquad i = 1, 2, \cdots, n. \tag{1.2}$$

If the migration and selection forces operate in reverse order then the transformation equations become

$$x_i' = f_i\left(\sum_{k=1}^{n} m_{ik} x_k\right), \qquad i = 1, 2, \cdots, n. \tag{1.3}$$

We abbreviate the transformation (1.2) by $\mathbf{x}' = T\mathbf{x}$. A tacit assumption underlying the models (1.2) and (1.3) is that the deme sizes are large and approximately of equal magnitude. (For variations and other facets of the general model concerned with population subdivision and selection migration interaction, we refer to Karlin [9].)

Owing to (1.1) we find that the frequency states $\mathbf{0} = (0, 0, \cdots, 0)$ and $\mathbf{1} = (1, \cdots, 1)$ are invariant points of T corresponding to fixation of the population consisting exclusively of the **a**-type and **A**-type, respectively. The gradient matrix of T at $\mathbf{0}$ (that is the local linear approximation valid about $\mathbf{0}$) reduces to

$$T'(\mathbf{0}) = MD \tag{1.4}$$

where

$$D = \operatorname{diag}\,(f_1'(0), f_2'(0), \cdots, f_n'(0)).$$

Manifestly, $f_i'(0) \ge 0$ since f_i is increasing. We will stipulate henceforth strict inequality (reflecting the usual phenomenon for selection forces, so that

in the previous notation $d_i = f_i'(0)$, $i = 1, 2, \cdots, n$. Because T is a monotone transformation (as each $f_i(x)$ is monotone), the *local instability of the equilibrium state* $\mathbf{0}$ *is assured if the spectral radius*

$$\rho(MD) = (\text{the spectral radius of } MD) \textit{ exceeds } 1. \tag{1.5}$$

In the presence of (1.5) for any $\mathbf{x}$ close to but distinct from $\mathbf{0}$ the iterates $T^k\mathbf{x}$ will depart from the neighborhood of $\mathbf{0}$. The same considerations show that $\mathbf{0}$ is locally stable if $\rho(MD) < 1$.

The delineation of natural conditions satisfied by the selection coefficients (covered in D) and their coupling to the migration rates embodied in M that imply $\rho(MD) > 1$ are of much interest. In fact where the spectral radius exceeds 1, then the A-type is "protected", i.e., this type is maintained in the population and can never approach extinction.

The gradient matrix at $\mathbf{0}$ for the model (1.3) is DM. In view of the familiar property that the eigenvalues of DM and MD coincide and are identical to the eigenvalues of $D^{\frac{1}{2}}MD^{\frac{1}{2}}$, all criteria for "protection" of the A-type is the same for the models (1.2) and (1.3). Thus the operational *order* of the selection and migration forces does not influence the occurrence of "A-protection" or "A extinction." (See Karlin [9] for other mathematical developments and extensive discussions of these models.)

The classical *inverse eigenvalue problems* have the following formulation. Let L be an ordinary differential operator on R = real line. Find a "potential" $g(x)$ such that the operator $Ki = (Lu)(x) + g(x)u(x)$ coupled with appropriate boundary conditions possesses a prescribed spectrum. Another version of the problem states: Determine a "density" $p(x)$ such that the operator $Ku = [Lu(x)/p(x)]$ with associated boundary conditions involves a prescribed spectrum. A voluminous literature exists on this subject, e.g., a good survey is contained in Chap. 1 of Hald [5]. Some discrete matrix analogs of these problems take the following form.

(i) Let an $n \times n$ matrix M be given. Determine a diagonal matrix D with the property that the spectrum of $M + D$ coincides with a prescribed set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. This problem is henceforth referred to as the *inverse additive eigenvalue problem* (I.A.E.P.).

(ii) If $M + D$ is replaced by MD then we are confronted with the inverse multiplicative eigenvalue problem (I.M.E.P.).

For some references and elaborations on certain aspects of these problems, see Friedland [3].

It is known that both inverse matrix problems generally do not admit unique solutions and if the allowable D matrices are required, say, to be positive definite (or real) then cases of no solution arise. In the complex domain (i.e., where D is permitted to be a diagonal complex matrix) generally $n!$ solutions are available. We will construct in Section 6 a class of special spectra Λ related to M for which problems (i) and (ii) admit a unique real solution.

With the motivation well rooted, we next highlight some of the principal results of this paper.

Let $M = \|m_{ij}\|_1^n$ be a non-negative matrix of order n. Denote by $\rho(M)$ the spectral radius of M. The classical Perron-Frobenius theorem tells us that $\rho(M)$ is an eigenvalue of M with the property that there exist non-trivial non-negative vectors $\mathbf{u} = (u_1, \cdots, u_n)$ and $\mathbf{v} = (v_1, \cdots, v_n)$ satisfying

$$M\mathbf{u} = \rho(M)\mathbf{u}, \qquad \mathbf{v}M = \rho(M)\mathbf{v}. \tag{1.6}$$

Moreover, if M is also irreducible then $\rho(M)$ is a positive simple eigenvalue and the associated eigenvectors $\mathbf{u}$ and $\mathbf{v}$ display only positive components.

With $\mathbf{u}$ *and* $\mathbf{v}$ *prescribed positive vectors and normalized to satisfy*

$$\sum_{i=1}^{n} u_i v_i = 1 \tag{1.7}$$

we denote by $\mathfrak{M}(\mathbf{u}, \mathbf{v})$ *the set of all non-negative matrices fulfilling* (1.6), *with* $\rho(M) = 1$.

Obviously $\mathfrak{M}(\mathbf{u}, \mathbf{v})$ is a convex polyhedral set.

In the special case $\mathbf{u} = \mathbf{v} = \mathbf{e} = (1/\sqrt{n}), (1, 1, \cdots, 1)$, (We shall employ later the same notation $\mathbf{e} = (1, 1, \cdots, 1)$ without the scalar factor $1/\sqrt{n}$. There should arise no confusion of the meaning of $\mathbf{e}$ from the context at hand.) then $\mathfrak{M}(\mathbf{e}, \mathbf{e})$ comprises precisely the set of all doubly stochastic matrices. It is a familiar fact (commonly referred to as the Birkhoff Theorem) that the *permutation matrices are the extreme points of* $\mathfrak{M}(\mathbf{e}, \mathbf{e})$.

The theorem now stated and proved in Section 3 is fundamental to a number of the applications

THEOREM 3.1. *Let M be an $n \times n$ irreducible non-negative matrix in $\mathfrak{M}(\mathbf{u}, \mathbf{v})$. Then for any positive definite diagonal matrix $D = \operatorname{diag}(d_1, \cdots, d_n)$, we have*

$$\rho(DM) = \rho(MD) \geq \prod_{i=1}^{n} d_i^{u_i v_i}. \tag{1.8}$$

If $\rho(M)$ is not normalized (equal to 1) then (1.8) *is replaced by*

$$\rho(DM) \geq \left(\prod_{i=1}^{n} d_i^{u_i v_i}\right)\rho(M). \tag{1.8$'$}$$

A basic inequality underlying (1.8) *is that for any positive vector $\mathbf{x} > 0$ and $M \in \mathfrak{M}(u, v)$, we have*

$$\prod_{i=1}^{n} \left[\frac{(M\mathbf{x})_i}{x_i}\right]^{u_i v_i} \geq 1. \tag{1.9}$$

Equality holds in (1.9) *if $\mathbf{x} = \alpha\mathbf{u}$ for any $\alpha > 0$. Where M is strictly positive then the equality in* (1.9) *entails $\mathbf{x} = \alpha\mathbf{u}$ for some $\alpha > 0$.*

A much simpler case of Theorem 3.1 concerns $\mathfrak{M}(\mathbf{e}, \mathbf{e})$ = *the class of doubly stochastic matrices.*

THEOREM 2.1. *Let $M \in \mathfrak{M}(\mathbf{e}, \mathbf{e})$ and D a positive definite diagonal matrix. Then*

$$\rho(DM) \geq \left(\prod_{i=1}^{n} d_i\right)^{1/n}. \tag{1.10}$$

If M is positive the inequality is sharp unless $D = \alpha I$ for some $\alpha > 0$. Equality holds in (1.10) *for the permutation matrix*

$$P = \|m_{ij}\|, \quad m_{i,i+1} = 1, \quad i = 1, 2, \cdots, n-1; \quad m_{n,1} = 1. \tag{1.11}$$

The proof of Theorem 2.1 set forth in Section 2 is elementary but the facts (1.8) and (1.9) are markedly deeper.

The inequality (1.8) can be improved by restricting further the class of matrices.

THEOREM 4.1.
(i) *Let $M \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$ be of the form $M = E_1 K E_2$ where E_1 and E_2 are diagonal positive definite matrices and K is positive semi-definite. Let D be as in Theorem* 3.1. *Then*

$$\rho(DM) \geq \sum_{i=1}^{n} u_i v_i d_i. \tag{1.12}$$

Moreover, for any $\mathbf{x} > \mathbf{0}$ and M irreducible, we have

$$\sum_{i=1}^{n} u_i v_i \frac{x_i}{(M\mathbf{x})_i} \leq \frac{1}{\rho(M)} \tag{1.13}$$

If $M \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$ is the specific rank one matrix, $M = \mathbf{u}'\mathbf{v} = \|u_i v_j\|$ then equality occurs in (1.12).

Note for this latter example that

$$\begin{aligned} &M = E_1 K E_2 \quad \text{where} \quad E_1 = \operatorname{diag}(u_1, u_2, \cdots, u_n), \\ &E_2 = \operatorname{diag}(v_1, v_2, \cdots, v_n), \quad \text{and} \quad K = \|k_{ij}\|, \\ &k_{ij} \equiv 1 \quad \text{for all} \quad i, j. \end{aligned} \tag{1.14}$$

The arithmetic geometric mean inequality implies

$$\sum_{i=1}^{n} u_i v_i d_i \geq \prod_{i=1}^{n} d_i^{u_i v_i}$$

((1.7) is needed here) which shows that (1.12) is usually a genuine sharpening of (1.8).

Another class of matrices to which (1.12) applies is described in Theorem 4.2 of Section 4. There is evidence for the conjecture that where $M \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$ is suitably totally positive, then (1.12) is also correct. On the other hand the inequality (1.12) cannot always apply for matrices of class $\mathfrak{M}(\mathbf{u}, \mathbf{v})$ as revealed by the example (1.11).

Section 5 is devoted to some extensions of (1.8) and (1.12). If M is appropriately totally positive, then by passing to the higher order compounds of M we extract estimates for the product of the largest k eigenvalues of MD, that is for the quantity $\prod_{i=1}^{k} \lambda_i(MD)$ where $\lambda_1(MD) > \lambda_2(MD) > \cdots > \lambda_n(MD)$ are the eigenvalues.

A variety of criteria emanating from (1.8) and (1.12) discerning instability of the fixed point **0** of the transformation (1.2) are discussed in Section 5. Finally, the evaluation of $\rho(MD)$ for some special important examples are set forth.

In Section 6 we apply the results of the earlier sections especially Theorems 3.1 and 4.1 to the study of certain matrix inverse eigenvalue problems. We describe here one of our main findings. Prior to doing this we prepare some background.

Let A be non-negative and irreducible. It can be shown that there exists a unique (up to a constant factor) positive definite diagonal matrix C such that

$$(1.15) \qquad CAC^{-1}\mathbf{e} = \mathbf{e}CAC^{-1}, \qquad \mathbf{e} = (1, \cdots, 1) \quad \text{(Lemma 6.1)}.$$

With (1.15) established, we readily produce a unique real matrix C_0 engendering

$$(1.16) \qquad (CAC^{-1} + C_0)\mathbf{e} = \mathbf{e}(CAC^{-1} + C_0) = \mathbf{e}.$$

Now we state our main result on the inverse additive eigenvalue problem (I.A.E.P.).

THEOREM 6.1. *Let A be non-negative and irreducible. Let $C = \operatorname{diag}(c_1, \cdots, c_n)$ and C_0 be determined conforming to* (1.15) *and* (1.16), *respectively. Denote by $\Lambda^* = \{\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*\}$ the spectrum of $A + C_0$. Let $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ be given and suppose the I.A.E.P. is solvable with respect to Λ, i.e., there exists at least one real diagonal D such that $A + D$ has spectrum Λ. Then Λ satisfies*

$$(1.17) \qquad (n-1)\lambda_1 \geq \sum_{\substack{i,j=1 \\ i \neq j}}^{n} c_i a_{ij} \frac{1}{c_j} + \sum_{i=2}^{n} \lambda_i .$$

Equality occurs in (1.17) *only for the prescription $\Lambda^*(\alpha) = \{\lambda_1^* + \alpha, \lambda_2^* + \alpha, \cdots, \lambda_n^* + \alpha\}$ for any real α. Where $\Lambda \equiv \Lambda^*(\alpha)$ then the I.A.E.P. is uniquely solvable.*

A version of Theorem 6.1 for the I.M.E.P. is the content of Theorem 6.2 and its corollary.

The final Section 7 is devoted to establishing a number of extensions of the previous matrix results to certain *integral operator analogs*. A convolution example is studied in detail.

2. Inequalities for the spectral radius for MD where M is doubly stochastic and D is positive diagonal.

We adhere to the notation of Section 1.

Throughout this section we focus on the collection of matrices $\mathfrak{M}(\mathbf{e}, \mathbf{e})$, $\mathbf{e} = (1, 1, \cdots, 1)$, (see after (1.7)), consisting of all doubly stochastic matrices. Recall that $\mathfrak{M}(\mathbf{e}, \mathbf{e})$ is a convex set spanned by the extreme points, the latter identified as all permutation matrices.

The next lemma is relevant to a number of contexts.

LEMMA 2.1. *Let M be doubly stochastic. Then*

(i)

$$(2.1) \qquad \min_{\mathbf{x}>0} \left[\frac{\prod_{i=1}^{n} (M\mathbf{x})_i}{\prod_{i=1}^{n} x_i} \right] \geq 1 \quad \textit{(the infimum extended over all positive vectors } \mathbf{x}).$$

Equality is always achieved independent of $\mathbf{x} > \mathbf{0}$ provided M is a permutation matrix or for all $M \in \mathfrak{M}(\mathbf{e}, \mathbf{e})$ if $\mathbf{x} = \alpha\mathbf{e}$, $(\alpha > 0)$. If M is positive, that is, all entries of M are positive, then the equality in (2.1) *persists only if $\mathbf{x} = \alpha\mathbf{e}$.*

(ii)

$$(2.2) \qquad \min_{x>0} \sum_{i=1}^{n} \left[\frac{(M\mathbf{x})_i}{x_i} \right] \geq n$$

with equality attained for $\mathbf{x} = \alpha\mathbf{e}$ or M = the identity matrix. If M is irreducible non negative then equality holds only if $\mathbf{x} = \alpha\mathbf{e}$, $\alpha > 0$.

Remark 2.1. The function $\phi(\xi_1, \xi_2, \cdots, \xi_n) = \prod_{i=1}^{n} \xi_i$, $(\xi_i > 0)$ is Schur-concave and accordingly (2.1) ensures by an elementary application of the theory of "majorization," (Hardy–Littlewood–Pōlya [6], Chaps. 1 and 2). To maintain this writing self contained, we provide a direct proof.

Proof.

(i) Since the log function is strictly concave and M is doubly stochastic we find

$$(2.3) \qquad \sum_{i=1}^{n} \log\left(\sum_{j=1}^{n} m_{ij}x_j\right) \geq \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \log x_j$$
$$= \sum_{j=1}^{n} (\log x_j)\left(\sum_{i=1}^{n} m_{ij}\right) = \sum_{j=1}^{n} \log x_j$$

and the resulting inequality is clearly synonomous with (2.1). The circumstances of equality can readily be discerned.

(ii) This is an immediate consequence of the arithmetic-geometric mean inequality imposed on the fact of (2.1) plus verification of the cases of equality. Indeed, we have

$$(2.4) \qquad \frac{1}{n} \sum_{i=1}^{n} \frac{(Mx)_i}{x_i} \geq \left(\prod_{i=1}^{n} \frac{(Mx)_i}{x_i} \right)^{1/n} \geq 1$$

and equality prevails throughout for $\mathbf{x} = \alpha\mathbf{e}$, α positive. Equality in (2.2) implies $(M\mathbf{x})_i = x_i$, $i = 1, \cdots, n$. As M is irreducible, the Perron-Frobenius theorem tells us that $\mathbf{x} = \alpha\mathbf{e}$.

LEMMA 2.2. *Let A be a non-negative matrix satisfying*

$$Ae = eA \quad \text{(i.e., the respective row and column sums agree).} \tag{2.5}$$

Then

$$\min_{x>0} \sum_{i=1}^{n} \frac{(Ax)_i}{x_i} = \sum_{i,j=1}^{n} a_{ij}. \tag{2.6}$$

Equality is achieved for $x = \alpha e$. Where A is irreducible nonnegative then equality holds only for $x = \alpha e$.

Proof. The condition (2.5) entails the existence of a positive diagonal matrix $E = \text{diag}(e_1, e_2, \cdots, e_n)$ such that $A + E = \gamma M$ where $\gamma > 0$ and $M \in \mathfrak{M}(e, e)$ is doubly stochastic. Then for $x > 0$

$$\sum_{i=1}^{n} \frac{(Ax)_i}{x_i} = \gamma \sum_{i=1}^{n} \frac{(Mx)_i}{x_i} - \sum_{i=1}^{n} e_i. \tag{2.7}$$

Since the minimum on the right is always achieved for $x = \alpha e$ independent of $M \in \mathfrak{M}(e, e)$, the conclusion (2.6) is valid.

An easy consequence of Lemma 2.1 is the result (1.10), now restated.

THEOREM 2.1. *Let $M \in \mathfrak{M}(e, e)$ and D be a positive definite diagonal matrix, $D = \text{diag}(d_1, d_2, \cdots, d_n)$. Then*

$$\rho(DM) = \rho(MD) \geq \left(\prod_{i=1}^{n} d_i\right)^{1/n} \tag{2.8}$$

(recall $\rho(A)$ denotes the spectral radius of A). If M is positive then the inequality is sharp unless $D = \alpha I$ for $\alpha > 0$. Equality holds for the permutation matrix (1.11).

Proof. Since the spectrum of DM and MD each coincides with the spectrum of $D^{\frac{1}{2}}MD^{\frac{1}{2}}$, we have in particular $\rho(DM) = \rho(MD)$.

Assume first that M is positive (all entries are positive). The Perron-Frobenius theorem guarantees the existence of some $x > 0$ satisfying

$$\rho(DM)x = DMx. \tag{2.9}$$

Multiplying components produces

$$[\rho(MD)]^n \prod_{i=1}^{n} d_i^{-1} = \prod_{i=1}^{n} \frac{(Mx)_i}{x_i} \geq 1 \tag{2.10}$$

where the final inequality results on the basis of (2.1) and thereby (2.8) is proved for M positive. By continuity, we infer that (2.8) is correct for all M in $\mathfrak{M}(e, e)$.

For the special permutation matrix $P = \|p_{ij}\|$ with $p_{n,1} = 1$, $p_{i,i+1} = 1$, $i = 1, \cdots, n-1$, a direct calculation yields $\rho(PD) = (\prod_{i=1}^{n} d_i)^{1/n}$ attesting to the fact that (2.8) cannot be sharpened without further restricting $\mathfrak{M}(e, e)$.

The proof of Theorem 2.1 is complete.

A modified version of the inequality (2.8) can be extended to the class of all positive matrices as follows. It is known (e.g., see Sinkhorn [17]) that if A is positive (or non-negative and fully indecomposable, concerning the latter concept see Brualdi, Parter and Schneider [2]) and Sinkhorn and Knopp [18]) then there exists unique (up to scalar multiples) positive diagonal matrices $E_1 = \text{diag}(\epsilon_1, \cdots, \epsilon_n)$ and $E_2 = \text{diag}(e_1, \cdots, e_n)$ such that

$$E_1AE_2 = M \text{ is doubly stochastic.} \tag{2.11}$$

Actually, a practical algorithm is available for constructing E_1 and E_2 where A is positive. In fact, normalize alternately the rows and columns to sum to 1. More specifically, determine a diagonal positive matrix C_n to satisfy $eA_nC_n = e$ and subsequently a diagonal positive B_n, $B_n\tilde{A}_ne = e$ where $\tilde{A}_n = A_nC_n$, $A_n = B_{n-1}\tilde{A}_{n-1}$, $n = 1, 2, \cdots$, $(A_1 = A)$. The convergence of

$$\prod_{i=1}^{n} C_i \to E_2, \qquad \prod_{i=1}^{n} B_i \to E_1,$$

leading to (2.11) is proved in Sinkhorn [17].

COROLLARY 2.1. *Let A be positive (or non-negative and fully indecomposable) and let E_1, E_2 be determined as in* (2.11).

Then

$$\rho(DA) \geq \left[\prod_{i=1}^{n} \frac{d_i}{\epsilon_i e_i}\right]^{1/n}.$$

If A is positive and symmetric then a known procedure for calculating E converting EAE = doubly stochastic, is by solving the following variational problem.

Determine a positive vector $\varepsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ which minimizes $g(\varepsilon) = (\varepsilon, A\varepsilon)$ subject to the constraint $\prod_{i=1}^{n} \epsilon_i = 1$.

3. Developments pertaining to Theorem 3.1 and related critical point theory.

The inequality embodied in (1.8) is more far reaching and deeper than the version in the doubly stochastic case covered in Theorem 2.1. The elementary technique of Lemma 2.1 does not carry over.

Let A be positive (it suffices to have A non-negative and irreducible) and suppose

$$Au = \rho(A)u, \qquad vA = \rho(A)v \tag{3.1}$$

and suppose u and v normalized to satisfy

$$\sum_{i=1}^{n} u_iv_i = 1. \tag{3.2}$$

The main objective of this section is to prove

(3.3) $$f(\mathbf{x}) = f(x_1, \cdots, x_n) = \sum_{i=1}^{n} u_i v_i \log \left(\frac{(A\mathbf{x})_i}{x_i}\right) \geq 0 \quad \text{for} \quad \mathbf{x} > 0.$$

The analysis is done through a series of lemmas.

LEMMA 3.1. *Let Δ be an open bounded connected domain contained in $\mathbf{R}^n$ (Euclidean n-space) and suppose $f(\mathbf{y})$ defined for $\mathbf{y} \in \Delta$ is real valued of continuity class $C^2(\Delta)$. Assume that $\lim_{k\to\infty} f(\mathbf{y}_k) = +\infty$ whenever $\mathbf{y}_k$ tends to $\partial\Delta$, the boundary of Δ. Assume furthermore that for every critical point $\mathbf{y}^0$ of f in Δ, the Hessian matrix $\|f_{ij}(\mathbf{y}^0)\|$, $(f_{ij}(\mathbf{y}^0) = [\partial^2 f/\partial y_i \partial y_j](\mathbf{y}^0))$ is strictly positive definite. Then there exists a unique critical point $\mathbf{y}^*$ in Δ and f achieves over Δ an absolute minimum at $\mathbf{y}^*$.*

Remark 3.1. The content of this lemma is undoubtedly known and would follow directly from the Morse critical point inequalities (the second of the series suffices for our purposes) except for the complication that Δ is open. Since f is infinite on $\partial\Delta$ the behavior near the boundary should be and indeed is irrelevant.

Proof. The stipulations of the lemma assure the existence of an absolute minimum $\mathbf{y}^* \in \Delta$ which, of course, is a critical point of f. We need to establish that $\mathbf{y}^*$ is the only critical point. Consider the differential equation system

(3.4) $$\frac{dy_i(t)}{dt} = -\frac{\partial f}{\partial y_i}(y_1(t), \cdots, y_n(t)), \qquad i = 1, 2, \cdots, n$$

with the initial conditions

(3.5) $$y_i(0) = y_i^0, \qquad \mathbf{y}^0 \in \Delta.$$

Direct evaluation produces

$$\frac{df(\mathbf{y}(t))}{dt} = -\sum_{i=1}^{n} \left[\frac{\partial f}{\partial y_i}(\mathbf{y}(t))\right]^2$$

and from this equation we infer that $f(\mathbf{y}(t))$ strictly decreases as t increases, $t > 0$, unless $\mathbf{y}^0$ is a critical point. We claim that $\mathbf{y}(t)$ converges as $t \to \infty$ to a critical point. Let $\tilde{\mathbf{y}}$ be a limit point of $\mathbf{y}(t)$ (i.e., $\tilde{\mathbf{y}} = \lim_{i\to\infty} \mathbf{y}(t_i)$) which necessarily exists in Δ since f is unbounded on $\partial\Delta$. Certainly, $f(\mathbf{y}(t))$ decreases to $f(\tilde{\mathbf{y}})$ since $f(\mathbf{y}(t_i))$ converges to $f(\tilde{\mathbf{y}})$ and $f(y(t))$ is decreasing. The presumption $\sum_{i=1}^n [(\partial f/\partial y_i)(\tilde{\mathbf{y}})]^2 = \alpha > 0$ implies the inequality $(df/dt)(\mathbf{y}(t_i)) \leq -\alpha/2$ for all i large enough which compels therefore $f(\tilde{\mathbf{y}}) = -\infty$, an absurdity. To avert this absurdity we must have that $\tilde{\mathbf{y}}$ (a conclusion applying to all such limit points) is a critical point. The hypothesis that $\|f_{ij}(\tilde{\mathbf{y}})\|$ is positive definite entails that $\tilde{\mathbf{y}}$ is locally stable with respect to the flow (3.4). It follows that $\lim_{t\to\infty} \mathbf{y}(t) = \tilde{\mathbf{y}}$ as previously claimed.

Consider now the domain of attraction $\Delta(\tilde{\mathbf{y}})$ to $\tilde{\mathbf{y}}$ consisting of all $\mathbf{y}$ fulfilling the conditions $\mathbf{y}(0) = \mathbf{y}$ and $\lim_{t\to\infty} \mathbf{y}(t) = \tilde{\mathbf{y}}$. Since $\tilde{\mathbf{y}}$ is locally stable it is clear that $\Delta(\tilde{\mathbf{y}})$ is open. Suppose $\Delta(\tilde{\mathbf{y}})$ is properly contained in Δ, then (because Δ is connected and open) we infer the existence of $\mathbf{z}$ in Δ, located on the boundary of $\Delta(\tilde{\mathbf{y}})$ and the previous analysis shows that $\lim_{t\to\infty} \mathbf{z}(t) = \tilde{\mathbf{z}}$ for some critical point $\tilde{\mathbf{z}} \neq \tilde{\mathbf{y}}$. But $\Delta(\tilde{\mathbf{z}})$ is an open domain for the same reason that $\Delta(\tilde{\mathbf{y}})$ is open, and a contradiction emerges as $\mathbf{z}$ lies on $\partial\Delta(\tilde{\mathbf{y}})$. To avoid this contradiction, we necessarily have $\Delta(\tilde{\mathbf{y}}) = \Delta$ and a unique critical point exists. The proof of Lemma 3.1 is complete.

We now recall the following concept.

DEFINITION 3.1. *A matrix $A = \|a_{ij}\|_1^m$ is said to be a (strict) M-matrix* (after Minkowski who introduced the pertinent classification) *if $a_{ij} \leq 0$ (<0) for all $i \neq j$ and there exists a positive vector $\mathbf{x}^0 > 0$ satisfying*

(3.6) $$A\mathbf{x}^0 \geq 0.$$

Such matrices occur widely in applications to econometric theory and in certain contexts of physical systems, e.g., see Karlin [10, chap. 8].

We review for easy reference some of the main characterizations and properties of M-matrices. *A (strict) M-matrix can be represented in the form*

(3.7) $$A = rI - B \text{ where } B \text{ is (strictly positive) nonnegative and } r \geq \rho(B).$$

Appealing to the Frobenius theory of positive matrices we find that if B is irreducible nonnegative then A possesses a simple positive eigenvalue λ_1 and all other eigenvalues satisfy

(3.8) $$\Re\, \lambda_i(A) > \lambda_1, \qquad i = 2, 3, \cdots, n.$$

If A is a symmetric strict M-matrix then (3.6) in conjunction with (3.8) imply that A *is positive semi definite* with the eigenvalue λ_1 having simple multiplicity.

We are now prepared for the next lemma which highlights a class of functions f fulfilling the conditions of Lemma 3.1.

LEMMA 3.2. *Let Δ be an open connected homogeneous domain in $\mathbf{R}_+^n$ (i.e., $\Delta \subset \alpha\Delta$ for any $\alpha > 0$). Assume that f is a homogeneous function of degree 0 defined on Δ of continuity class $C^{(2)}(\Delta)$. Consider $f(\mathbf{y})$ restricted to the bounded region $S = \Delta \cap \Sigma$ where*

$$\Sigma = \{\mathbf{y} \mid \mathbf{y} \in R_+^n, \Sigma \mathbf{y}_i = 1\}.$$

Suppose further that $f(\mathbf{y}) \to +\infty$ whenever $\mathbf{y} \in S$ tends to the boundary ∂S. If

(3.9) $$\frac{\partial^2 f}{\partial y_i\, \partial y_j}(\mathbf{y}) < 0 \quad \text{for} \quad i \neq j \quad \text{and all } \mathbf{y} \in S,$$

then $f(\mathbf{y})$ on S admits a unique critical point located at the absolute minimum of f.

Proof. Since f is homogeneous of degree 0, it satisfies the Euler equation

(3.10) $$\sum_{i=1}^{n} y_i \frac{\partial f}{\partial y_i}(\mathbf{y}) = 0.$$

Differentiating (3.10) in y_j prduces $\sum_{i=1}^n y_i\, \partial^2 f/\partial y_i \partial y_j(\mathbf{y}) + (\partial f/\partial y_j)(\mathbf{y}) = 0$ $j = 1, 2, \cdots, n$ and at a critical point $\mathbf{y}^0$ these relations reduce to

(3.11) $$\mathbf{y}^0 F = 0 \quad \text{where} \quad F = F(\mathbf{y}^0) = \left\| \frac{\partial^2 f}{\partial y_i\, \partial y_j}(\mathbf{y}^0) \right\|.$$

The assumption (3.9) in conjunction with (3.11) establish that F is a strict M-matrix (Definition 3.1). By virtue of (3.8) we know that the eigenvalues of $F(\mathbf{y}^0)$ can be arranged in the order

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$$

where the unique eigenvector (apart from scalar multiples) for $\lambda_1 = 0$ is $\mathbf{y}^0$. We infer accordingly that the second variation (the Hessian) of f at $\mathbf{y}^0$ with respect to variations *in* S is positive definite. Thus, f confined to S satisfies the requirements of Lemma 3.1. Applying the conclusion of that lemma, the desired result is achieved.

Proof of (3.3). The function

(3.12) $$f(\mathbf{x}) = \sum_{i=1}^n u_i v_i \log \left(\frac{(A\mathbf{x})_i}{x_i} \right)$$

defined on

(3.13) $$\Sigma = \{\mathbf{x} = (x_1, \cdots, x_n) > 0,\ |\mathbf{x}| = \Sigma x_i = 1\}$$

is plainly homogeneous of degree 0. Also $f(\mathbf{x})$ tends to ∞ if at least one component of $\mathbf{x} \in \Sigma$ goes to zero. Clearly $f \in C^{(2)}(\Sigma)$.

A direct calculation produces

$$\frac{\partial^2 f}{\partial x_j\, \partial x_k} = -\sum_{i=1}^n \frac{u_i v_i a_{ij} a_{ik}}{(A\mathbf{x})_i^2} \quad \text{for} \quad j \neq k$$

which is certainly negative since A is a positive matrix.

Straightforward computation relying on the relations (3.1) verifies that the vector $\mathbf{u}/|\mathbf{u}|$ is a critical point of f. On the basis of Lemma 3.2 we know that $\mathbf{x} = \alpha\mathbf{u}$ provides the unique absolute minimum of (3.12) and this value is manifestly 0.

Proof of Theorem 3.1. The inequality (1.9) is obviously equivalent to (3.3). The statements discerning the possibilities of equality in (3.3) are decided on the basis of the uniqueness results affirmed in Lemma 3.2.

We turn to the proof of (1.8). Invoking the Perron-Frobenius theory, we obtain

$$\rho(DM)x_i = d_i \sum_{j=1}^n m_{ij} x_j, \qquad i = 1, 2, \cdots, n \quad \text{for some } \mathbf{x} > \mathbf{0}.$$

Therefore

(3.14) $$\rho(DM) \prod_{i=1}^n d_i^{-u_i v_i} = \prod_{i=1}^n \left[\frac{(M\mathbf{x})_i}{x_i} \right]^{u_i v_i}$$

where $\mathbf{u} = (u_1, \cdots, u_n)$ and $\mathbf{v} = (v_1, \cdots, v_n)$ satisfy (1.6) and (1.7). Applying (1.9) in (3.14) entails (1.8). The proof of Theorem 3.1 is complete if $M > \mathbf{0}$. For $M \geq 0$, apply the conclusion to $M(\epsilon) = (1 - \epsilon)M + \epsilon\, \|u_i v_j\|_1^n$, and then let $\epsilon \to 0$.

Remark 3.2. It can be proved that if merely M is non-negative, a-periodic and irreducible then equality holds in (1.9) iff $\mathbf{x} = \alpha\mathbf{u}$ and equality in (1.8) iff $D = \alpha I$.

COROLLARY 3.1. *Let M, $\mathbf{u}$, $\mathbf{v}$ satisfy the conditions of Theorem* 3.1, *then*

(3.15) $$\sum_{i=1}^n u_i v_i \frac{(M\mathbf{x})_i}{x_i} \geq 1.$$

Equality holds for $\mathbf{x} = \alpha\mathbf{u}$.

Proof. The generalized arithmetic—geometric mean inequality implies

$$\sum_{i=1}^n u_i v_i \frac{(M\mathbf{x})_i}{x_i} \geq \prod_{i=1}^n \left[\frac{(M\mathbf{x})_i}{x_i} \right]^{u_i v_i}$$

and the right side is not smaller than 1 by (1.9).

To illustrate further the usefulness of Lemma 3.1 we present the following extention of a theorem of Sinkhorn [17] on constructing doubly stochastic matrices related to prescribed non-negative matrices.

THEOREM 3.2. *Let $M = \|m_{ij}\|$ be irreducible non-negative and $m_{ii} > 0$ for all i. Let $\mathbf{u}$ and $\mathbf{v}$ be positive vectors. Then there exist unique* (up to scalar multiples) *positive diagonal matrices*

$$E_1 = \text{diag}\,(\epsilon_1, \cdots, \epsilon_n) \text{ and } E_2 = \text{diag}\,\{e_1, \cdots, e_n\}$$

such that $E_1 M E_2$ belongs to $\mathfrak{M}(\mathbf{u}, \mathbf{v})$.

Proof. Let $f(\mathbf{x}, M)$ be the function defined by (3.12) ($A = M$). The condition $m_{ii} \geq \alpha > 0$ entails $(M\mathbf{x})_i/x_i \geq \alpha$ for any $\mathbf{x} > \mathbf{0}$ and if $\mathbf{x}$ tends to the boundary of $\Delta = \{\mathbf{x} : |\mathbf{x}| = 1, \mathbf{x} > \mathbf{0}\}$ manifestly, $f(\mathbf{x}, M) \to \infty$. By Lemma 3.1, let $\xi(M) = (\xi_1(M), \cdots, \xi_n(M))$ denote the unique critical point (up to a scalar multiple) of $f(\mathbf{x}, M)$.

Let $D = \text{diag}\,\{d_1, \cdots, d_n\}$ be a positive definite diagonal matrix. It is elementary to deduce the relations

(3.16) $$\xi(DM) = \xi(M) \quad \text{and} \quad \xi(MD^{-1}) = D\xi(M).$$

Indeed, (3.16) is clear on the basis of the identities

$$f(\mathbf{x}, DM) = f(\mathbf{x}, M) + \sum_{i=1}^n u_i v_i \log d_i$$

and

$$f(D\mathbf{x}, MD^{-1}) = f(\mathbf{x}, M) - \sum_{i=1}^n u_i v_i \log d_i .$$

Now, determine E_2 by the prescriptions

$$e_i = \frac{\xi_i(M)}{u_i} \qquad i = 1, 2, \cdots, n \tag{3.17}$$

($E_2 = \text{diag}\,(e_1, \cdots, e_n)$) and $E_1 = \text{diag}\,(\epsilon_1, \cdots, \epsilon_n)$ to satisfy

$$E_1ME_2\mathbf{u} = \mathbf{u} \tag{3.18}$$

It follows from (3.16) that

$$\xi(E_1ME_2) = \mathbf{u} \tag{3.19}$$

Also, the relations

$$\frac{\partial}{\partial x_i} f(\mathbf{x}, E_1ME_2)|_{\mathbf{u}} = 0, \qquad i = 1, 2, \cdots, n$$

since $\mathbf{u}$ is the appropriate critical point, reduce to

$$\mathbf{v}E_1ME_2 = \mathbf{v}.$$

Suppose now that $\tilde{E}_1M\tilde{E}_2 \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$, By virtue of (3.16) we have

$$\mathbf{u} = \alpha\tilde{E}_2^{-1}\xi(M) \text{ for some } \alpha > 0.$$

Comparing to (3.17) we see that

$$\tilde{E}_2 = \alpha^{-1}E_2 \quad \text{and then} \quad \tilde{E}_1 = \alpha E_1$$

is clear. The proof of the Theorem is complete.

The natural generalizations of Corollary 3.1 ensues.

COROLLARY 3.2. *Let M be an $n \times n$ matrix as in Theorem 3.2. Let $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_n)$ be a positive vector normalized by the condition*

$$\sum_{i=1}^{n} \omega_i = 1. \tag{3.20}$$

Then there exists a diagonal positive definite matrix

$$D(\boldsymbol{\omega}) = \text{diag}\,(d_1(\boldsymbol{\omega}), \cdots, d_n(\boldsymbol{\omega}))$$

such that

$$\rho(DA) \geq \prod_{i=1}^{n} \left(\frac{d_i}{d_i(\boldsymbol{\omega})}\right)^{\omega_i} \tag{3.21}$$

for any diagonal positive definite matrix D. The equality sign holds if $D = \alpha D(\boldsymbol{\omega})$ for some $\alpha > 0$.

Proof. Take $D = E_1E_2$, where E_1 and E_2 are defined in Theorem 3.2 and $\mathbf{u} = \mathbf{e}$ and $\mathbf{v} = \boldsymbol{\omega}$.

Remark 3.3. According to Brualdi, Parter and Schneider [2, Lemma 2.3] the conditions of Theorem 3.2 imply that M is fully indecomposable. Thus

in case $\mathbf{u} = \mathbf{v} = \mathbf{e}$ Theorem 3.2 goes back to Brualdi, Parter and Schneider and Sinkhorn & Knopp [18]. Finally, in view of this lemma it follows that the results of Theorem 3.2 remain valid if we assume that M is a fully indecomposable matrix.

4. Derivation of the inequality (1.12) and ramifications.

We begin with some preliminaries

LEMMA 4.1. *Let $M = E_1KE_2$ where E_1 and E_2 are diagonal positive definite matrices and K is positive definite. Furthermore, assume M is irreducible and nonnegative. Let $\mathbf{u}$ and $\mathbf{v}$ be positive vectors satisfying*

$$M\mathbf{u} = \rho(M)\mathbf{u} \qquad \mathbf{v}M = \mathbf{v}\rho(M) \tag{4.1}$$

(for simplicity of writing take $\rho(M) = 1$)

and normalized to satisfy

$$\sum_{i=1}^{n} u_iv_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} u_i^2 = 1. \tag{4.2}$$

Define the positive vector $\mathbf{y}^0$ by the equation

$$E_1^{\frac{1}{2}}E_2^{-\frac{1}{2}}\mathbf{y}^0 = \mathbf{u}. \tag{4.3}$$

Then

$$E_1^{-\frac{1}{2}}E_2^{\frac{1}{2}}\mathbf{y}^0 = \alpha\mathbf{v} \quad \textit{for some} \quad \alpha > 0. \tag{4.4}$$

Proof. We evaluate $(E_1^{-\frac{1}{2}}E_2^{+\frac{1}{2}}\mathbf{y}^0)M^{-1}$ heavily exploiting the fact that E_1 and E_2 are positive diagonal matrices. We obtain

$$\begin{aligned}(E_1^{-\frac{1}{2}}E_2^{+\frac{1}{2}}\mathbf{y}^0)M^{-1} &= \mathbf{y}^0E_1^{-\frac{1}{2}}E_2^{+\frac{1}{2}}(E_2^{-1}K^{-1}E_1^{-1}) = \mathbf{y}^0E_1^{+\frac{1}{2}}E_2^{-\frac{1}{2}}(E_1^{-1}K^{-1}E_2^{-1})E_2E_1^{-1} \\ &= \mathbf{u}(E_1^{-1}K^{-1}E_2^{-1})E_2E_1^{-1} = \mathbf{u}(M')^{-1}E_2E_1^{-1} = \mathbf{u}E_2E_1^{-1} \\ &= \mathbf{y}^0E_1^{\frac{1}{2}}E_2^{-\frac{1}{2}}E_2E_1^{-1} = E_1^{-\frac{1}{2}}E_2^{\frac{1}{2}}\mathbf{y}^0.\end{aligned}$$

Comparing the outside equations we see that $\mathbf{z} = E_1^{-\frac{1}{2}}E_2^{\frac{1}{2}}\mathbf{y}^0$ is a left eigenvector of M^{-1} with eigenvalue 1, i.e., $\mathbf{z}M^{-1} = \mathbf{z}$. We infer by the Perron-Frobenius theorem since $\rho(M) = 1$ is a simple eigenvalue that

$$\mathbf{z} = E_1^{-\frac{1}{2}}E_2^{\frac{1}{2}}\mathbf{y}^0 = \alpha\mathbf{v} \quad \text{for some positive } \alpha. \tag{4.5}$$

The proof of the lemma is complete.

Proof of (1.12), consult section 1. Assume first that K is positive definite and positive. Because K is positive definite and E_1, E_2 and D are diagonal, the eigenvalues of $DM = DE_1KE_2$ coincide with E_2DE_1K and $K^{\frac{1}{2}}(E_1E_2)^{\frac{1}{2}}\cdot D(E_1E_2)^{\frac{1}{2}}K^{\frac{1}{2}}$. In particular

$$\rho(DM) = \rho(K^{\frac{1}{2}}(E_1E_2)^{\frac{1}{2}}D(E_1E_2)^{\frac{1}{2}}K^{\frac{1}{2}}).$$

On the basis of a familiar characterization of the largest eigenvalue of a symmetric matrix and taking account of the fact that all component matrices involved are nonnegative, we find that

$$(4.6)\quad \rho(DM) = \sup_{\mathbf{x}>0} \frac{(K^{\frac12}(E_1E_2)^{\frac12}D(E_1E_2)^{\frac12}K^{\frac12}\mathbf{x}, \mathbf{x})}{(\mathbf{x}, \mathbf{x})} = \sup_{\mathbf{y}>0} \frac{(D\mathbf{y}, \mathbf{y})}{(K^{-1}(E_1E_2)^{-\frac12}\mathbf{y}, (E_1E_2)^{-\frac12}\mathbf{y})}$$
$$\geq \frac{(D\mathbf{y}^0, \mathbf{y}^0)}{(K^{-1}(E_1E_2)^{-\frac12}\mathbf{y}^0, (E_1E_2)^{-\frac12}\mathbf{y}^0)}$$

where $\mathbf{y}^0$ is determined as in (4.3). Substituting for $\mathbf{y}^0$ and relying on the fact of (4.4) the final denominator in (4.6) reduces to $\alpha(\mathbf{u}, \mathbf{v}) = \alpha \sum_{i=1}^n u_iv_i$ while the numerator becomes $\alpha(D\mathbf{u}, \mathbf{v}) = \alpha \sum_{i=1}^n d_iu_iv_i$.

The case that K is non-negative and positive semi-definite is handled by standard perturbation procedures. Apply the result to $K_\epsilon = (1 - 2\epsilon)K + \epsilon I + \epsilon E_1^{-1}[u_iv_j]_1^n E_2^{-1}$ and let $\epsilon \to 0+$. The proof of (1.12) is complete.

Proof of (1.13). Application of the Perron Frobenius theorem gives

$$(4.7)\qquad \rho(DM)x_i = d_i(M\mathbf{x})_i \qquad i = 1, 2, \cdots, n$$

for some $\mathbf{x} > 0$. Therefore

$$(4.8)\qquad \rho(DM) \sum_{i=1}^n u_iv_i \frac{x_i}{(M\mathbf{x})_i} = \sum_{i=1}^n d_iu_iv_i\ .$$

The inequality $\sum_{i=1}^n d_iu_iv_i \leq \rho(DM)$ established as (1.12) together with (4.8) (since $\rho(DM) > 0$ as M is irreducible and nonnegative) obviously imply

$$(4.9)\qquad \sum_{i=1}^n u_iv_i \frac{x_i}{(M\mathbf{x})_i} \leq 1.$$

The inequality (4.9) is validated in this analysis only for $\mathbf{x}$, an eigenvalue of MD for $\rho(MD)$. Now given an arbitrary $\mathbf{y} > 0$, we determine the vector $\mathbf{d} = (d_1, \cdots, d_n)$ by the relation $d_i = y_i/(M\mathbf{y})_i$ and obviously this $\mathbf{y} > \mathbf{0}$ qualifies as the eigenvector for $\rho(DM) = 1$. Thus, (4.9) holds for all $\mathbf{x} > \mathbf{0}$.

Where $\rho(M) \neq 1$ then (4.9) should read as,

$$(4.10)\qquad \sup_{\mathbf{x}>0} \sum_{i=1}^n u_iv_i \frac{x_i}{(M\mathbf{x})_i} \leq \frac{1}{\rho(M)}.$$

With (1.12) and (1.13) in hand, the proof of Theorem 4.1 is complete.

The next theorem describes another class of matrices for which (1.12) and (1.13) hold.

Theorem 4.2. *Let* $M \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$. *Assume furthermore that* M^{-1} *exists as an M-matrix* (see Definition 3.1). *Then*

$$(4.11)\qquad \sup_{\mathbf{x}>0} \sum_{i=1}^n u_iv_i \frac{x_i}{(M\mathbf{x})_i} \leq 1 \quad \text{and}$$

for any positive diagonal matrix D,

$$(4.12)\qquad \rho(MD) \geq \sum_{i=1}^n u_iv_id_i$$

Proof. We have available the representation

$$(4.13a)\qquad M^{-1} = rI - B \quad \text{where} \quad B \geq 0 \qquad r > \rho(B)$$

and

$$(4.13b)\qquad B\mathbf{u} = \rho(B)\mathbf{u}, \qquad \mathbf{v}B = \rho(B)\mathbf{v}.$$

(Pertaining to (4.13a), see the discussion following Definition 3.1). Clearly $M^{-1}\mathbf{u} = [r - \rho(B)]\mathbf{u}$

$$(4.14)\qquad \text{so that} \quad 1 = \rho(M) = r - \rho(B).$$

Obviously for any $\mathbf{x} > \mathbf{0}$ and $\mathbf{y} = M\mathbf{x}$,

$$\sum_{i=1}^n u_iv_i \frac{x_i}{(M\mathbf{x})_i} = \sum_{i=1}^n u_iv_i \frac{(M^{-1}\mathbf{y})_i}{y_i} = r \sum_{i=1}^n u_iv_i - \sum_{i=1}^n u_iv_i \frac{(B\mathbf{y})_i}{y_i}$$
$$= r - \sum_{i=1}^n u_iv_i \frac{(B\mathbf{y})_i}{y_i}.$$

The result of Theorem 3.1 applies to give

$$\sum_{i=1}^n u_iv_i \frac{(B\mathbf{y})_i}{y_i} \geq \left[\prod_{i=1}^n \frac{(B\mathbf{y})_i}{y_i}\right]^{u_iv_i} \geq \rho(B)$$

and therefore (4.11) holds.

Backtracking over the analysis of Theorem 4.1 we can check that the inequality (4.11) entails (4.12). The proof is complete.

Remark 4.1. If there exists a positive number $\beta > 0$ such that $-M^{-1} + \beta I$ is irreducible nonnegative then the equation sign holds in (4.12) iff $D = \alpha I$ for some $\alpha > 0$. This statement is readily confirmed by a careful scrutiny of the details of the analysis.

Remark 4.2. Consider a matrix M of the form $M = E_1KE_2$ where the component matrices have the properties enunciated in Theorem 4.1 and assume M is positive. Let D_1 and D_2 be diagonal positive matrices such that D_1MD_2 is doubly stochastic. We adopt the notation $D_1 = \text{diag}\,(d_1^{(1)}, \cdots, d_n^{(1)})$, $D_2 = \text{diag}\,(d_1^{(2)}, \cdots, d_n^{(2)})$. Obviously

$$D_1MD_2 = \tilde{E}_1K\tilde{E}_2\,, \qquad \tilde{E}_i = D_iE_i\,. \qquad i = 1, 2$$

The inequality (1.12) in terms of this reduction becomes

$$(4.15)\qquad \rho(DM) \geq \frac{1}{n} \sum_{i=1}^n \frac{d_i}{d_i^{(1)}\, d_i^{(2)}}.$$

5. Extensions, examples and applications.

I. *Bounds for lower order eignvalues; compound matrices*

Let A be an *oscillatory* matrix meaning that every compound of A, $A_{[p]}$ ($p = 1, 2, \cdots, n$) is nonnegative and that $A_{[p]}{}^m$ is strictly positive for some integer m, m can depend on p. Recall that the elements of $A_{[p]}$ consist of all pth order minors where the indices can be identified with all p tuples of integers and these arranged in lexicographic order. (Thus if $A = \|a_{ij}\|_1^n$ then the elements of $A_{[p]}$ in row $\boldsymbol{\alpha} = (i_1, i_2, \cdots, i_p)$ column $\boldsymbol{\beta} = (j_1, \cdots j_p)$ ($1 \leq i_1 < i_2 < \cdots < i_p \leq n; 1 \leq j_1 < j_2 < \cdots < j_p \leq n$) is $A_{[p]}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \det(\|a_{i_\nu j_\mu}\| \nu, \mu = 1)^p$.

In particular, of A is oscillatory then $A_{[p]}$ is a nonnegative and irreducible matrix of size $\binom{n}{p} \times \binom{n}{p}$. Oscillatory matrices have remarkable properties and we record a number of them for ready reference. (Further discussion and validations can be found in Gantmacher and Krein [4], see also Karlin [11], for extensions and applications).

(i) All the eigenvalues of A are positive and distinct. Arranging them in decreasing order gives

$$\lambda_1(A) > \lambda_2(A) > \cdots > \lambda_n(A) > 0 \tag{5.1}$$

Thus

$$Au^{(i)} = \lambda_i u^{(i)} \qquad v^{(i)}A = \lambda_i v^{(i)} \tag{5.2}$$

The multiplying factors for the eigenvectors can be chosen in a manner that the wedge product vectors

$$\mathbf{u}^{(1)} \wedge \mathbf{u}^{(2)} \wedge \cdots \wedge \mathbf{u}^{(k)} > 0, \mathbf{v}^{(1)} \wedge \mathbf{v}^{(2)} \wedge \cdots \wedge \mathbf{v}^{(k)} > 0 \tag{5.3}$$

(are positive) $\qquad k = 1, 2, \cdots, n.$

Recall that if $\mathbf{x}^{(i)} = (x_1^{(i)}, \cdots, x_n^{(i)})$ then $\mathbf{x}^{(1)} \wedge \mathbf{x}^{(2)} \wedge \cdots \wedge \mathbf{x}^{(k)}$ can be concretely expressed as the vector whose $\boldsymbol{\alpha} = (j_1, j_2, \cdots, j_k)$ coordinate is $\det(\|x_{j_\nu}{}^{(i)}\|_{i=1}^{k}{}_{,\nu=1}^{k})$. This arrangement of the coordinates is consistent with the lexicographic ordering used in representating $A_{[k]}$.

The vectors listed in (5.3) are right and left eigenvectors, respectively for the spectral radius of the corresponding compound matrix $A_{[k]}$ so that,

$$\rho(A_{[k]}) = \prod_{i=1}^{k} \lambda_i. \tag{5.4}$$

and

$$A_{[k]}(\mathbf{u}^{(1)} \wedge \cdots \wedge \mathbf{u}^{(k)}) = \left(\prod_{i=1}^{k} \lambda_i\right)(\mathbf{u}^{(1)} \wedge \cdots \wedge \mathbf{u}^{(k)}).$$

We will further scale the eigenvectors in (5.2) to satisfy

$$\sum_{\alpha=1}^{\binom{n}{k}} (\mathbf{u}^{(1)} \wedge \mathbf{u}^{(2)} \wedge \cdots \wedge \mathbf{u}^{(k)})_\alpha (\mathbf{v}^{(1)} \wedge \mathbf{v}^{(2)} \wedge \cdots \wedge \mathbf{v}^{(k)})_\alpha = 1 \tag{5.5}$$

$k = 1, 2, \cdots, n.$

Let $\mathbf{x} \circ \mathbf{y}$ *denote the Schur product of the vectors* $\mathbf{x}$ and $\mathbf{y}$, i.e., $\mathbf{x} \circ \mathbf{y} = (x_1y_1, x_2y_2, \cdots, x_ny_n)$.

Let D be, as usual, a diagonal matrix with positive diagonal entries. If A is oscillatory then AD is manifestly also oscillatory. Recall the elementary fact that $(AD)_{[k]} = A_{[k]}D_{[k]}$ and $D_{[k]}$ is obviously again a positive diagonal matrix of order $\binom{n}{k}$. The results of Theorem 3.1 and 4.1 carry over, mutatis mutandis, to the compounds of A.

THEOREM 5.1. (1) *Let A be an oscillatory matrix. Assume the eigenvectors are normalized to satisfy the conditions* (5.3) *and* (5.5). *Let D be a positive diagonal matrix. (For* $\mathbf{w} = (w_1, \cdots, w_n)$, $w_i > 0$, *we introduce the notation* $|D|^{\mathbf{w}} = \prod_{i=1}^{n} d_i^{w_i}$.) *Then*

$$\prod_{i=1}^{k}\left(\frac{\lambda_i(DA)}{\lambda_i(A)}\right) \geq |D_{[k]}|^{(\mathbf{u}^{(1)}\wedge\cdots\wedge\mathbf{u}^{(k)})\circ(\mathbf{v}^{(1)}\wedge\cdots\wedge\mathbf{v}^{(k)})} \tag{5.6}$$

$k = 1, 2, \cdots, n.$

(ii) *If $A = E_1KE_2$ where E_1 and E_2 are positive diagonal and K is oscillatory and symmetric, then*

$$\prod_{i=1}^{k}\left(\frac{\lambda_i(DA)}{\lambda_i(A)}\right) \geq (\mathbf{v}^{(1)} \wedge \cdots \wedge \mathbf{v}^{(k)}, D_{[k]}(\mathbf{u}^{(1)} \wedge \cdots \wedge \mathbf{u}^{(k)})). \tag{5.7}$$

On the right above appears the inner product of the vectors

$$\mathbf{v}^{(1)} \wedge \cdots \wedge \mathbf{v}^{(k)} \text{ and } D_{[k]}(\mathbf{u}^{(1)} \wedge \cdots \wedge \mathbf{u}^{(k)}).$$

II. *Upper bounds for the spectral radius* $\rho(DM)$.

The following classical characterizations of the spectral radius for nonnegative matrices will serve us in Theorem 5.2 below and in the discussion of a number of examples.

Let A be a nonnegative matrix. Then

$$\rho(A) = \sup_{\mu \in \Gamma} \mu \quad \text{where} \quad \Gamma = \{\mu \mid \mu \text{ real and } A\mathbf{x} \geq \mu\mathbf{x} \text{ for some } \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}\}. \tag{5.8}$$

(The notation $\mathbf{z} \geq \mathbf{w}$ signifies that $\mathbf{z} - \mathbf{w}$ is a nonnegative vector.)

(5.9) If there exists $\mathbf{y} > \mathbf{0}$ (strictly positive) satisfying $A\mathbf{y} \leq \gamma\mathbf{y}$, $\gamma > 0$ then $\rho(A) \leq \gamma$.

The characterization (5.8) due initially to Wielandt can be cast in the form of a minimax expression. Proofs of (5.8) and (5.9) can be found in innumerable sources, e.g., see the appendix of Karlin [10].

THEOREM 5.2. (i). *Let M be nonnegative and irreducible and D a positive definite diagonal matrix,* $D = \operatorname{diag}(d_1, \cdots, d_n)$, *then*

(5.10) $$\rho(DM) \leq \rho(M) \max_{1 \leq i \leq n} (d_i).$$

(ii) *If M is totally positive of order 2 then* $\rho(MD) \leq \sum_{i=1}^n m_{ii} d_i$.

Proof. (i). Since M is irreducible there exists $\mathbf{u} > \mathbf{0}$ satisfying $M\mathbf{u} = \rho(M)\mathbf{u}$. Therefore

$$DM\mathbf{u} \leq \mathbf{u}\rho(M) \max_{1 \leq i \leq n} d_i .$$

The result (5.10) now follows by direct appeal to (5.9) as $\mathbf{u}$ is strictly positive.

(ii). The proof of (ii) is found in Karlin [12, Th. 10.11].

In Theorem 4.1 (consult also (4.10)) we uncovered the bound

$$\sup_{\mathbf{x}>0} \sum_{i=1}^n u_i v_i \frac{x_i}{(M\mathbf{x})_i} \leq \frac{1}{\rho(M)}.$$

For comparison purposes we record the following fact. Let A be an *oscillatory matrix of order* 2 (designated OS_2), i.e., A is irreducible nonnegative and the second order compond $A_{[2]}$ is also nonnegative and irreducible. Assume $\alpha_i > 0$, $i = 1, 2, \cdots, n$, then

(5.11) $$\min_{\mathbf{x}>0} \sum_{i=1}^n \alpha_i \frac{x_i}{(A\mathbf{x})_i} = \frac{1}{\max_{1 \leq i \leq n} \left[\frac{a_{ii}}{\alpha_i}\right]}.$$

This finding is implicit in Karlin [12, Eq. (10.8)]. For completeness we review the setting from which (5.11) emanates. Let A be nonnegative irreducible and let D be a diagonal matrix where the vector $\mathbf{d}$ of the diagonal components satisfies $d_i \geq 0$, $\Sigma d_i = |\mathbf{d}| = 1$. We wish to evaluate $\max_{\mathbf{d} \geq 0, |\mathbf{d}|=1} \rho(DA)$ where $D = \operatorname{diag} \mathbf{d}$. Where A is of class OS_2 we obtain

(5.12) $$\max_{\mathbf{d} \geq 0, |\mathbf{d}|=1} \rho(DA) = \max_{1 \leq i \leq n} a_{ii}.$$

Obviously, for $A > 0$ and each $\mathbf{d} > 0$, $|\mathbf{d}| = 1$ there exists $\mathbf{x} > 0$ satisfying $DA\mathbf{x} = \rho(DA)\mathbf{x}$ or equivalently $[d_i/\rho(DA)] = x_i/(A\mathbf{x})_i$ and therefore $[1/\rho(DA)] = \sum_{i=1}^n x_i/(A\mathbf{x})_i$. Since $\mathbf{d}$ is arbitrary subject to the constraints $\mathbf{d} \geq 0$, $|\mathbf{d}| = 1$ and $\mathbf{x}$ is determined up to a scalar multiple we deduce that

(5.13) $$\inf_{\mathbf{x}>0} \sum_{i=1}^n \frac{x_i}{(A\mathbf{x})_i} = \frac{1}{\max_{\mathbf{d} \geq 0, |\mathbf{d}|=1} \rho(DA)} = \frac{1}{\max_{1 \leq i \leq n} a_{ii}}$$

the last equation resulting by virtue of (5.12). By absorbing a diagonal matrix composed of the α_i into A we find that (5.11) agrees with (5.13).

III. *Applications of the inequalities* (1.8) *and* (1.12) *to local stability analysis.*

For the population migration models of (1.2) and (1.3) it is of interest to ascertain conditions on the migration matrix M and the matrix of selection coefficients D implying

(5.14) $$\rho(DM) > 1.$$

Where the above inequality holds, the connotation is (for the model (1.2)) that the A-type in the population can never tend to extinction and will always be represented with moderate to high frequency.

Obviously from Theorems 3.1 and 4.1 we secure simple sufficient conditions insuring (5.14). More specifically, if

(5.15) $$M \in \mathfrak{M}(\mathbf{u}, \mathbf{v}) \quad \text{and} \quad M = E_1 K E_2$$

(E_1, E_2 positive diagonal, K positive semi-definite) then (5.14) prevails if

(5.16) $$\sum_{i=1}^n u_i v_i d_i > 1.$$

Generally, already the equation $\sum u_i v_i d_i = 1$ entails (5.14). However we pointed out earlier in (1.14) that $M = \|u_i v_j\| \in M(\mathbf{u}, \mathbf{v})$ carries the equation sign

(5.17) $$\rho (DM) = \sum_{i=1}^n u_i v_i d_i$$

and in this case (5.14) and (5.16) are trivially equivalent. It is elementary to check that if $M \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$ is of type (5.15), then M^k for each integer $k \geq 1$ is also of type (5.15). Therefore, in the presence of (5.16), we have

(5.18) $$\rho(DM^k) > 1 \quad \text{for all } k.$$

Conversely suppose (5.18) holds for all k (or even for infinitely many k) with M irreducible and $\in \mathfrak{M}(\mathbf{u}, \mathbf{v})$. We know from a standard ergodic theorem that $M^k \to \|u_i v_j\|$ as $k \to \infty$. With this convergence and the stipulation of (5.18) we infer the inequality $\sum_{i=1}^n u_i v_i d_i \geq 1$.

IV. *Discussion of some examples.*

(a) Consider a tridiagonal (Jacobi) nonnegative matrix

(5.19) $$M = \begin{pmatrix} r_1 & p_1 & 0 & 0 & 0 \\ q_2 & r_2 & p_2 & 0 & \vdots \\ 0 & q_3 & r_3 & p_3 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdot & & q_{n-1} & r_{n-1} & p_{n-1} \\ 0 \cdots & & & q_n & r_n \end{pmatrix}$$

with $p_i > 0$, $q_i > 0$ and $r_i \geq 0$. Assume also $q_i + r_i + p_i = 1$ for all i so that M is a stochastic matrix. Choosing ϵ_i positive to satisfy $\epsilon_i/\epsilon_{i+1} = p_i/q_{i+1}$ $i = 1, \cdots, n-1$ and defining $E_2 = \operatorname{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ we obtain $ME_2 = K$ where K is a symmetric Jacobi matrix. Where the upper bordered principal determinants of M are positive we deduce that K is positive definite. More concretely, if

$$(5.20)\qquad \begin{vmatrix} r_1 & p_1 & 0 & \cdots & 0 \\ q_2 & r_2 & p_2 & \cdots & \vdots \\ 0 & q_3 & r_3 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 \cdots 0 & \cdots q_i & r_i & & p_i \end{vmatrix} > 0, \qquad i = 1, 2, \cdots, n.$$

then M is of type E_1KE_2 where E_i are positive definite diagonal matrices and K is positive definite.

Moreover, it is easy to verify that $M \in \mathfrak{M}(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} = (1, 1, \cdots, 1)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ has the form

$$(5.21)\qquad v_i = \frac{\pi_i}{\sum_{j=1}^n \pi_j}, \qquad \pi_1 = 1 \text{ and } \pi_j = \frac{p_{j-1}p_{j-2}\cdots p_1}{q_jq_{j-1}\cdots q_2}, \qquad j \geq 2.$$

Provided (5.20) holds then application of Theorem 4.2 affirms that

$$(5.22)\qquad \rho(DM) > 1 \quad \text{if} \quad \frac{\sum_{i=1}^n d_i\pi_i}{\sum_{i=1}^n \pi_i} \geq 1.$$

The special doubly stochastic matrix

$$(5.23)\qquad M = \begin{vmatrix} 1-m & m & 0 & \cdots & 0 \\ m & 1-2m & m & \cdots & \vdots \\ 0 & m & 1-2m & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & m & 1-2m & m \\ 0 \cdots\cdots & \cdots\cdots & 0 & m & 1-m \end{vmatrix}$$

is positive definite provided $m \leq \frac{1}{4}$ independent of the size of the matrix. Then conforming to (5.22) we have

$$\rho(DM) > 1 \quad \text{if} \quad \frac{1}{n}\sum_{i=1}^n d_i \geq 1$$

(b) We concentrate next on a nonnegative matrix M of the form

$$(5.24)\qquad M = E + R \text{ with } E = \operatorname{diag}(e_1, \cdots, e_n),\ e_i > 0 \text{ and } R = \|u_iv_j\|_{i,j=1}^n.$$

An important special case occuring in the theory of geographical subdivisions of populations has

$E = (1 - \alpha)I$ (I = identity matrix) and where $v_i > 0$,

$$\sum_{i=1}^n v_i = 1, \qquad u_i \equiv \alpha$$

so that

$$(5.25)\qquad m_{ij} = (1-\alpha)\delta_{ij} + \alpha v_j \cdot (\delta_{ij} = \text{Kronecker delta}), \quad \sum_{j=1}^n v_j = 1.$$

The determination of $\rho(DM)$ depends on the following known formula for calculating the inverse of (5.24). Consider a matrix $C = B + R$ where R is of rank 1 as in (5.24). If B^{-1} exists then provided $c = 1 + (\mathbf{v}, B^{-1}\mathbf{u}) \neq 0$, C^{-1} has the form

$$(5.26)\qquad C^{-1} = B^{-1} - \left(\frac{1}{c}\right)S \text{ where } S \text{ of rank 1 has the form } \|s_ir_j\| \text{ and specifically } \mathbf{r} = \mathbf{v}B^{-1},\ \mathbf{s} = B^{-1}\mathbf{u} \text{ with } M \text{ as in (5.24).}$$

We can also represent $MD - \lambda I$ in the same form, viz

$$MD - \lambda I = (ED - \lambda I) + \tilde{R} \text{ with } \tilde{R} = \|u_iv_jd_j\|.$$

We examine two cases.

Case (i). Suppose $e_{i_0}d_{i_0} \geq 1$ for some index i_0, $1 \leq i_0 \leq n$. Let $\mathbf{z}^{(i_0)} = (0, 0 \cdots 0, 1, 0 \cdots 0)$ with the component 1 appearing only at the i_0-th coordinate. Since MD is nonnegative then by virture of $e_{i_0}d_{i_0} \geq 1$ we find

$$(5.27)\qquad MD\mathbf{z}^{(i_0)} \geq \mathbf{z}^{(i_0)}$$

and the two vectors in (5.27) are not equal. Invoking the characterization of (5.8) implies

$$\rho(MD) > 1.$$

Case (ii). Suppose $e_id_i < 1$ for all $i = 1, 2, \cdots, n$. Then for each $\lambda \geq 1$, $(ED - \lambda I)^{-1}$ exists. By the prescription of (5.26), $(MD - \lambda I)^{-1}$ exists if $c(\lambda) = 1 + (\mathbf{v}\, D, (ED - \lambda I)^{-1}\mathbf{u}) \neq 0$ that is provided

$$c(\lambda) = 1 + \sum_{i=1}^n \frac{v_id_iu_i}{e_id_i - \lambda} \neq 0.$$

Obviously $c(\infty) = 1$ and since $e_id_i < \lambda$ for all $\lambda \geq 1$ the monotone increasing function $c(\lambda)$ vanishes somewhere on the range $1 \leq \lambda < \infty$ and then $\rho(DM) > 1$ if and only if $0 > c(1) = 1 + \sum_{i=1}^n [v_id_iu_i/(e_id_i - 1)]$ or equivalently

$$(5.28)\qquad \sum_{i=1}^n \frac{v_id_iu_i}{1 - e_id_i} > 1$$

To sum up, if M has the form (5.24), then

(5.29) $\rho(DM) > 1$ iff either $e_id_i \geq 1$ for some i or

$$\sum_{i=1}^n \frac{v_id_iu_i}{1 - e_id_i} > 1 \text{ holds}$$

Where $e_id_i < 1$ for all i and (5.28) prevails then the actual value of $\rho(DM)$ can be calculated as the unique solution λ of the equation

$$\sum_{i=1}^{n} \frac{d_i u_i v_i}{\lambda - e_i d_i} = 1$$

located in the interval $\max_{1 \le i \le n} e_i d_i < \lambda < \infty$. The preceding discussion interpreted for the special example (5.25) leads to the following characterization: $\rho(DM) > 1$ iff either $(1 - \alpha)d_i > 1$ for some i or

$$\alpha \sum_{i=1}^{n} \frac{v_i}{\frac{1}{d_i} - 1 + \alpha} > 1 \text{ holds}$$

(c) A hybrid version of the previous example combined with a permutation matrix is as follows. Consider M of the form

(5.30) $M = E + R$ where R decomposes into block matrices explicit structure

$$R = \begin{bmatrix} 0 & \mathbf{R}_{12} & 0 \cdots & 0 \\ 0 & 0 & \mathbf{R}_{23} & 0 \\ \vdots & & & \mathbf{R}_{p-1p} \\ \mathbf{R}_{p1} & 0 & 0 & 0 \end{bmatrix} \tag{5.31}$$

where $\mathbf{R}_{kk+1} = \|u_i^{(k)} v_j^{(kn)}\|_{i,j=1}^{n_k, n_{k+1}}$ is a rank one matrix of order $n_{k+1} \times n_k$, $k = 1, 2, \cdots, p$ (interpret $p + 1$ as 1) and $\sum_{k=1}^{p} n_k = n$, $E = \operatorname{diag}(e_1, e_2, \cdots, e_n)$ is a positive diagonal matrix. It is convenient to view $(e_1, \cdots, e_n) = \hat{\mathbf{e}}$ as a juxtaposition of k vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \cdots, \mathbf{e}^{(p)}$ where $\mathbf{e}^{(1)}$ consist of the first n_1 components of $\hat{\mathbf{e}}$, $\mathbf{e}^{(2)}$ the next n_2 components, etc.

The complete result of instability corresponding to the matrix DM ($D = \operatorname{diag}(d_1, \cdots, d_n)$ is a positive diagonal matrix as usual) supplementing (5.29) is as follows: (We decompose the vector $\mathbf{d}$ analogously to $\hat{\mathbf{e}}$.)

(5.32) $\rho(DM) > 1$ iff one of the following two conditions hold: either $e_i d_i \ge 1$ for some i or

$$(-1)^p \prod_{k=1}^{p} \left(\sum_{i=1}^{nk} \frac{u_i^{(k)} v_i^{(k)} d_i^{(k)}}{e_i^{(k)} d_i^{(k)} - 1} \right) > 1$$

6. Some classes of inverse eigenvalue problems.

We will apply the relations (2.6) and (2.8) to set forth general necessary conditions for the solution of certain inverse eigenvalue problems. The uniqueness characterizations underlying (1.9) will serve to secure unique solutions for special classes of inverse eigenvalue problems. At this point it would be helpful to consult the background material of the introductory section. We start our deliberations with the inverse additive eigenvalue problem (I.A.E.P.) whose formulation is as follows

(i) Let $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ consist of n values in R^1 and let a matrix A be prescribed. The problem is to ascertain criteria for the existence and de-

termination of at least one real valued diagonal matrix D with the property that the spectrum of $A + D$ coincides with Λ. In our context, the I.A.E.P. is said to be solvable for the prescriptions A and Λ if there exists at least one real diagonal matrix D such that $A + D$ has spectrum Λ.

Our main finding on problem (i) is summarized in Theorem 6.1 highlighted in the introductory section. We proceeed to the proof. The following lemma will be needed.

LEMMA 6.1. *Let A be an irreducible nonnegative matrix. There exists a unique (up to a constant factor) diagonal matrix $C = \operatorname{diag}(c_1, c_2, \cdots, c_n)$, $\mathbf{c} = (c_1, \cdots, c_n) > \mathbf{0}$, satisfying*

$$CAC^{-1}\mathbf{e} = \mathbf{e}CAC^{-1} \qquad \mathbf{e} = (1, 1, \cdots, 1) \tag{6.1}$$

Proof. Consider the function

$$f(\mathbf{x}) = \sum_{i=1}^{n} \frac{(A\mathbf{x})_i}{x_i} \quad \text{defined for } \mathbf{x} > \mathbf{0}, \qquad |\mathbf{x}| = \Sigma x_i = 1. \tag{6.2}$$

Straightforward adaptation of the reasoning of Lemma 3.2 establishes the existence of a unique critical point $\boldsymbol{\xi}$ for $f(\mathbf{x})$. Thus $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ satisfies the equations

$$\frac{-(A\boldsymbol{\xi})_j}{\xi_j^2} + \sum_{i=1}^{n} \frac{a_{ij}}{\xi_i} = 0 \qquad j = 1, 2, \cdots, n. \tag{6.3}$$

Defining $C = \operatorname{diag}(1/\xi_1, 1/\xi_2, \cdots, 1/\xi_n)$ the equations (6.3) are synonymous with (6.1). Conversely if $\mathbf{c}^* = \operatorname{diag}(c_1^*, c_2^*, \cdots, c_n^*)$ qualifies for (6.3), with $\mathbf{c}^*$ normalized by $\sum_{i=1}^{n} 1/c_i^* = 1$ then we can backtrack and verify that the vector $\boldsymbol{\xi}^* = (1/c_1^*, \cdots, 1/c_n^*)$ is a critical point of $f(\mathbf{x})$. The uniqueness assertion of the lemma ensues since a single critical point exists. The proof of lemma 6.1 is complete.

With C constructed conforming to the relations (6.1), we next determine C_0 as the unique diagonal matrix satisfying

$$(CAC^{-1} + C_0)\mathbf{e} = \mathbf{e}(CAC^{-1} + C_0) = \mathbf{e}. \tag{6.4}$$

We are now prepared to deal with Theorem 6.1 which for convenience is restated.

THEOREM 6.1. *Let A be nonnegative and irreducible. Let C and C_0 be determined fulfilling (6.1) and (6.4). Denote by $\Lambda^0 = \{\lambda_1^0, \lambda_2^0, \cdots, \lambda_n^0\}$ the spectrum of $A + C_0$ (since $A \ge 0$ is irreducible and C_0 is real it follows from the Perron Frobenius theorem that λ_1^0 is certainly real and $\lambda_1^0 > \operatorname{Re} \lambda_i^0$, $i = 2, 3, \cdots, n$). Suppose the I.A.E.P. with respect to A and the prescribed spectrum $\Lambda = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$ having $\lambda_1 > \operatorname{Re} \lambda_i$, $i = 2, 3, \cdots, n$ is solvable such that the spectrum of $(A + D) = \Lambda$ with D real diagonal. Then Λ satisfies*

$$(n - 1)\lambda_1 \ge \sum_{\substack{i,j=1 \\ i \ne j}}^{n} c_i a_{ij} \frac{1}{c_j} + \sum_{i=2}^{n} \lambda_i \tag{6.5}$$

The equality sign holds only for the spectral set $\{\lambda_1^0 + \beta, \lambda_2^0 + \beta, \cdots, \lambda_n^0 + \beta\}$ *for any real* β.

Proof. Suppose the spectrum of $A + D$ is Λ. Because C and D are diagonal, the spectrum of $CAC^{-1} + D$ is likewise Λ. On the basis of the Perron Frobenius theorem we obtain

$$(CAC^{-1} + D)\mathbf{x} = \lambda_1 \mathbf{x} \quad \text{for} \quad \mathbf{x} > 0.$$

Therefore,

$$(6.6) \qquad n\lambda_1 = \sum_{i=1}^{n} \frac{((CAC^{-1} + D)\mathbf{x})_i}{x_i} = \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} \frac{(CAC^{-1}\mathbf{x})_i}{x_i}$$

Invoking Lemma 2.2 we find that the last sum is bounded below by

$$\sum_{i,j=1}^{n} c_i a_{ij} \frac{1}{c_j}$$

and equality is attained only if $\mathbf{x} = \alpha\mathbf{e}$. The equation Trace $(A + D) = \sum_{i=1}^{n} \lambda_i$ written out is

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} d_i \quad \text{or} \quad \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} a_{ii} .$$

Substituting for $\sum d_i$ the inequality (6.5) is achieved with the equation sign possible only if

$$(6.7) \qquad (CAC^{-1} + D)\mathbf{e} = \mathbf{e}(CAC^{-1} + D) = \alpha\mathbf{e} \qquad \text{prevails.}$$

But the fulfillment of (6.7) in view of the results of Lemma 6.1 and the nature of the determination of C_0 imply $D = D_0 + \beta I$ for some appropriate real β. For $D_0 = C_0 + \beta I$ the spectrum of $A + D_0$ is precisely $\{\lambda_1^0 + \beta, \cdots, \lambda_n^0 + \beta\}$. The proof of Theorem 6.1 is complete.

COROLLARY 6.1. *Let the hypothesis and notation of Theorem* 6.1 *prevail. Consider the I.A.E.P. for A with prescribed spectrum* $\Lambda = \{\lambda_1^0 + \alpha, \cdots, \lambda_n^0 + \alpha\}$, α *real and fixed. This inverse additive eigenvalue problem has a unique real solution given by* $D_0 = C_0 + \alpha I$.

We develop next the analog of Theorem 6.1 for the inverse multiplicative eigenvalue problem (I.M.E.P.)

(ii) Given A nonnegative and irreducible and the spectrum Λ as in problem (i) we seek to solve or characterize a diagonal matrix D with nonnegative entries such that the spectrum of AD is precisely Λ.

THEOREM 6.2. *Let A be a positive nonsingular matrix. Let* D_1 *and* D_2 *be diagonal positive matrices rendering* D_1AD_2 *doubly stochastic (in this connection see the discussion culminating section* 2). *The matrices* D_1 *and* D_2 *are unique up to constant multipliers. Define* $D_0 = D_1D_2$ *and denote the spectrum of* AD_0 *by* $\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_n^0\}$, $\lambda_1^0 = \rho(AD_0)$. *If the I.M.E.P. is solvable for* Λ *with* $\lambda_1 = \rho(AD)$, *then we have*

$$(6.8) \qquad \prod_{i=2}^{n} \lambda_i^0 \geq \prod_{i=2}^{n} \frac{\lambda_i}{\lambda_1}$$

Equality holds only if $\lambda_i = \alpha\lambda_i^0$, $i = 1, 2, \cdots, n$, *for some* α.

Proof. Consider that $\lambda_i(AD) = \lambda_i$ with $\lambda_1(AD) = \rho(AD)$. If $d_i = 0$ for some i then $\prod_{i=2}^{n} \lambda_i = 0$ and (6.8) is trivially true. Suppose now that $d_i > 0$, $i = 1, \cdots, n$. Invoking the inequality (2.8) (since D_2AD_1 is doubly stochastic) yields

$$(6.9) \qquad \lambda_1^n = [\rho(AD)]^n = [\rho(AD_0DD_0^{-1}]^n \geq \left[\prod_{i=1}^{n} d_i\left(\prod_{1}^{n} (d_i^0)^{-1}\right)\right]$$

Plainly

$$(6.10) \qquad \prod_{i=1}^{n} d_i^0 = \left(\prod_{i=1}^{n} \lambda_i^0\right)(\det A)^{-1}, \qquad \prod_{i=1}^{n} d_i = \left(\prod_{i=1}^{n} \lambda_i\right)(\det A)^{-1}.$$

Of course, $\lambda_1^0 = 1$ since D_2AD_1 is doubly stochastic. Combining (6.10) into (6.9) produces (6.8).

The equation sign in (6.8), as A is strictly positive compels the case of equality in Lemma 2.2 and then necessarily $DD_0^{-1} = \alpha I$. The proof is finished.

COROLLARY 6.2. *Let A satisfy the assumptions and notation of Theorem* 6.2. *We prescribe* $\Lambda^0 = \{\beta\lambda_1^0, \beta\lambda_2^0, \cdots, \beta\lambda_n^0\}$ *for* $\beta > 0$. *Then the I.M.E.P. has a unique solution* $D = \beta D_0$.

Another result supplementary to Theorem 6.2 relies on (1.12) instead of (2.8). Accordingly we obtain

THEOREM 6.3. *Let A be a doubly stochastic matrix such that* $a_{ii} = a > 0$ *(with constant diagonal.) Assume furthermore that either* A^{-1} *is an M-matrix (see Definition* 3.1) *or* $A = E_1KE_2$ *where* K, E_1 *and* E_2 *the latter two diagonal, are all positive definite. If for some positive diagonal D the spectrum of DA is the set* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *with* $\lambda_1 > \operatorname{Re} \lambda_i$, $i = 2, \cdots, n$, *then*

$$(6.11) \qquad \lambda_1 \geq \frac{1}{na} \sum_{i=1}^{n} \lambda_i .$$

The equality sign is maintained if and only if $D = \alpha I$ *for some* $\alpha > 0$.

Remark 6.1. The uniqueness feature of the inverse problems of Corollaries 6.1 and 6.2 are akin or perhaps discrete analogs to the uniqueness result demonstrated for the system of eigenvalues occurring in the case of a vibrating membrane, see Kac [7] and Kac and Van Moerbeke [8].

7. Inequalities for the spectral radius of some integral operators.

By a standard discretization akin to the analysis of Fredholm kernel operators, approximating an integral operator by a sequence of matrix operators we can secure extensions of (1.8) and (1.12) to certain classes of integral operators. This procedure will be described succinctly. Let Ω be a compact set in R^m. As usual, denote by $C(\Omega)$ the collection of all real valued continuous functions on Ω. The subset of nonnegative functions of $C(\Omega)$ is designated by $C_+(\Omega)$ and by $C_+^+(\Omega)$, the positive function. Throughout, let

(7.1) *$K(x, y)$ be a continuous nonnegative kernel on $\Omega \times \Omega$.*

Let σ be a prescribed nonnegative regular finite measure on Ω so that $\sigma(\Omega) < \infty$. Consider the linear operator

$$K_\sigma : C(\Omega) \to C(\Omega) \tag{7.2}$$

defined explicitly by the integral kernel mapping

$$K_\sigma(f)(x) = \int_\Omega K(x, y)f(y)\, d\sigma(y) \qquad f \in C(\Omega). \tag{7.3}$$

The operator K_σ is obviously compact, even transforming a bounded set of measurable functions into an equi-continuous family of functions. The assumption (7.1) entails that K_σ preserves the cone $C_+(\Omega)$. The theory of operators leaving invariant a cone (dubbed positive operators) is well developed, e.g., see Krein and Rutman [15], Karlin [13], Sawashima [16], Krasnoselskii [14, Chap. 2], Anselone and Lee [1] among others. Most of the Frobenius theory of nonnegative matrices carries over to the case of compact nonnegative operators. More specifically,

(i) The spectral radius $\rho(K_\sigma) > 0$ and $\rho(K_\sigma)$ is a simple eigenvalue of K_σ

(ii) Suppose $\Omega = \bar{\Omega}^0$ ($\bar{\Omega}^0$ is the closure of its interior), then there exists a unique (up to scalor multipliers) functions $u(x)$ and $v(x) \in C_+(\Omega)$ with $u(x)$ and $v(x)$ strictly positive on Ω^0 satisfying

$$\rho(K_\sigma)u(x) = \int_\Omega K(x, y)u(y)\, d\sigma(y) \quad \text{(indicating that } u \text{ is an eigenvector for } K_\sigma \text{ corresponding to } \lambda_0 = \rho(K_\sigma)) \tag{7.4}$$

$$\rho(K_\sigma)v(y) = \int_\Omega v(x)K(x, y)\, d\sigma(x) \quad \text{(signifying } v \text{ is an eigenvector for } K_\sigma^*\text{; the adjoint to } K_\sigma)$$

(ii′) If Ω^0 is empty, then we stipulate that u and $v \in C_+^+(\Omega)$. In the presence of (ii) and (ii′) we normalize u and v to satisfy

$$\int_\Omega u(y)\, d\sigma(y) = 1 \quad \text{and} \quad \int_\Omega u(y)v(y)\, d\sigma(y) = 1 \tag{7.5}$$

We call an operator *endowed with the properties (i) and (ii) irreducible.*

A number of useful sufficient conditions implying that K_σ is irreducible are subsumed in the following two cases. Thus, K_σ is nonnegative irreducible if either of the following two hypotheses holds:

(a) $K(x, x) > 0$ for $x \in \Omega$ (K is strictly positive on the diagonal), Sawashima [16];

(7.6) (b) $K(x, y) = r(x)G(x, y)s(y)$ where r and $s \in C_+^+(\Omega)$ and G is symmetric such that G induces a positive definite operator.

We now state the first result which extends the inequality (1.8).

THEOREM 7.1. *Let K_σ be defined as in* (7.3) *satisfying* (7.1) *and either* (7.6a) *or* (7.6b). *(Thus K_σ is irreducible.) Let $p(x) \in C_+^+(\Omega)$. Denote by $(Kp)_\sigma$ the integral operator* (7.3) *induced by the kernel $K(x, y)p(y)$. Then*

$$\rho((Kp)_\sigma) \geq \rho(K_\sigma) \exp\left[\int_\Omega u(x)v(x)(\log p(x))\, d\sigma(x)\right]. \tag{7.7}$$

Proof. We will be brief. Let $X \in \{x_i\}_1^\infty$ be a dense set of points in Ω. The span of a convex set by its extreme point (Krein-Milman Theorem) provides the existence of a sequence of discrete measures $\sigma^{(k)}$ concentrating at a finite sub-set of X say $X^k = \{x_i\}_1^k$ with $\sigma^{(k)}$ converging in the weak $*$ topology to σ, that is

$$\lim_{k\to\infty} \int_\Omega f\, d\sigma^{(k)} = \int_\Omega f\, d\sigma \quad \text{for all} \quad f \in C(\Omega)$$

The integral operators K_σ and $(Kp)_\sigma$ can be approximated in the weak operator topology by a sequence of operators with finite range induced by the matrix mappings.

$$K_\sigma^{(k)} : \int K(x_i\,, y)f(y)\, d\sigma_k \quad \text{and} \quad Kp_\sigma^{(k)} : \int K(x_i\,, y)p(y)f(y)\, d\sigma_k \tag{7.8}$$

$$i = 1, 2, \cdots, k.$$

It can be checked that each of the operators (7.8) is nonnegative and irreducible. The inequality (1.8) is in force involving the eigenvectors $\mathbf{u}_k = \{u_k(x_i)\}_1^k$ and $\mathbf{v}_k = \{v_k(x_i)\}_1^k$ appropriately normalized. A routine compactness argument, involving the Fredholm determinants, validates that we can extract an appropriate subsequence of $\mathbf{u}_k$ converging uniformly to $u(x)$ on X and similarly, $\mathbf{v}_k \to v(x)$. The inequality (7.7) ensues by proceeding to the limit in k applying at each stage the corresponding matrix inequality. The proof is complete.

By completely similar means, we deduce

THEOREM 7.2. *Let the assumptions of Theroem 7.1 hold. Assume also that*

(7.9) *$K(x, y) = r(x)G(x, y)s(y)$ where r and $s \in C_+^+(\Omega)$ and $G(x, y)$ is a positive definite symmetric kernel, i.e.,*

$$\iint_{\Omega\times\Omega} G(x, y)f(x)f(y)\, d\sigma(x)\, d\sigma(y) > 0 \quad \textit{for} \quad f \in C(\Omega) \quad \textit{and} \quad f \not\equiv 0.$$

Let $p(x) \in C_+^+(\Omega)$. Then

$$\rho((Kp)_\sigma) \geq \rho(K_\sigma) \int_\Omega u(x)v(x)p(x)\, d\sigma(x) \tag{7.10}$$

Remark 7.1. A more delicate analysis shows that equality holds in (7.10) if and only if $p(x) \equiv \alpha > 0$ on Ω. For this purpose, we rely on the minimax characterization of the spectral radius for symmetrizable operators.

An application

Consider a translation kernel $K(x, y) = k(x - y)$. Take $\Omega = (-\pi, \pi)$. Assume $k(x)$ is periodic, symmetric with respect to 0 (an even function) and $\in C(\Omega)$. Accordingly, the Fourier expression of k has the form

$$k(x) \sim \sum_{n=0}^{\infty} a_n \cos nx \tag{7.11}$$

Assume further that the matrix

(7.12) $\|k(x_i - x_j)\|_1^p$ is nonnegative definite for every collection $-\pi < x_1 < x_2 < \cdots < x_p < \pi$, p arbitrary.

The celebrated Bochner theorem applied to $k(x)$ guarantees

$$a_n \geq 0, \qquad n = 0, 1, 2, \cdots, \sum_{n=0}^{\infty} a_n < \infty \tag{7.13}$$

and the fact of (7.13) entails that

(7.14) $k(x) = \sum_{n=0}^{\infty} a_n \cos nx$ with absolute and uniform convergence present

We also assume henceforth

(7.15) $k(x) > 0$ for $-\pi \leq x \leq \pi$ and normalized such that

$$k(0) = 1 = \sum_{n=0}^{\infty} a_n .$$

Let K be the operator

(7.16) $K(f) = \int_{-\pi}^{\pi} k(x - y)f(y)\, dy$ defined for $f \in C(\Omega)$ or even for $f \in L_2(\Omega)$.

The facts of (7.13) and (7.14) imply that K on $L_2(\Omega)$ is a *trace* (nuclear) *operator*. Obviously, the eigenvalues of K are $\{a_n\}_{n=1}^{\infty}$. In particular,

$$\rho(K) = 2\pi\, a_0 = \int_{-\pi}^{\pi} k(x)\, dx \tag{7.17}$$

and the associated eigenfunctions are the constant functions

$$u(x) = v(x) \equiv \frac{1}{\sqrt{2\pi}}$$

Let $p(x) \in C_+^+(\Omega)$ and consider the operator on $L_2(\Omega)$ or $C(\Omega)$

$$Kp(f) = \int_{-\pi}^{\pi} k(x - y)p(y)f(y)\, dy \tag{7.18}$$

Denote the spectrum of Kp by $\{\lambda_i(p)\}_1^{\infty}$ arranged so that $\lambda_1(p) = \rho(Kp)$. Standard theory tells us that Kp is also a nuclear (trace class 1) operator and therefore $\sum_{i=1}^{\infty} |\lambda_i(p)| < \infty$. Since Kp is a trace operator we find recalling the normalization $k(0) = 1$ that

$$\int_{-\pi}^{\pi} k(x - x)p(x)\, dx = \int_{-\pi}^{\pi} p(x)\, dx = \sum_{i=1}^{\infty} \lambda_i(p) \tag{7.19}$$

Application of theorem 7.2 gives

THEOREM 7.3. *Let $k(x)$ satisfy the assumptions stated prior to* (7.11), *and also* (7.12) *and* (7.15). *Let $p(x) \in C_+^+(\Omega)$. Then*

$$\rho(Kp) \geq \left(\int_{-\pi}^{\pi} k(x)\, dx\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} p(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(x)\, dx \cdot \sum_{i=1}^{\infty} \lambda_i(p) \tag{7.20}$$

(notice that $1/2\pi \int_{-\pi}^{\pi} p(x)\, dx = \int_{-\pi}^{\pi} p(x)u(x)v(x)\, dx$ and $\int_{-\pi}^{\pi} k(x)\, dx = \rho(K)$ by (7.17).)

Equality sign holds iff $p(x) \equiv \alpha > 0$.

From (7.15) and (7.13) we have that $1/2\pi \int_{-\pi}^{\pi} k(x)\, dx = a_0 < 1$, so that (7.20) is equivalent to

$$\rho\left(\frac{1 - a_0}{2\pi} Kp\right) \geq \frac{a_0}{2\pi} \sum_{i=2}^{\infty} \lambda_i(p).$$

Since the spectrum of the integral operator with kernel $k(x - y)p(y)$ coincides with the operator having kernel $\sqrt{p(x)}\, k(x-y) \sqrt{p(y)}$ we deduce that $\lambda_i(p) \geq 0$, $i = 1, 2, \cdots$. As a consequence we have

THEOREM 7.4. *Consider the multiplicative inverse eigenvalue problem to determine p belonging to $C_+(\Omega)$ with the prescribed spectrum $\Lambda = \{\lambda_i\}_1^{\infty}$. Assume that the spectrum is $\alpha\Lambda^0$ where $\Lambda^0 = \{a_i\}_0^{\infty}$, $\alpha > 0$. Then $p(x) = \alpha$.*

REFERENCES

1. P. M. ANSELONE, AND J. W. LEE, *Spectral properties of integral operators with non-negative kernels*, Linear Algebra and Appl., 9(1974), 67–87.
2. R. A. BRUALDI, S. V. PARTER, AND H. SCHNEIDER, *The diagonal equivalence of a non-negative matrix to a stochastic matrix*, J. Math. Anal. Appl. 16(1966), 31–50.
3. S. FRIEDLAND, *Inverse eigenvalue problems*, submitted.
4. F. R. GANTMACHER AND M. G. KREIN, *Oscillating Matrices and Kernels and Small Vibrations of Mechanical Systems*, Moscow, 1950.
5. O. H. HALD, *On discrete and numerical inverse Sturm–Liouville problems*, Uppsala Univ., Dept. Computer Sci., Report No. 42.
6. G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, England, 1946.
7. M. KAC, *Can one hear the shape of a drum?*, Bulletin Amer. Math. Soc. 73(1966), 1–23.
8. M. KAC AND P. VAN MOERBEKE, *On some isospectral second order differential operators*, Proc. Nat. Acad. Sci. 71(1974), 2350–2351.
9. S. KARLIN, *Environmental selection gradients and gene flow interaction*. Theor. Pop. Biol., to appear.

10. ———, *Mathematical Methods and Theory in Games, Programming and Economics*, Addison Wesley, Cambridge, Massachusetts, 1959.
11. ———, *Total Positivity*, Vol. I., Stanford University Press, California, 1968.
12. ———, *Some extremal problems for eigenvalues of certain matrix and integral operators*, Advances in Mathematics 9(1972), 93–136.
13. ———, *Positive operators*, J. Math. Mech. 8(1959), 907–937.
14. M. A. Krasnosel'skii, *Positive Solution of Operator Equations*. Noordhoff LTP, The Netherlands, 1964.
15. M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Translation #26, (1950).
16. F. Sawashima, *On non-support positive operators*, Jap. Journal of Math. (1966), 186–199.
17. R. Sinkhorn, *A relationship between arbitrary positive matrices and doubly stochastic matrices*, Ann. Math. Statist. **35**(1964), 876–879.
18. R. Sinkhorn and P. Knopp, *Concerning non-negative and doubly stochastic matrices*, Pacific J. Math. **21**(1967), 343–348.

Friedland: Department of Mathematics, Stanford University, Stanford, California 94305. The Institute for Advanced Study, Princeton, N. J. 08540.

Karlin: Mathematics Department, The Weizmann Institute of Science, Rehovot, Israel. Department of Mathematics, Stanford University, Stanford, California 94305