Logistic Map: Cobweb Plots and the Time Domain

In this worksheet, you'll be exploring the dynamics of the logistic map:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n) \tag{1}$$

Point your browser to:

`joshuagarland.com`

and then navigate to the "Dynamics Sandbox" and click on the magnifying glass in the cobweb diagram image.

For the following exercises we will be using the two different plots in this app: the time series on the right and the "cobweb plot" on the left, which plots x_{n+1} versus x_n. Here is a brief explanation about the controls of this application.

Cobweb and Time Domain Plot Controls:

Inputs:

The "Initial Condition x_0" text field changes the point from which the trajectory begins. This input is only well defined between 0 and 1. The "Parameter (r)" text field changes the logistic map's r parameter. Alternatively, you can also change r by clicking your mouse on the cobweb plot. The top of the parabola–defined by Equation —will be moved to the point you clicked, effectively adjusting the r parameter. The r parameter currently selected will appear in the cobweb plot's legend, as well as in the "Parameter (r)" text field. The "Number of Initial Iterates" field changes how many iterates of the logistic map are plotted initially, equivalently, how long the initial trajectory is.

Plot Controls: After manually changing the text fields press "Restart Simulation" to update the plots. You can add single iterates to the end of the trajectory or remove a single iterate from the beginning of the trajectory using the respective buttons; these two buttons can be helpful for understanding the mechanics of the cobweb plot and for removing transient behavior. Finally, you can animate both the cobweb plot and time series by clicking "Start Animation". This animation process is stopped by clicking "Stop Animation".

Exercises

(1) Begin by building some intuition about the cobweb plot and its correspondence to the time domain. Start by creating a zero-point trajectory from $x_0 = 0.2$ $r = 3.5$. (Remember to click "Restart Simulation" after changing text fields to update the simulation.) Click the "Add Iterate" button, this will add a single iteration of the logistic map to the cobweb plot and time series. Continue adding iterates one at a time until you feel comfortable with how iterates are added to a cobweb plot and the relationship between the cobweb and time series plot.

(2) As you may have just discovered these cobweb plots can get very cluttered. Sometimes initial behavior—known as the "transient behavior"—can hide what is really going on. Create a 50-point trajectory from $x_0 = 0.2$ with $r = 3.5$. Here the transient behavior is cluttering the cobweb plot. Clean up the cobweb, by clicking "Remove Iterate" until the dynamics in both the cobweb plot and time domain are clear.

(3) Generate a ten-point trajectory of the logistic map with parameter value $r = 2.5$. What kind of dynamics is this? Now change $r = 2.99$. What kind of dynamics is this? What if you increase the number of iterates—do the dynamics look the same? This question can be explored by (a) manually adjusting the "Number of Initial Iterates" using the text field, (b) adding and removing single iterates, or (c) by simply clicking "Start Animation." Vary the initial condition $x_0 \in [0, 1]$. Does this change the dynamics? What is happening here?

(4) Generate a 50-point trajectory of the logistic map starting at $x_0 = 0.2$ using parameter value $r = 3.68725$. What kind of dynamics is this? What if you click "Start Animation" and watch for a while, does your conclusion change? What is the take away here?

(5) Set $r = 3.828$ and plot 50 iterates, now click "Start Animation". The cobweb plot will be a mess, but the time-domain plot should show some interesting patterns. Raise r slowly to 3.8285. For $r \in (3.828, 3.8285)$ the dynamics are very deceiving—be patient! Describe & explain what you see, if you don't see anything interesting you aren't being patient enough. How does this relate or not relate to problem 4?

(6) Find a two-cycle and the onset of chaos. Now find an n-cycle, where n is as large as you can find.

(7) There appears to be a period 15 orbit at $r = 3.8521738$. What do you think? Can you verify or disprove this?

Homework

- Code up the logistic map yourself and plot x_n vs n for some of the interesting r values.
- (for experts) The "second-return map" of the logistic map is defined as

$$x_{n+2} = (f \circ f)(x_n).$$

  Where $f(x) = rx(1-x)$. Code up your own cobweb diagram using the second-return map and play around with it. How is this different than the cobweb plot in this app? How do the features of the second-return map align with known features of the original logistic map? Find a period 2 orbit in the original map and then use that same r value in the second-return map? What kind of dynamics are these? Does this make sense?