

LINEAR SOCIAL INTERACTIONS MODELS

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ABSTRACT

This paper provides a systematic analysis of identification in linear social interactions models. This is a theoretical and econometric exercise as the analysis is linked to a rigorously delineated model of interdependent decisions. We develop an incomplete information game that describes individual choices in the presence of social interactions. The equilibrium strategy profiles are linear. Standard models in the empirical social interactions literature are shown to be exact or approximate special cases of our general framework, which provides a basis for understanding the microeconomic foundations of those models. We consider identification of both endogenous (peer) and contextual social effects under alternative assumptions regarding the analyst's *a priori* knowledge of social structure or access to individual-level or aggregate data. Finally, we discuss potential ramifications for identification of endogenous group selection.

JEL Codes: C21, C23, C31, C35, C72, Z13

Key Words: social interactions, identification, games of incomplete information

...it is said by some that men will think and act for themselves; that none will disuse spirits or anything else, merely because his neighbors do; and that moral influence is not the powerful engine contended for...Let me ask the man who would maintain this position most stiffly, what compensation he will accept to go to church some Sunday and sit during the sermon with his wife's bonnet upon his head? Not a trifle, I'll venture. There would be nothing irreligious in it...Then why not? Is it not because there would be something egregiously unfashionable about it? Then it is the influence of fashion; and what is the influence of fashion but the influence that other people's actions have on our own actions, the strong inclination each of us feels to do as we see our neighbors do? Nor is the influence of fashion confined to any particular thing or class of things. It is just as strong on one subject as another.

Abraham Lincoln,
February 22, 1842

Address to the Washington Temperance Society of Springfield, Illinois

1. Introduction

Although the proposition that individuals are subject to social influence states the obvious, the study of social influences on individual behavior in economics is a relatively recent phenomenon.¹ In the last two decades, however, a rich theoretical, econometric, and empirical literature in social economics has emerged (see Benhabib, Bisin, and Jackson (2011a,b) for a comprehensive overview). While each of these three dimensions — the theoretical, econometric, and empirical — has made important advances, it is fair to say that they are yet to be well integrated. By this we mean that the theoretical models used to study social interactions are distinct from the econometric environments in which identification is studied, while empirical work generally does not systematically exploit the implications of theory and econometrics for the formulation of data analyses.

The objective of this paper is to facilitate the integration of the theoretical, econometric, and empirical sides of the social interactions literature through a systematic investigation of linear social interactions models. Linear models are the workhorse of empirical research and have been the primary subject of econometric work on the identification of social interactions since

¹Of course there are exceptions to this claim. Becker (1974) is an example of theoretical analysis that predate the modern literature, and Henderson, Mieszkowski, and Sauvageau (1978) and Datcher (1982) are early and seminal contributions in the empirical study of neighborhood effects. Examples of recent empirical applications of social interactions models include Conley and Udry (2010) on the diffusion of technology; Nakajima (2007) on smoking; Sirakaya (2006) on crime; Rege, Telle, and Votruba (2012) on the take up of welfare programs; and Topa (2001) and Bayer, Ross, and Topa (2008) on labor market outcomes. Another major area of social interactions work is education, which we discuss throughout the paper.

Manski (1993). Our analysis provides rigorous microfoundations for a broad class of linear social interactions models.

The central, and fundamentally optimistic, message of this paper is that in most cases, linear social interactions models *are* identified. We employ a theoretically grounded model to understand the conditions under which social interaction effects are or are not identified. The identification problem is shown to depend on three factors: the prior knowledge available to an analyst on the social structure characterizing direct interactions between individuals, the type of data available to the analyst — whether aggregated or individual level — and the implications of endogenous network formation for the conditional expectations of unobserved heterogeneity given the social structure. The onus on the empiricist lies in establishing what they know about social interactions *a priori* and, conditional on this information, verifying that their social interactions model satisfies the conditions needed for identification, many of which are provided in this paper. The conditions we describe do not involve adding stronger assumptions than have appeared in previous papers. Rather, we show that in some cases, non-identification results are artifices of strong assumptions and in others, we establish identification under weaker assumptions than have been previously employed.

We start by providing rigorous microfoundations that either exactly nest or approximate the many linear econometric models that have appeared in the social interactions literature. This is useful for empiricists because it permits a structural interpretation of regression parameter estimates, thereby allowing particular studies to shed light on more general contexts. Further, these microfoundations allow one to assess whether empirical formulations are sensible when one considers them as equilibrium strategy profiles that emerge from a noncooperative game of incomplete information.

We translate this theoretical framework into an econometric one, which we use as a basis to study identification. The main purpose of our identification results is to provide a series of conditions that empiricists can readily check, depending on their particular empirical application. Identification of utility parameters obviously depends on the researcher's *a priori* knowledge of

social structures. Without any such prior knowledge, identification fails. This is the first basic identification result we establish.

Our second set of results considers the case most commonly assumed in the applied literature, where a researcher has full prior knowledge of the social structure. We show that when the researcher has access to individual data, structural parameters are almost always identified. This casts non-identification results that have been at the center of much of the conventional econometrics literature in a new light, since they pertain to a narrow class of models that have no obvious theoretical rationale. With full prior knowledge of social structure but access to only aggregate data, first moments do not enable identification. However, building on approaches proposed by Glaeser, Sacerdote, and Scheinkman (1996, 2003) and formalized by Graham (2008), we show that second moments do.

The assumption of full prior knowledge of social structure, although routinely imposed in empirical work, may be conceptually untenable. Our third set of identification results explore how far one can get with partial prior knowledge of social structure. These results, motivated in part by the availability of social network data, indicate that when *a priori* information regarding the intensity of social ties between individuals is absent, prior knowledge of the mere existence (or absence) of ties between individuals enables identification. Identification for this case bears a conceptual resemblance to classical rank and order conditions for identification in linear simultaneous equations models (cf. Fisher 1966), but the structure of the social interactions framework means that there are interesting differences from the standard results. Our results indicate that much more general models of social interactions can be employed in empirical work than has been done previously, when individual-level data are available. At the same time, we argue that there are limits to identification when data are comprised of individual observations and group level averages.

Finally, we discuss the issues of endogenous network formation, and the presence of public variables observable to those in the network but unobservable to the researcher. We treat endogenous network formation as the first stage of a two-stage game in which our general linear

social interactions model describes payoffs from choices in the second stage. We argue that the implications for identification of endogenous network formation entirely depend on the information available to agents at the time of network formation, so that for a number of interesting cases endogeneity does not matter. For the case where our results no longer apply, dealing with endogeneity involves constructing a variant of the control function invented in Heckman (1979) and extended in Heckman and Robb (1986).

Two previous studies are relatively close to this one. (We discuss others in the context of our results later on.) Bramoullé, Djebbari, and Fortin (2009) consider identification for known social structures. We provide a generalization of their results by allowing for distinct social structures for contextual and endogenous effects, i.e. the effect of network members' exogenous characteristics and endogenous behaviors, respectively, on individual behavior. Further, they do not study identification when one does not know the complete social structure and when it is endogenous. Blume et al. (2011) anticipates some of our analysis but here, we employ a more general preference structure that allows for different types of social interactions. Our results on identification under partial knowledge of the social structure are completely new as are our results on identification under aggregation and our discussion of endogeneity and information asymmetries between the analyst and the agents under study.

Throughout the paper, we will employ social interactions effects between students as an example in order to interpret assumptions and findings. The evidence for social interactions in education is well surveyed in Epple and Romano (2011) and Sacerdote (2011). This empirical literature is large, exploring social influences on educational and other outcomes. It includes a range of environments that fall into the general framework we study. For example, it is common to assume that individual outcomes are determined by unweighted averages of peer outcomes and/or characteristics, with definitions of peer groups ranging from self-identified friendships (Patacchini, Rainone, and Zenou 2012) to classmates (Graham 2008) to schoolmates (Bifulco, Fletcher, and Ross 2011) to Zip Codes (Corcoran et al. 1992).

In section 2 we develop a social interactions game of incomplete information whose Bayes-Nash equilibrium produces linear strategy profiles. Section 3 introduces additional assumptions needed to study these equilibrium strategy profiles as econometric models of individual outcomes. Section 4 studies identification based on complete knowledge of the social structure that connects agents in the population. Section 5 provides conditions under which identification will hold for partial knowledge of social structure. Section 6 considers the implications of alternate formulations of unobserved heterogeneity due to endogenous network formation and differences between the information sets of agents and the analyst. Section 7 concludes.

2. Microfoundations

In this section, we set up a theoretical model from which the econometric model we subsequently study is directly derived. We consider a Bayesian game – a social interactions game – in which the population of network members is a set V containing $N < \infty$ members. Each individual i is described by a vector of characteristics (x_i, z_i) , where $x_i \in \mathbf{R}$ is a publicly observed characteristic, and $z_i \in \mathbf{R}$ is a private characteristic observable only to individual i .² An individual's type is a vector $(x, z_i) \in \mathbf{R}^{N+1}$, which details i 's observable and unobservable characteristics, and the observable characteristics of everyone else. The vector of players' types is $(x, z) \in \mathcal{T} = \mathbf{R}^{2N}$. The *a priori* distribution of types is an exogenous probability distribution ρ on \mathcal{T} . Knowledge of ρ is common to all individuals, and each individual's beliefs about the types of others is a conditional distribution of ρ given the individual's type.

Utility depends on an individual's own action and characteristics as well as network members' actions and characteristics. Individual i chooses an action $\omega_i \in \mathbf{R}$ to maximize utility:

$$U_i(\omega_i, \omega_{-i}) = \left(\gamma x_i + z_i + \delta \sum_j c_{ij} x_j \right) \omega_i - \frac{1}{2} \omega_i^2 - \frac{\phi}{2} \left(\omega_i - \sum_j a_{ij} \omega_j \right)^2.$$

Utility is separable into two components. The first two terms denote the private component of utility and the last is the social component. Both the private and social components are strictly

²We restrict attention to one observable and one unobservable characteristic in order to simplify notation. All of our results are easily extended to the case when each of these is a vector.

concave in individual i 's action. Marginal private utility is linear in individual i 's own observable characteristic x_i and private characteristic z_i . The term $\delta \sum_j c_{ij} x_j$ captures *contextual effects* — the direct influence of others' characteristics on i 's choices. It is a weighted average of the characteristics of members of a *contextual-effects network*. The middle term captures convex costs of action. In our model, *endogenous*, or *peer effects*, come from social pressure, that is, social norms. This is described in the last term as the squared distance between individual i 's behavior ω_i and the average $\sum_j a_{ij} \omega_j$ of the behaviors of his peers in an *peer-effects network*. The parameter ϕ determines the marginal rate of substitution between the private and social components of utility.

The matrices A and C , whose elements a_{ij} and c_{ij} determine peer- and contextual-effects, are *weighted adjacency matrices* or *weighted sociomatrices* for the peer- and contextual-effects network, respectively. Each has dimension $N \times N$ and the magnitudes of the matrix elements measures the strength of social ties. The networks themselves can be described by graphs: the peer-effects network \mathcal{A} has vertex set V and edge set $E = \{(i, j) : a_{ij} > 0\}$. The contextual-effects network \mathcal{C} is defined similarly with C instead of A . Because of assumptions made in the next section (axioms E.2 and E.3, we will be able to represent these networks by undirected graphs. We write $i \sim_A j$ if there is an edge between i and j in the peer-effects network, and $i \sim_C j$ if the edge exists in the contextual-effects network.

To illustrate how this model may be translated to an empirical application, consider a school. The network is the population of N students. Each student i chooses a level of effort. Observable characteristics include indicators of socio-economic status, such as family income. Contextual effects emerge because families with resources may contribute public goods such as school supplies or volunteer time to the classroom. The unobserved type z_i may capture individual characteristics such as ability and family values concerning education, or common influences such as teacher quality. The peer effect is understood here to be a pressure to conform. Peer- and contextual-effects networks may differ, then, because the entire classroom may benefit from such things as parent volunteers, while peer effects come only from a student's friends.

Our utility function nests examples in the literature such as Davezies, d'Haultfoeuille, and Fougère (2009) and Manski and Mayshar (2003). Our model is closest to Blume et al. (2011) but strictly nests it because we allow for distinct peer-effect and contextual-effect sociomatrices and because we work with much weaker error restrictions than the i.i.d. assumption made by the earlier paper. This weakening is important in moving from the theoretical to the econometric model.

Since all actions are chosen simultaneously, an equilibrium concept is required. We see this as an incomplete information game, and look for a Bayes-Nash equilibrium. That is, individuals choose an action to maximize their expected utility given their type and the public types of others. The Bayesian game formalism assumes that the description of the game $\langle U, \gamma, \delta, \phi, A, C, \rho \rangle$ is common knowledge among individuals. Furthermore, we assume that x is common knowledge, and each individual alone observes his private z_i . Equilibrium beliefs are constructed from the individuals' strategy functions and the common prior belief. The following axioms ensure the existence of a Bayes-Nash equilibrium:

- T.1.** $\phi \geq 0$. A and C are non-negative, for each $i \in V$, $\sum_j a_{ij}$ is either 0 or 1, and similarly for C . For all $i \in V$, $a_{ii} = c_{ii} = 0$.
- T.2.** Second moments of ρ exist.

Axiom *T.1* has several parts. The restrictions $\phi \geq 0$ and non-negative A together impose a preference for behavioral conformity. Hence, a student is more likely to exert effort if classmates also exert effort. The analogous restriction on C means the effect of exogenous characteristics is proportionate to the strength of a tie. So the age of a student's friends may matter more than that of acquaintances and may be positive or negative, but the sign of the contextual effects will be the same as the sign of δ . The restriction that row sums of the peer- and contextual-effects is either 0 or 1 means that individuals are either "loners", i.e. individuals who do not experience either type of social interaction effect, or that they care about the weighted averages of actions and characteristics of network members with whom they interact. The formulation means that loners and others share common γ 's and ϕ 's, which is convenient for the derivations. The restriction

$a_{ii} = c_{ii} = 0$ ensures that the peer- and contextual-effects sociomatrices measure only the effects of *others* on each individual. “Own-effects” are captured elsewhere in the model. Axiom T.2 is necessary to ensure that expected utility is well-defined for a large class of strategies.

Formally, a strategy for individual i is a function $f_i : \mathbf{R}^{N+1} \rightarrow \mathbf{R}$ that assigns a choice ω_i to each of his possible types (x, z_i) . Denote by \mathcal{F} the set of all strategy profiles $f(x, z) = (f_1(x, z_1, \dots, z_N), \dots, f_N(x, z_1, \dots, z_N))$ such that for each x and i , $f_i(x, \cdot) : z_i \mapsto \mathbf{R}$ is in L^2_ρ .³ A Bayes-Nash equilibrium of the game is a vector of strategy profiles $f(x, z)$ such that each f_i maximizes $E(U_i(\omega_i, \omega_{-i}) | x, z_i)$, where the expectation is taken with respect to the strategies f_{-i} and the common prior ρ .

Theorem 1. *If the Bayesian game satisfies axioms T.1 and T.2, it has a unique Bayes-Nash equilibrium. The equilibrium strategy profile can be written*

$$f(x, z) = \frac{1}{1 + \phi} \left(I - \frac{\phi}{1 + \phi} A \right)^{-1} (\gamma I + \delta C)x + \mu(x, z) + \frac{1}{1 + \phi} z \quad (1)$$

where $\mu_i(x, z)$ depends only on x and z_i . If z is independent of x , then $\mu_i(x, z)$ depends only on z_i . If the elements of the z are all pairwise independent, then $\mu_i(x, z)$ depends only on x . If both are true,

$$\mu(x, z) = \frac{1}{1 + \phi} \left(\left(I - \frac{\phi}{1 + \phi} A \right)^{-1} - I \right) E(z), \quad (1a)$$

a constant vector.

Theorem 1 provides sufficient conditions for the existence of a unique pure strategy Bayes-Nash equilibrium to the game, whose individual strategy profiles obey the linear structure commonly assumed in the empirical literature. The first term in equation (1) describes endogenous, contextual, and direct own-effects of public types x . This is the focus of empirical research. The third term expresses the direct effect on equilibrium behavior of individuals’ private types z . The second term, $\mu(x, z)$, is the effect of higher-order beliefs — individuals’ expectations of others’ private types, their expectation of others’ expectations of their beliefs, etc. In general, i ’s higher

³This means that the squared integral of $f_i(x, \cdot)$ with respect to z_i exists and is finite, ensuring that preferences over strategies for the Bayesian game are well-defined.

order beliefs may be a function of (x, z_i) , others' characteristics and i 's private type. This may be important when network membership is endogenous and x and the z_i 's are correlated. We consider this case in section 6. When x and z are independent, however, $\mu(x, z) = \mu(z)$, i.e. each individual's expectation of others' private types depends only on his own private type. Our econometric model will make this assumption (E.4 below), and we will sweep these higher-order beliefs into the unobserved term in the regression equation.

There are, generally speaking, two kinds of endogenous social interactions models. We have chosen to model social interactions as emerging from social norms. We model this as a conformity effect. Another source of social interactions comes through strategic complementarities in production. In the education literature papers such as Epple and Romano (1998) and Calvó-Armengol, Patacchini, and Zenou (2009) take this approach in modeling peer effects.⁴ The interpretation of the peer effects in such papers is that the marginal cost of educational achievement is affected by peer composition. A quadratic utility function embodying this approach is

$$U_i(\omega_i, \omega_{-i}) = \left(\gamma x_i + z_i + \delta \sum_j c_{ij} x_j \right) \omega_i + \phi \sum_j a_{ij} \omega_i \omega_j - \frac{1}{2} \omega_i^2$$

Here the first two terms describe a production function that maps effort to an educational outcome. The second term is a strategic complementarity. The hard work of other students spills over to increase the marginal product of student i 's effort. This is plausible for all kinds of reasons (see Sacerdote 2011). The third term is the disutility of effort. The proof of theorem 1 applies to this model too, and an equilibrium exists and is unique for $0 \leq \phi < 1$. If the row sums of the sociomatrix A are identical, this model is observationally equivalent to the social-interactions model; the only difference between the two is that in the Bayes-Nash equilibrium of the complementarity game, A is multiplied by the parameter ϕ rather than by $\phi / (1 + \phi)$.⁵ This renders moot the issue

⁴Epple and Romano (2011, sec. 2.1) surveys different models of externalities between students.

⁵The complementarity model becomes more complicated when $\phi \geq 1$, and so this case is assumed away in the literature.

of identifying the source of endogenous social interaction. If the row sums of A are not identical, the complementarity model is observationally equivalent to the social-interactions model with i -specific ϕ . Our existence proof covers this case as well.

3. From a Theoretical to an Econometric Model

An econometric evaluation of our theoretical model requires additional assumptions. In this section we introduce these assumptions and discuss their role in identification of the utility parameters. Loosely speaking, parameters are “identified” if the map from utility parameters into the joint distribution of regressors and outcomes is one-to-one. The following definition, due to Koopmans (1953), is useful in translating our theoretical framework to an econometric one.

Definition 1. *A structure s for the linear social interactions model is a list $\langle \gamma, \delta, \phi, A, C, \rho \rangle$, where γ, δ and ϕ are utility parameters, A and C are peer- and contextual-effects sociomatrices, and ρ is the a priori probability distribution on $\mathbf{R} \times \mathbf{R}$. A model is a set of structures.*

The empiricist is interested in whether the utility parameters γ, δ and ϕ are identified in a model in which a number of sometimes implicit restrictions have been imposed on A, C and ρ . Here, we make these restrictions explicit by concerning ourselves with the model \mathcal{M} , all of whose structures satisfy $T.1$ – $T.2$ and the following assumptions:

- E.1.** The support of the marginal distribution of x has dimension N .
- E.2.** For all i and j , $a_{ij} > 0$ iff $a_{ji} > 0$. For some i and j , $a_{ij} > 0$.
- E.3.** For all i and j , $c_{ij} > 0$ iff $c_{ji} > 0$. For some i and j , $c_{ij} > 0$.
- E.4.** For all $i, j \in V$, x_j and z_i are uncorrelated.
- E.5.** At least one of γ and δ is nonzero.

$E.1$ ensures that the $N \times N$ matrix of parameters post-multiplied by x in equation (1) is unique. Assumptions $E.2$ and $E.3$ place additional restrictions on the sociomatrices. While these are not

necessary conditions in our theorems, they greatly simplify derivations. Each is, in itself, weak and only serves to eliminate knife-edge cases. The first parts of *E.2* and *E.3* require that when j exerts social influence on i , i also exerts social influence on j . These are weak assumptions because nonzero elements of the sociomatrices are allowed to be arbitrarily small and the strength of ties between two individuals may be asymmetric. We do not require these two assumptions for most of our results, including those which rely on holes in the network for identification. However, we prefer to maintain them because they greatly simplify proofs. The second parts of *E.2* and *E.3* require, respectively, that there exist at least one pair of agents who exert peer-effects on each other and one pair of agents who exert contextual-effects on each other. If it is known that one or the other of the sociomatrices is the zero matrix, identification is straightforward without all the complications that we take up in section 4. *E.4* is a standard exogeneity condition with respect to x . In the context of our theory model, it means that higher order beliefs depend only on own-types, so $\mu(x, z) \equiv \mu(z)$. We will relax this assumption in section 6. *E.5* eliminates the special case in which the x 's have no effect on the outcomes. This case is discussed in theorem 2.

For what follows, it will be convenient to define

$$\begin{aligned}\mu &= \text{E} \left(\mu(z) + \frac{1}{1+\phi} z \right), \\ \varepsilon &= \mu(z) + \frac{1}{1+\phi} z - \mu, \\ B_\phi(s) &= \frac{1}{1+\phi} \left(I - \frac{\phi}{1+\phi} A \right)^{-1},\end{aligned}$$

and

$$B(s) = B_\phi(s)(\gamma I + \delta C). \quad (2)$$

For structures in models satisfying assumptions *T.1–T.2* and *E.1–E.4*, the equilibrium equation system of theorem 1 becomes

$$\omega = \mu + B(s)x + \varepsilon, \quad (3)$$

With this change of variables, the residual term ε has unconditional mean 0. Although it should not be forgotten that both μ and the distribution of ε depend upon ρ , we shall not be using either for identification, except in section 4.2. Instead, we will identify parameters through the matrices $B(s)$. We index these matrices by s to emphasize that it is from the structure that we will recover utility parameters.

Equation (3) may be contrasted with a purely statistical model of the form

$$\omega = \alpha + Bx + e, \tag{4}$$

in which e is constructed to be orthogonal to $(1, x)$. Viewing this statistical model through the prism of the game of section 2 and the econometric assumptions of this section has three advantages. First, it imposes some parameter restrictions on the model (e.g. the row sums of B will be identical). Second, it facilitates the interpretation of parameter values in terms of commonly accepted models of interactive decision making. Third, it allows for causal conclusions from parameter estimates because it makes clear what environmental perturbations leave the structure unchanged.

From E.1 it is immediately clear that α, B and $\text{Var}(e)$, the covariance matrix of the reduced form errors from equation (4), summarize the relevant information for identification via the first and second moments of the data, and that each is unique. For purposes of identification, these moments are the objects that the data provide to the analyst.

Most of this paper is concerned with identification of the utility parameters from the matrices $B(s)$ of equation (3), which describe how equilibrium strategy profiles vary with characteristics x .

Definition 2. *Utility parameters γ, δ and ϕ are identified in a model \mathcal{M} by B if for all $s, s' \in \mathcal{M}$, if $B(s) = B(s')$ then $(\gamma, \delta, \phi) = (\gamma', \delta', \phi')$.*

Our identification definition ignores the constant term because, in comparing the equilibrium strategy profile (3) with the statistical model (4), without restrictions on $\rho_{\varepsilon|x}$ (the marginal prior

of ε given x), the individual-specific constant terms cannot provide additional information on γ, δ and ϕ .

Since B depends on the structure only through γ, δ, ϕ, A and C , identification of the utility parameters will obviously depend on what is known about A and C *a priori*. Without *a priori* information about structures, identification will fail since the inverse image of a matrix B under the map $s \mapsto B(s)$ could contain structures with very different sociomatrices. In section 4.1 we assume that the pair (A, C) is known *a priori*, while in section 5.1, *a priori* knowledge only pertains to C . Let $\mathcal{M}(A, C)$ and $\mathcal{M}(C)$ denote the sets of all $s \in \mathcal{M}$ with fixed sociomatrices A and C , and with fixed contextual-effects sociomatrix C , respectively. These should be thought of as *sub-models* of \mathcal{M} . For instance, when A and C are known *a priori*, the identification exercise is that of identifying the utility parameters in the set of structures $\mathcal{M}(A, C)$. It follows from these definitions that anything identified in \mathcal{M} is identified in $\mathcal{M}(C)$ for every contextual-effects matrix sociomatrix C , and anything identified in $\mathcal{M}(C)$ is identified in $\mathcal{M}(A, C)$ for every peer-effects sociomatrix A .

We first establish a basic identification result. This result supposes the following structure on observations:

K. 1. For all i , the analyst observes (ω_i, x_i)

Assumption K.1 requires that the analyst observe both the outcomes and characteristics of each member of the population. Theorem 2 says that without any *a priori* knowledge other than T.1–T.2, E.1–E.4 and K.1, the reduced form parameters $B = B(s)$ and the sum $\beta = \gamma + \delta$ are nonetheless identified; that is, they are identified in \mathcal{M} . This is critical. The remainder of the paper is concerned with the unpacking of $B = B(s)$ to recover utility and social interactions parameters. The theorem goes on to state that the parameter set $\gamma = \delta = 0$ is identified, where “identified” here means “identified in \mathcal{M} ”. A third result states that with an additional piece of *a priori* information, the set $\delta = \phi = 0$ is identified, and in this case, by virtue of the first result, γ is identified as well.

Theorem 2. *Assume T.1–T.2, E.1–E.4, and K.1.*

- i. The matrix $B(s)$, μ , and the sum $\beta = \gamma + \delta$ are identified from the joint distribution of ω and x without any additional a priori information.*
- ii. $E(\omega|x)$ is independent of x if and only if $\delta = \gamma = 0$.*
- iii. If it is known a priori that either $A \neq C$ or $\gamma + \delta \neq 1$, then for all i , $E(\omega_i|x) = E(\omega_i|x_i)$ if and only if $\delta = \phi = 0$. In this case, γ is identified as well.*

Otherwise, the parameters $\gamma, \delta, \phi, A, C$ are not identified without additional a priori information.

These results do not require E.5. But having dispensed with this case, it is convenient for expository purposes to maintain E.5 for the remainder of the paper. The condition $\gamma + \delta \neq 1$ is required to ensure that the contextual and endogenous peer effects do not cancel each other out.

Theorem 2 is a negative result from the perspective of identifying social interactions. The non-identifiability of δ and ϕ means that the structural parameters representing the two possible social effects, peer and contextual, cannot be recovered given the assumptions we have made so far. To understand why, consider the following econometric specification, which is delivered from the first-order conditions for expected utility maximization in the Bayes-Nash equilibrium:

$$\omega_i = \frac{\gamma}{1+\phi}x_i + \frac{\delta}{1+\phi}\sum_j c_{ij}x_j + \frac{\phi}{1+\phi}\sum_j a_{ij}E(\omega_j|x) + \frac{1}{1+\phi}\varepsilon_i \quad (5)$$

This system of N equations is just a classic simultaneous equations system except that expectations of endogenous variables appear on the right hand side of the equation, rather than realizations. In fact one can interpret two stage least squares as making exactly this substitution. The non-identification of this simultaneous equations system is a classical result — one that is unaffected by the row summability of A and C . From this vantage point, identification failure stems from the absence of exclusion restrictions in the system. (See Bramoullé, Djebbari, and Fortin (2009) for elaboration of this intuition.)

We close this section by showing how a number of existing models constitute special cases of our general framework. The social interactions literature has focused on equation (5), the first order conditions for expected utility maximization, rather than the equilibrium strategy profiles. Hence our first two examples focus on econometric models that may be interpreted as special cases of (5). Our third example illustrates how our models instantiates the idea of weak versus strong ties, a sociological distinction which is important for a variety of economic network analyses.

Example 1: linear-in-means models. In many social interactions models, individuals are partitioned into non-overlapping groups g . Let n^g denote the size of group g . In the linear-in-means model, an individual's behavior depends on his average group characteristics and average group behavior. This amounts to imposing the following constraints on the sociomatrices:

$$\begin{aligned} c_{ij} &= \frac{1}{n^g - 1} \text{ if } i, j \in g, \\ a_{ij} &= \frac{1}{n^g - 1} \text{ if } i, j \in g, \\ c_{ij} &= a_{ij} = 0 \text{ if } i \in g, j \notin g \end{aligned} \quad (6)$$

Combined with the assumption that $E(\varepsilon|x) = 0$, the first order conditions (5) may be rewritten:

$$\omega_i = \frac{\gamma}{1 + \phi} x_i + \frac{\delta}{(1 + \phi)(n^g - 1)} \sum_j x_j + \frac{\phi}{(1 + \phi)(n^g - 1)} \sum_{j \neq i} E(\omega_j|x) + \frac{1}{1 + \phi} \varepsilon_i \quad (7)$$

Manski's (1993) study of identification of social effects is based upon a large sample approximation of this model, in which for all $i, n^g \rightarrow \infty$. In the limit

$$\omega_i = \frac{\gamma}{1 + \phi} x_i + \frac{\delta}{1 + \phi} \bar{x}^g + \frac{\phi}{1 + \phi} E(\bar{\omega}^g|x) + \frac{1}{1 + \phi} \varepsilon_i \quad (8)$$

where \bar{x}^g and $\bar{\omega}^g$ are group-level averages of the respective variables.

The unweighted averaging assumed in the linear-in-means model does not have a theoretical justification but rather reflects a modeling choice made for simplicity, or because of limits on what is observable about the groups. It is trivial to think of contexts in which weights will not be equal.

For high school students, one could easily imagine differences in sociomatrix elements that reflect relative popularity, strong versus weak friendships, and the like. Goldsmith-Pinkham and Imbens (2013) in fact report evidence of violations of the linear-in-means social structure for high school students. One message of this paper will be that it is not necessary to rely on the simplification of unweighted averaging. While some prior information on the sociomatrices A and C is necessary for identification of the utility parameters, the necessary information is less than that assumed in the linear-in-means model.

Our framework can also be used to assess the interpretability of different variations of (5) with respect to rigorous microfoundations. For example, a major empirical study of educational peer effects is Sacerdote (2001), which examines roommate pair interactions at Dartmouth. Sacerdote assumes that each student i 's grade point average depends on his own ability and the ability and grade point average of his roommate j . Sacerdote is careful to allow for measurement error in ability. We ignore this for simplicity, since in its absence his model reduces to

$$\omega_i = d_0 + d_1x_i + d_2x_j + d_3\omega_j + \zeta_i.$$

Sacerdote follows the theoretically appropriate formulation of endogenous social effects by employing the average of each individual's roommates, which for pairs is simply the outcome of the other roommate. Further, given that there is only a single roommate, there is no issue of the restriction of the linear-in-means model A matrix. On the other hand, as in Lee (2007), the inclusion of ω_j rather than $E(\omega_j)$ begs the question of what information sets are available to agents, since a roommate's grades are not observable contemporaneously. That said, there is a simple reinterpretation of this model as

$$\omega_i = d_0 + d_1x_i + d_2x_j + d_3 E(\omega_j) + d_3(\omega_j - E(\omega_j)) + \zeta_i$$

which is isomorphic to our equilibrium best response function when roommates are playing a Bayes-Nash game. Does this do violence to Sacerdote's analysis? We argue that it does not, since instrumenting for ω_j is equivalent to replacing this variable with $E(\omega_j)$. Note that in this

specification, d_2 and d_3 are not separately identified if $E(\omega_j)$ is determined by a linear combination of x_i , x_j and a constant. Identification of $d_2 + d_3$ holds and is a special case of theorem 2.⁶

Example 2: linear-in-means models based on neighborhoods. A second common approach to analyzing social effects has extended the linear-in-means model by exploiting observed network data to locate individuals in neighborhoods, and using these neighborhoods to generate sociomatrices. One example of this strategy is De Giorgi, Pellizari, and Redaelli (2010) who employ administrative data from university students to explore peer effects among classmates, where interactions are determined by overlapping classroom enrollments. A similar approach is employed in Calvó-Armengol, Patacchini, and Zenou (2009), using Add Health data, which we discuss later.

Formally, let i 's neighborhood h be the set of other agents to whom he is connected and let n^h be the number of agents in this set. Note that $i \notin h$. The weights associated with a linear-in-means model based on neighborhoods correspond to

$$\begin{aligned} c_{ij} &= \frac{1}{n^h} \text{ if } j \in h, \\ a_{ij} &= \frac{1}{n^h} \text{ if } j \in h, \\ c_{ij} = a_{ij} &= 0 \text{ if } j \notin h \end{aligned}$$

The reduced form regression that is generated by the addition of these assumptions to our framework is

$$\omega_i = d_0 + d_1 x_i + d_2 \bar{x}^h + d_3 E(\bar{\omega}^h) + \zeta_i \quad (9)$$

where \bar{x}^h and $\bar{\omega}^h$ denote averages for neighborhood h . While equation (9) may resemble equation (5), it in fact implies a much richer structure for social interactions. Unlike the linear-in-means model, agents are no longer partitioned into non-overlapping groups; an agent to whom many are

⁶Sacerdote (2001) shows that identification can hold under restrictions on the unobservables in his model. In our formulation of his model, this would require that $\omega_j - E(\omega_j)$ and ζ_i are uncorrelated, which Sacerdote (2001) notes involves the very stringent and arguably non-credible requirement that ζ_i and ζ_j be uncorrelated.

connected has a larger influence than one to whom few are connected because of differences in the number of neighborhoods the respective agents inhabit.

However, the formulation is still restrictive relative to our general C and A sociomatrix formulation as (9)'s generalization of (5) involves the relaxation of the block diagonality assumption of the linear-in-means model but retains equal values of the non-zero elements of each row of the implied sociomatrix. This could be an inaccurate, not to mention excessively restrictive, representation of social interactions.

In this example too, our microfoundations can be used to evaluate the statistical formulations of (9). For example, De Giorgi, Pellizari, and Redaelli (2010) use $\bar{\omega}^h$ rather than $E(\bar{\omega}^h)$. As argued above, our Bayes-Nash formulation is more natural and does no violence if employed to interpret their regression.

Calvó-Armengol, Patacchini, and Zenou (2009) make a more substantial deviation from our framework. They develop a complete-information social interaction game where individuals respond to the choices of peers in their social network. Individuals utilities are additively separable in two choice variables, private effort and peer-induced effort. These are perfect substitutes in the production of observed output (which is not an argument of the utility function). Their model differs from ours in two important way. First, they assume that individual characteristics (including contextual effects) affect only the utility of private effort, and so they elide the identification problem since individual characteristics do not feed back into peer effects. So, for instance, if a student in a classroom had an exogenous improvement in health status that reduced his cost of effort, and so he chose to work more, this would have no effect on others' effort levels or outcomes. Second, although individual characteristics are not an argument of peer effort utility, there is heterogeneity in equilibrium peer effort nonetheless, because *ceteris paribus* the marginal utility of peer effort is assumed to scale linearly with the number of connections one has — individuals with more contacts are assumed to be more susceptible to peer pressure. These unusual modeling choices serve a purpose. Calvó-Armengol, Patacchini, and Zenou make a direct connection between the

peer-effort choice and sociological measures of centrality. They claim that the equilibrium peer effect equals the Katz-Bonacich centrality vector.

This and other measures can be derived from our A matrix. Both models have the advantage that the attenuation rate of influence, a key parameter, can be derived from the marginal rate of substitution between private and social components of utility.

Example 3: strong and weak ties Empirical work by economists on social interactions has largely concerned networks with only one type of connection between agents. Sociologists, on the other hand, have recognized that social connections may have different manifestations, and that the distribution of different kinds of connections in a social network has an impact on network outcomes. Perhaps the most well-known distinction among connections is that of strong and weak social ties. Granovetter (1973) argued that weak ties play an important role in job search because they relay useful job information more frequently. Lin (2002) suggests that weak ties are useful because weak-tie job referrals are drawn from a different and often better distribution of openings. Montgomery (1994) has embedded simple two-edge-type social networks into job search models to investigate the impact of the distribution of weak versus strong ties on employment rates and wage distributions. While the labor-market literature extolls the virtues of weak ties, in other aspects of economic life strong ties may be more important. Some ethnographic work suggests that strong ties have more value to poor individuals than weak ties. The suggestion is that the poor, lacking access to markets, rely more on reciprocity in their social networks for the provision of credit and a variety of commodity flows (see Granovetter 1973, pp. 209–13.)

The flexibility of weighted sociomatrices allows for the empirical distinction between strong and weak ties. Suppose that an individual i has n_{iS} strong ties and n_{iW} weak ties. Suppose too that the ratio of the strength of strong to weak ties is θ . Define elements of the peer-effects sociomatrix as

$$a_{ij} = \begin{cases} 1/(n_{iW} + \theta n_{iS}) & \text{if } j \text{ is weakly tied to } i; \\ \theta/(n_{iW} + \theta n_{iS}) & \text{if } j \text{ is strongly tied to } i; \\ 0 & \text{otherwise.} \end{cases}$$

A statistical model with this kind of network structure can be estimated from survey data that includes information on tie strength or data on ties that would allow a researcher to infer the nature of the tie. The Add Health data set is an obvious candidate for constructing weak versus strong ties.⁷ Patacchini, Rainone, and Zenou (2012) in fact explore strong versus weak ties by assuming that the weights on friends report in the data are linearly declining in the order listed by each student and by studying the differences between friendships reported in two survey waves versus one.

4. Identification with known sociomatrices

In this section, we describe identification of the primitive utility parameters γ , δ and ϕ when the sociomatrices A and C are known to the analyst. We do not take a stance on the source of this *a priori* knowledge. It may be the case that the matrices are empirical constructions or chosen for theoretical reasons. Formally, we augment the assumptions made in sections 2 and 3 with

K.2. A and C are exogenous and known to the analyst *a priori*.

Assumption *K.2*, that the analyst knows the values of the sociomatrices, is strong, and we believe that standard approaches to generating *a priori* values of A and C are often theoretically unjustified. However, since this is in fact how the bulk of the social interactions and networks literatures has proceeded, it is important to understand identification for such contexts. We begin by maintaining assumption *K.1*, which said that the analyst observes individual outcomes and

⁷The Add Health data set originated in 1994, when over 90,000 subjects in a randomly selected set of schools across the U.S. were asked to name their five best school friends, many of whom were included as subjects in the sample. This allows researchers to observe a friendship network with friend characteristics. The school survey was later supplemented by a household survey performed on a randomly selected sub-sample of 20,545 subjects, tracked over four waves up to 2005. In the second wave (1996), subjects were asked to list their friends again, which allows researchers to look at the evolution of the friendship network and make inferences on the strength of ties over time.

characteristics. We will relax this assumption when we consider identification with aggregated data in section 4.2.

One major result of this section is that when individual data are available, there is a precise sense in which identification of the primitive utility parameters of the linear social interactions models is “typically” the case. Identification is “generic” in that the set of sociomatrix pairs (A, C) for which utility parameters are *not* identified is a lower-dimensional subset of $M_A \times M_C$, which is the set of all (A, C) pairs that satisfy the relevant assumptions placed on our theoretical and econometric models. Our results indicate that concerns over simultaneity as a source of non-identification of social effects are misplaced when A and C are known, unless one has a justification for working with a model from the non-generic (small) set of linear social interactions models in which identification fails.

We further consider identification when data are aggregated. Consideration of this case was initiated by Glaeser, Sacerdote, and Scheinkman (2003) and formalized in Graham (2008). They employ versions of the linear-in-means model and focus on identifying the equivalent of the contextual-effects parameter δ under the assumption that $\phi = 0$, i.e. peer effects are not present. Here we provide identification results that generalize the cases these authors studied. We show that identification is possible when both effects are present and for social structures other than the linear-in-means specification.

4.1 Individual-level data. We begin by generalizing an important result due to Bramoullé, Djebbari, and Fortin (2009), which places conditions on the sociomatrices that are sufficient for identification of the parameters. Our generalization accounts for distinct peer- and contextual-effects sociomatrices. The result provides conditions such that the matrix B in the statistical model (4), when identified with $B(s)$ in the structural equation (3), can be used to back out the values of γ, δ and ϕ . Since A and C are known *a priori*, “is identified” means in this subsection “is identified in $\mathcal{M}(A, C)$ ”. We provide conditions on the structure of the underlying networks that guarantee identification of the utility parameters. One of these conditions, namely that networks overlap, is

defined below. The results in this section can be seen as demonstrating how exclusion restrictions enable identification.

Definition 3. *Given two networks \mathcal{N}_1 and \mathcal{N}_2 , The network \mathcal{N}_1 overlaps the network \mathcal{N}_2 if every component of \mathcal{N}_2 contains an i and a j who are connected in \mathcal{N}_1 .*

Theorem 3. *Assume T.1–T.2, E.1–E.5, and K.1–K.2. Linear independence of the four matrices I , A , C and AC is necessary for identification of the utility parameters γ, δ and ϕ . Suppose that $A \neq C$, and that the contextual-effects and peer-effects networks overlap. If $\gamma + \delta \neq 0$, it is sufficient as well.*

Theorem 3 says that the failure of identification implies the existence of a non-zero solution in α, β, θ and τ of the following equation system:

$$\begin{aligned} \alpha + \tau \sum_j a_{ij} c_{ji} &= 0 \text{ for all } i, \\ \beta c_{ij} + \theta a_{ij} + \tau \sum_j a_{ij} c_{ji} &= 0 \text{ for all } i \neq j \\ \alpha + \beta + \theta + \tau &= 0 \end{aligned}$$

These linear dependence conditions implicitly define the set of matrices A and C such that identification fails. The conditions in the theorem can be checked, but they are admittedly abstract. The following corollary gives an easily checked sufficient condition. Moreover, it is clear that the sufficient condition is “almost always” satisfied. The utility parameters are identified for all pairs of sociomatrices outside of a lower-dimensional set.⁸ The reflection problem is an artifact of a particular specification; it is not the general case.

⁸Our results may be seen as a complement to McManus (1992) who established generic identification for parametric nonlinear models. He studies a space of nonlinear functions each of which is indexed by a parameter vector and employs a slightly different notion of genericity.

Corollary 1. *A sufficient condition for identification is that there exist two individuals i and j such that $\sum_k a_{ik}c_{ki} \neq \sum_k a_{jk}c_{kj}$.*

To place this corollary in the education context, suppose that social interactions are confined to students in a given classroom. Student i has a direct peer effect on student k , and student k in turn has a contextual effect on student i . The sum $\sum_k a_{ik}c_{ki}$ measures the “self indirect contextual effect”. Corollary 1 requires that this effect be different for two different students. If peer relationships are at all asymmetric and if contextual effects are not distributed uniformly in the classroom, this is likely to be the case.

Beyond giving an easily verifiable condition for identification, Corollary 1 shows how rare failure of identification is when the sociomatrices are known. For any A , the set of C for which identification fails is lower-dimensional in M_C . The corollary displays the natural intuition that *a priori* knowledge of A and C radically decreases the number of unknown parameters in the best response function in equation (2). Identification fails when the sociomatrices are so symmetric that distinct relationships within the network become redundant.

Our finding that identification is generic for known sociomatrices contrasts with much of the conventional wisdom in the econometric literature on the identification of social effects. In particular, since Manski (1993), there has been a recognition that for linear-in-means models identification can fail. Manski’s (1993) demonstration that the parameters of such social interactions model are not identified for the large sample approximation (8) immediately follows from the fact that $E(\bar{\omega}^g|x)$ is linearly dependent on 1 and \bar{x}^g . Manski dubbed this identification failure “the reflection problem”, and it has dominated econometric work on social effects ever since. The following theorem indicates how central the linear-in-means assumption is to this traditional non-identification result.

Theorem 4. *Assume T.1–T.2, E.1–E.5, and K.1–K.2. Suppose the contextual effects matrix contains only one component.⁹*

⁹If not, identification is typical. For instance, if the peer- and effects- components are the same, and two are of different size, then I , A , C and AC are independent and so the utility parameters are identified. If they are not, then

- i. If $A \neq C$, C is a linear-in-means sociomatrix, then the utility parameters are not identified.
- ii. If $A \neq C$, A is a linear-in-means sociomatrix, the peer-effects network is connected, and C is bistochastic, then the utility parameters are not identified.
- iii. If $A = C$ and $\gamma + \delta \neq 0$, the utility parameters are not identified if and only if C is a linear-in-means sociomatrix.

Theorem 4 shows that non-identification comes not from the fact that the sociomatrices for peer- and contextual-effects are the same, but from the extreme symmetry imposed by the linear-in-means structure. It further demonstrates that when peer- and contextual-effects networks are the same, identification fails only in the linear-in-means model. This theorem expands on a result of Bramoullé, Djebbari, and Fortin (2009), who show in an econometric specification similar (but not identical) to ours, that linear independence of the zeroeth, first and second powers of A is a necessary and sufficient condition for identification. We establish this result in our model, and then go on to prove that among sociomatrices satisfying $T.1$ (and not necessarily $E.3$), only the linear-in-means matrix has this property.

Generally speaking, identification will be determined by the specific values of the elements in the sociomatrices A and C . From theorem 4, however, we can derive criteria for identification which depend only on the shape of the network, the network topology, and the location of 0's in the sociomatrices.

Corollary 2. *Assume that there are distinct individuals i and j who are connected by a sequence of edges, some in the peer-effects network and some in the contextual-effects network, and who are not connected by a path in either the peer-effects network or the contextual-effects network alone. Then γ , δ and ϕ are identified.*

This corollary says that utility parameters are identified except possibly when each component of the peer-effects network is the union of components of the contextual-effects network.

the matrices are independent unless the sums of a_{ij} over j in the intersections of the components are related in a particular way, etc.

It demonstrates how restrictions on merely the qualitative structure of social interactions imply identification. For instance, if families attending a given school deliver contextual effects at the school level (e.g. they provide public goods) but peer effects do not cross classrooms or grade levels, then the utility parameters are identified. This sufficient condition for identification cannot arise if the peer- and contextual-effects networks are the same, and in particular, if $A = C$. This illustrates how the existence of richer peer and contextual social structures can facilitate rather than hamper identification.

Taken as whole the results in this section thus far show, in our judgment, that concerns about non-identification with *a priori* knowledge of A and C are misplaced. Of course, this does not mean that a given model is identified. Rather, these results say that if the utility parameters are not identified, then the researcher's choice of A and C is a very special case relative to the set of matrices that are consistent with the behavioral model we have described. It is always possible that a researcher has a principled reason for choosing sociomatrices under which identification fails. Our message is simply that such a reason needs to be present to conclude that the presence of social effects in preferences cannot be uncovered by the data.

Comparing corollaries 1 and 2 to previous work on identification and the linear-in-means model highlights the fragility of the reflection problem. Previous work has already produced variations of the linear-in-means model in which, unlike Manski's formulation, identification holds. Lee (2007), Lee, Liu, and Lin (2010), Bramoullé, Djebbari, and Fortin (2009) and Davezies, d'Haultfoeuille, and Fougère (2009) provide positive identification results based on (7), the exact linear-in-means model as opposed to the large sample approximation. Bramoullé, Djebbari, and Fortin (2009) and Davezies, d'Haultfoeuille, and Fougère (2009) study a version of Lee (2007), and find that if there are at least 2 groups of different size, identification holds for the exact model. Further, Bramoullé, Djebbari, and Fortin (2009) and Blume et al. (2011) show that the Manski non-identification result will hold if, contrary to our theoretical reasoning, a_{ii} is non-zero, which in the linear-in-means case implies that if each agent reacts to an unweighted average of the expected choice, the reflection problem reemerges even if groups are finite — a conclusion that was anticipated in Moffitt (2001).

4.2. Aggregate Data. Individual-level data on social interactions are often unavailable, or are incomplete due to sampling gaps. However, aggregate statistics are widely available (e.g., average standardized test scores at the school-level, city-level crime incidence, county-level unemployment rates, etc.). One approach that takes advantage of such data, originating with Glaeser, Sacerdote, and Scheinkman (1996, 2003) and later extended by Graham (2008), focuses on the informational content of cross-sectional data on group level averages.

To see how such data can be related to the linear social interactions model we have developed, consider data drawn from $G + 1$ non-overlapping groups numbered $g = 0, \dots, G$. Each group g contains n^g members. We assume that the primitive utility parameters γ, δ and ϕ are constant across the groups in order to render the use of aggregate data interpretable, but that sociomatrices are group-specific, so each group g is associated with a distinct set of sociomatrices A^g and C^g . For many contexts, heterogeneity in social structure seems natural across groups, even when populations are of the same size. One example is school classrooms, where one would naturally expect different social structures, even for classrooms of a given size.

Denote by ω_i^g, x_i^g and ε_i^g the outcomes, observed characteristics, and unobserved characteristics, respectively, of individual i in group g and let $\bar{\omega}^g, \bar{x}^g$ and $\bar{\varepsilon}^g$ be the group-level averages of these variables. The model is such that assumptions $T.1$ – $T.2$, $E.1$ – $E.5$ hold at the group level. In addition, we assume:

K.1.' For all g , A^g and C^g are exogenous and known to the analyst *a priori*.

K.2.' For all g , the analyst observes $(\bar{\omega}^g, \bar{x}^g)$.

Assumption $K.1'$ establishes what, in this section, is assumed to be the analyst's *a priori* knowledge. Assumption $K.2'$, that the analyst observes only group-level averages of s and ω , replaces the assumption that the analyst observes individual-level data. For each group, let $(\omega^g, x^g, \varepsilon^g)$ denote the vectors of outcomes and observed and unobserved group characteristics for group g , respectively, and let \tilde{n}^g denote the vector each of whose elements is $1/n^g$. Given equations (3)

and (2), $\bar{\omega}^g$, \bar{x}^g , and $\bar{\varepsilon}^g$ are related to the individual-level variables by

$$\bar{\omega}^g \equiv \bar{n}^g \omega^g = \bar{n}^g \mu^\varepsilon + \bar{n}^g B_\phi^g (\gamma x^g + \delta C^g x^g) + \frac{1}{1 + \phi} \bar{n}^g \varepsilon^g \quad (10)$$

A natural starting point for many empiricists would be to estimate a linear-in-means model, which amounts to imposing the restriction on (10) that the rows and columns of A^g and C^g each sum to 1. This yields

$$\bar{\omega}^g = \mu^\varepsilon + (\gamma + \delta) \bar{x}^g + \frac{1}{1 + \phi} \bar{\varepsilon}^g \quad (11)$$

It is easy to see from this equation that separate identification of the structural parameters from the joint distribution of $\bar{\omega}^g$ and \bar{x}^g is not possible: δ and γ enter the joint distribution only through the sum, and ϕ cannot be untangled from the variance of $\bar{\omega}$ under our current assumptions.

While the first moments do not permit identification, the key insight of Glaeser, Sacerdote, and Scheinkman (2003) and Graham (2008) is that, under further assumptions, second moments may. Glaeser, Sacerdote, and Scheinkman pointed out that conditional on \bar{x}^g , variation in $\bar{\omega}^g$ is consistent with the variation that would be predicted in averaging i.i.d. random variables. Their argument, which is heuristic, is that $\text{Var}(\bar{\omega}^g | \bar{x}^g)$ will reveal social interactions by comparing the sample variances for different group sizes to one in which $\omega_i - E(\omega_i | \bar{x}^g)$ is i.i.d. within and across groups.

Following Glaeser, Sacerdote, and Scheinkman (1996) and the case in Graham (2008) where group level effects are absent, we add the following additional constraint on the model in this section.¹⁰

E. 6. For all i, j x_i^g and x_j^g are i.i.d. and ε_i^g and ε_j^g are i.i.d.

¹⁰Graham (2008) has additional identification results in the case where random group effects are present.

This is a stronger i.i.d. assumption than we have had so far, namely that for all i, j, x_i^g and ε_i^g are i.i.d. Together with E.6, equation (10) yields:¹¹

$$\text{Var}(\bar{\omega}^g) = \frac{1}{(n^g)^2} \sum_{j \in g} \left(\sum_{i \in g} B_{ij}^g \right)^2 \sigma_x^2 + \left(\frac{1}{1 + \phi} \right)^2 \frac{1}{n^g} \sigma_\varepsilon^2 \quad (12)$$

Setting one group, $g = 0$, as the baseline, define the following statistic for the remaining groups $g = 1, \dots, G$, where $\text{Var}(x) = \sigma_x^2$ is observed

$$v^g = \frac{n^g \text{Var}(\bar{\omega}^g)}{\sigma_x^2} - \frac{n^0 \text{Var}(\bar{\omega}^0)}{\sigma_x^2}$$

The following theorem shows that v^g can be used for identification. Define M_A^g and M_C^g to be the set of peer- and contextual-effects sociomatrices for group g ; that is, they satisfy the matrix assumptions T.1, E.2 and E.3.

Theorem 5. *Assume T.1–T.2, E.1–E.6 and K.1'–K.2'. Assume also that δ, ϕ , and $\beta = (\gamma + \delta) \neq 0$. If*

- i. There are at least 5 groups,*
- ii. 5 or more groups have at least 3 members, and*
- iii. A^g is not bistochastic,*

then the utility parameters γ, δ and ϕ are identified from v^1, \dots, v^5 , except on a closed lower dimensional set of matrices $(C^1, \dots, C^5) \subset \prod_{g=1}^5 M_C^g$.

Theorem 5 says when data are in the form of group averages, second moments can be used to identify the utility parameters for a generic set of contextual-effects matrices, under some additional conditions. The analyst needs to observe at least 5 groups, 5 of which have at least 3 members. Moreover, there needs to be some heterogeneity in the peer-effects sociomatrix within

¹¹Equation (12) follows directly from

$$\bar{\omega}^g = \mu^\varepsilon + \frac{1}{n^g} \sum_{j \in g} \left(\sum_{i \in g} B_{ij}^g \right) x_j^g + \frac{1}{1 + \phi} \frac{1}{n^g} \sum_{i \in g} (\varepsilon_i^g - \mu^\varepsilon).$$

each group: the row sums and column sums of each A^g cannot all be 1. These conditions effectively provide enough variation to allow each group to provide distinct second moments from which the utility parameters can be backed out.

The theorem builds on Graham (2008), who explores the case in which peer effects are effectively absent (for all $g, A^g = 0$), and contextual effects are characterized by a linear-in-means structure where for all $g, c_{ii} = 0$ (all of which are allowed by $T.1-T.2$ and $E.1-E.5$). The linear-in-means assumption reduces the number of required groups with distinct C^g 's to 3 relative to the 5 in theorem 5. Our result indicates that the logic of Graham's analysis extends beyond the linear-in-means model. Moreover, it does not require the absence of a peer effect, and neither does it require different group sizes, which in his analysis generates the necessary variation for identification. However, in our formulation, this comes at a cost: data loss in moving from individual to group-average observations necessitates *a priori* information on characteristic covariances. Although we conjecture that similar statistics can be constructed with more complicated covariance structures, here, we have assumed independence.

4.3. Mixed individual and aggregate data. We conclude this section by considering linear social interactions models that are based on a combination of individual-level and aggregate data. A number of studies, including many in the important first generation of empirical social interactions research, combine individual-data from the Panel Study of Income Dynamics (PSID) with aggregate data from, say, the ZIP-code or census-tract level.¹²

The sampling scheme for the PSID, when combined with aggregate information, produces regressions of the form

$$\omega_i = b_0 + b_1 x_i + b_2 \bar{x}^g + \eta_i \quad (13)$$

¹²The PSID is a unique data set: it has been tracking a sample of 5,000 households since 1968, expanding over time to follow respondents as households structure changes and recording rich individual and family-level measures for both parents and offspring. However, it has limited information on residential neighborhoods. As a result, researchers interested in studying social interactions have resorted to generating neighborhood information by matching aggregate data sets to the PSID via respondent Zip codes (Datcher 1982, Corcoran et al. 1992) or their census-tract, (Campbell, Haveman, and Wolfe (2011), Plotnick and Hoffman (1999) and Sharkey and Elwert (2011)). Hence PSID-based social interactions studies involve the individual aggregate mix we describe.

where g denotes the relevant level of aggregation. This regression, to be interpretable as an equilibrium strategy profile, implies assumptions akin to the linear-in-means model in equation (6). Since the sampling scheme we describe provides no information on A^g and C^g , this equation represents an information reduction relative to the row describing ω_i in equation (4), which we showed in theorem 2 is not identified when these matrices are unknown. Relative to the ω_i row found in equation (4), equation (13) represents a misspecified regression so the parameters in (13) will depend on the underlying parameters $\gamma, \delta, \phi, A^g$ and C^g . The one positive use of (13) is that if $b_2 = 0$, then neither peer- nor contextual-effects are present in the preferences of agents.

5. Identification with Partial Information on Sociomatrices

Theorem 2 states that without prior knowledge on the sociomatrices beyond what is necessary for the existence of a Bayes-Nash equilibrium in the quadratic-payoff game, there is little that can be learned about the preference parameters which constitute the primitives of the behavioral model. Section 4 explored the polar opposite case that is employed in most empirical applications, namely identification when these matrices are (assumed to be) known. We now explore the degree to which parameters can be identified with only partial knowledge of social interactions.

There are many ways in which one can model partial knowledge of A and C . Two forms of partial knowledge are, in our view, particularly salient. First, the analyst may have *a priori* knowledge about C without any *a priori* knowledge about A beyond $T.1$ and $E.2$. This is a natural case to consider because peer effects embodied in A represent a primitive psychological proclivity to behave similarly to others, for which theory provides no guidance. Such guidance may, however, exist for C . Classrooms provide a simple example. If students supply some goods which are partially public, for example musical instruments, then average parental income may be plausibly assumed to determine the level of such goods, which constitute a contextual effect in our model. Alternatively, data sets exist in which parental involvement in a classroom is measured (e.g. Bassani (2008) and Sui-Chu and Willms (1996)). The total level of parental involvement can represent a public good analogous to the musical instrument expenditures example. If social units

produce public goods, the decision mechanism will implicitly define contextual effects; this occurs in Calabrese et al. (2006).

A second type of partial knowledge of sociomatrices may come from data sets in which individuals are asked to identify those to whom they are connected. These data sets, leaving aside imperfections such as limits on the number of friends that can be named, represent cases in which the analyst has information about connections between individuals but not the sociomatrices themselves. In such contexts, a researcher needs to make a judgment as to the interpretation of the data on direct connections in term of the sociomatrices. Knowledge of the presence or absence of ties between individuals in the network creates a close parallel between identification of social interaction parameters and classical results on the identification of simultaneous equations systems, since holes in the network (i.e. the absence of edges) in essence provide exclusion restrictions that can be exploited.

Throughout subsection 5.1 we impose two additional constraints.

E.6. $-\gamma/\delta$ is not an eigenvalue of C .

E.7. $\phi > 0$.

These constraints differ from those in section 4 since they do not pertain to objects observed *a priori*. Assumption E.6 ensures that B is non-singular. Assumption E.7 is largely for convenience. If $\phi = 0$, then the identified matrix B is a linear combination of I and C . The converse is true if I , A , C and AC are independent. This is generically true, but we would like to do better since A is unobservable. Results such as corollary 2 show that this can be guaranteed with modest additional *a priori* information about the peer-effects network. Such is the case, for example, in theorem 6 below.

5.1. Unknown peer-effects sociomatrix A . Peer networks are notoriously hard to measure, and so here we investigate identification when the contextual-effects sociomatrix C is known, but the analyst has either partial information, or none at all, on the peer-effects sociomatrix A . For

the case in which the researcher knows C and the topology of the peer-effects network, one can establish identification under weak conditions, as seen in theorem 6.

Theorem 6. *Assume T.1–T.2, E.1–E.7, and K.1. Assume also that the contextual-effects sociomatrix C is known, the peer-effects sociomatrix A is unknown, and the peer-effects network is known a priori. If*

- i. $N \geq 3$,*
- ii. there are two distinct individuals j and i who are known to be unconnected in the peer-effects network, that is, $a_{ij} = a_{ji} = 0$, and*
- iii. $B_{ij}^{-1} \neq \sum_k c_{ik} B_{kj}^{-1}$,*

then the utility parameters γ , δ and ϕ are identified from the conditional mean of ω given x .

Theorem 6 demonstrates that even when the peer-effects sociomatrix A is unknown, limited, qualitative information about the peer-effects network suffices for identification of the utility parameters. In a school class of with three or more members, for example, where the researcher knows C but not A , the simple knowledge that two students do not exert peer influence on one another may be sufficient for identification. This is possible because, although A is not observed, the matrix B is still identified and can be used to back out utility parameters, provided the conditions of the theorem are satisfied. This clearly fails if $\delta = 0$, since in this case $a_{ij} = 0$ implies that $B_{ij}^{-1} = 0$, which violates condition *iii* of the theorem. Otherwise, a derivative argument shows that if $N \geq 3$, then for each $C \in M_C$ there is a generic subset $S_A \subset M_A$ such that if $A \in S_A$, then γ , δ and ϕ are also identified in the set of models satisfying T.1–T.2, and E.1–E.7 by the conditional mean of ω given x .

5.2. Identification with a priori qualitative network knowledge. As noted in section 2, data sets with network data, such as Add Health do not furnish the sociomatrices A and C : survey respondents indicate to whom they are connected, but not the weights. We can interpret such data as providing information about exclusion restrictions, the location of 0's in the A and C

matrices. No survey we know of distinguishes between peer- and contextual-effects networks. We suggest that data collection, even if measures of interaction intensity cannot be constructed, allow for distinct sociomatrices.

How can such knowledge facilitate identification? In this section we show that if networks are sufficiently sparse, there is a path to identification which is analogous to classical linear simultaneous equations results, which link identification to exclusion restrictions. Interestingly, we can state the requirements for identification in terms that are analogous to the necessary order condition rather than the necessary and sufficient rank condition for simultaneous equation identification, which is a consequence of the social interactions structure.

Theorem 7. *Assume T.1–T.2, E.1, E.4, E.5, and K.1. Suppose that the only a priori information about A and C is, for some given individual i , the sets $\{j : j \sim_A i\}$ and $\{j : j \sim_C i\}$. For an individual i , consider the following three conditions.*

1. $\#\{j \not\sim_C i\} + \#\{j \not\sim_A i\} \geq N - 1$.
2. $N - 1 > \#\{j \not\sim_C i\} \geq \#\{j \not\sim_A i\}$.
3. $\#\{j \not\sim_A i\} \geq \#\{j \not\sim_C i\}$.

If conditions 1 and 2 are satisfied, then for each γ and δ there is a generic set of contextual-effects matrices C such that the utility parameters are identified. If conditions 1 and 3 hold, then there is a generic set of peer-effects matrices A such that the utility parameters are identified.

The results in sections 5.1 and 5.2 have important implications for the interpretation of surveys that measure social interactions. In terms of interpretation, Theorems 6 and 7 demonstrate the importance of network structure in generating identification. The key conditions across our results is *a priori* knowledge of 0's in the sociomatrices. Survey data on social networks do not provide information on the intensity of bilateral interactions. Rather they provide information on whether or not a bilateral interaction is present. Our emphasis on the importance of "holes" in the social structure extends the argument in Bramoullé, Djebbari, and Fortin (2009) that 0's in a known

sociomatrix allow for instruments. Our results show that these 0's can facilitate identification even when, unlike in Bramoullé, Djebbari and Fortin, the sociomatrices are unknown.

Our results also suggest a potentially serious limitation in current surveys, specifically Add Health, which is arguably the most popular data set for the study of social interaction effects. Its main draw is that high school students in its nationally representative sample are interviewed not only about the usual demographic and outcome variables of interest, but also about who their friends are. Unfortunately, the data set's friendship questions are restricted in that each student is allowed to name up to 5 friends of each gender. This has important ramifications in view of the result in theorem 6, which indicates that it is more useful to know who is *not* someone's friend rather than who *is*. Moreover, the restriction on the number of friends means that the failure to identify someone as a friend does not mean that there is a corresponding zero in the associated sociomatrices. The limitation on the number of friends that could be named in the interviews has long been understood as inducing measurement error in network structure. However, as far as we know, the effects of this limitation on identification per se have not been recognized.¹³

Our results provide a substantial generalization of Lee, Liu, and Lin (2010) as we do not need to assume that each agent equally weights others to whom he is directly connected. Lee, Liu, and Lin assume that the sociomatrices are functions of the common adjacency matrix, which clearly does not need to be the case.

5.3. Identification with aggregated social network data. We conclude this section with an analysis of a different type of partial knowledge, namely partial knowledge that reflects the absence of individual level data with which to evaluate social effects. Section 5.2 provided some positive results on inference of structural parameters when social interaction effects data are aggregated. Here we show that these effects disappear when individuals are sampled across groups and paired with group level averages. For data sets employing the PSID, for example, it is common to see models in which individual outcomes are assumed to depend on individual

¹³Another concern is that the failure to identify someone as a friend is consistent with a negative entry in one or both of the sociomatrices we have employed. While we have assumed that all elements of A and C are non-negative (axiom T.1) negative values are certainly empirically plausible. We thank Jesse Naidoo for this observation.

characteristics and certain census tract aggregates. We provide a link to this type of empirical analysis by considering the case where data are of the form $\omega_i, x_i, \bar{\omega}^g, \bar{x}^g$, where $\bar{\omega}^g$ and \bar{x}^g denote group level average outcomes and characteristics, respectively, of i 's group g . We make two knowledge assumptions. First, we assume that C is known because otherwise information would be lost relative to theorem 2 and identification would obviously fail. Second, the analyst only observes one individual per group, whom we denote as 1.

K.1''. C is exogenous and known to the analyst.

K.2''. For all g , the analyst observes $(\bar{\omega}^g, \bar{x}^g, \omega_1, x_1)$.

Finally, we place a restriction on the nature of observed heterogeneity, namely, that it is i.i.d. across members of the same group.

E.8. For each g , x_i is i.i.d. within g .

For each individual i in the sample we observe that individual's record and his group averages. The presumption is that the individual's social network is confined to the group. The individual can be netted out of the group average, so from equation (3) we derive two relationships: one for the behavior of everyone but individual i , and one for the behavior of individual i .

$$\begin{aligned} E(\bar{\omega}^g | x_i, \bar{x}^g) &= \mu^g + b^g E(x | x_i, \bar{x}^g) + b_{gi} x_i \\ E(\omega_i | x_i, \bar{x}^g) &= \mu_i + b_{-i} E(x | x_i, \bar{x}^g) + b_{ii} x_i \end{aligned} \tag{14}$$

where the bars denote group averages exclusive of individual i , and variables with a i subscript refer to individual i . The coefficients $b^g = (1/(N-1)) \sum_{j,k \in g, k \neq i} B_{kj}$, $b_{gi} = \sum_{k \in g, k \neq i} b_{ki}$, and $b_{-i} = \sum_{j \in g, j \neq i} b_{ij}$ are all sums of terms in the matrix B .

Theorem 8. *Assume T.1–T.2, E.1–E.4, E.8, and K.1''–K.2''. Assume also that $E(\bar{\omega}^g | x_i)$ and $E(\omega_i | x_i, \bar{x}^g)$ are known. Then $\beta = \gamma + \delta$ is identified, and γ, δ, ϕ are not identified.*

Returning to our schooling example, suppose the researcher observes a sample of student test scores ω_i and individual characteristics x_i , as well as classroom average characteristics and

test scores \bar{x}^g and $\bar{\omega}^g$. Theorem 8 shows that in this case utility parameters cannot be identified because the assumption of a linear-in-means structure entails too great a loss of information. As in other cases, if the projection of ω_i onto x_i and x_{-i} differs from the projection of ω_i onto x_i , then all one can say is that some sort of social interaction is present. This is a cautionary message given the ubiquity of these models in empirical practice. Our results provide a complement to Davezies, d'Haultfoeuille, and Fougère (2009) who consider the problem of identification for a linear-in-means model in which the analyst does not have data on the group aggregate variables, but does know the group sizes. Identification is shown to hold when there are groups of at least 3 distinct sizes. Our relatively negative result stems from the heterogeneity in sociomatrices across groups. This precludes our use of that paper's approach, whereby observed means of others can be treated as mismeasured true means. For our context, the mismeasurement involves loss of information on the weights of the sociomatrix as well as the values of x and ω .

6. Endogeneity of Social Structure

A standard concern in uncovering social interactions is the endogeneity of the social structure. The issue is straightforward: does a correlation between high ability friends and an individual student's educational performance reflect a social interaction of the type we have modeled or does it occur because the student's unobserved type is correlated with his friendship choices? This concern has generated interest in randomized assignment to groups, as in Sacerdote (2001), as well as cases in which a "natural experiment" alters group composition, e.g. Cipollone and Rosolia (2007).¹⁴ A focus on data in which exogenous social structure is present delimits the domain of environments that may be studied, so it is important to understand how endogeneity should be understood and accounted for in more general settings.

A natural way to extend our model of social interactions to network formation is to formulate a two-stage game, in which networks are formed in the first stage and actions are determined in the second. For each possible network there is a unique second-stage equilibrium, and each

¹⁴Blume et al. (2011) discuss how quasi-experiments may not satisfactorily resolve self-selection problems in identifying social interactions.

individual's expected utility of this second-stage equilibrium is a value function for the network which gives payoffs for the first-stage game.

While this abstract conceptualization is useful in understanding the implications of endogeneity, it is not one that can be directly implemented in the context of an econometric model of network formation and subsequent choices. The reason for this is that there simply does not exist a viable general theoretical model of network formation. Networks for business relations, job search and classroom friendships are formed according to very different rules, and vary greatly in the degree to which they are instrumental for the second-stage game. While network formation games have been devised for particular contexts, they do not even include pair-specific weights in the decision process.

An alternative approach is to imagine conditions that should be properties of equilibrium outcomes for many different games. This path, first travelled by Gale and Shapley (1962), leads to network stability concepts such as pairwise stability (Jackson and Wolinsky 1996) and pairwise-Nash stability (Calvó-Armengol and Ilkiliç 2009). A network is pairwise-Nash stable if and only if a) no individual wants to drop any edges, and b) there is no missing edge that if added would, *ceteris paribus*, be a Pareto improvement for the individuals it connects. It is neither a strictly cooperative nor a strictly non-cooperative concept. Stability expresses the idea that breaking relations is a non-cooperative activity while forming new relations involves mutual consent.

While this approach has been employed in a few recent studies,¹⁵ it is not a panacea. The basic problems are threefold. First, stable networks may not exist. Non-existence, however, can be circumvented by introducing random stable networks, that is, probability distributions on graphs which satisfy an expectation based concept of stability. Specifically, one can imagine a probability distribution on graphs for which the inequalities in the stability definition are satisfied

¹⁵Badev (2013) studies the coevolution of friendship networks and smoking behaviors in an environment where agents make myopic friendship decisions among $k - 1$ randomly selected others. This model is shown to converge to a k -stable Nash network, which means that no agent wishes to deviate by simultaneously altering $k - 1$ friendship statuses as well as his choice. Sheng (2012) uses pairwise stability in the context of identification of a network formation game but omits choices that are affected by network structure. Hsieh and Lee (2012) employ a two stage approach to generate joint estimation of the likelihood of a network and associated outcomes; their approach implicitly assumes perfect information and does treat network formation and subsequent outcomes as the solutions that appear in a dynamic decision problem.

in expectation. Existence can easily be shown in two cases: if ε is observed only just prior to the second stage or the support of the marginal distribution of ε is finite.¹⁶ In the first case, selection is not an issue because private types are not observed until after the network is formed. In the second case, discreteness of the set of possible z 's rules out many common econometric models. It is quite possible, however, that an existence proof can be provided for more general classes of models.

A second problem for both pairwise-stable and random pairwise-stable networks is that factors other than the utility of second-stage choices may play a role in determining the utility of a given network. The sociology literature is replete with descriptions of such payoffs. For instance, there might be an independent value to homophily — associating with people similar to oneself — which is distinct from the value of the game outcome.

Structural estimation of these models, then, involves specifying these additional factors. This requirement may be impossible to realize.

A third problem is that the set of pairwise stable random graphs will typically not be a singleton. Thus partial-identification techniques will come into play, and it may be that the set of pairwise stable random graphs is too large to impose useful first-stage restrictions.

For these reasons, we believe it makes more sense to address endogeneity by considering its effects on inference from data on the second stage of the game. This involves returning to our model and asking how endogeneity can invalidate our assumptions. From this vantage point, the implications of endogeneity depend on the information available to agents when networks form.

If either the public types or the private types relevant for the second-stage choice are not observed at the time the network is formed, then the missing variable cannot enter into the first-stage interim payoff functions. In this case, the linear structure of the second stage is maintained and endogeneity is not an issue. By contrast, suppose that x and z_i are available to agent i at

¹⁶For both cases, consider Myerson's (1991) network formation game. In the first case, the first-stage game is a complete-information game, and a correlated equilibrium will satisfy the needed inequalities. In the second case, a perfect direct correlated equilibrium (Dhillon and Mertens 1996) of Myerson's (1991) network formation game is a pairwise-Nash stable random graph, and since Myerson's game is finite, these equilibria exist.

the outset of the first stage. The expected second state payoff will depend upon both of these variables, and so both will influence individuals' first-stage choices. Consequently, an individual i , observing that he is connected to j , can, with knowledge of x_j , make an inference about the value of z_j that is dependent on x_j . Thus E.4 is violated. In this case $\mu(x, z)$ is not independent of x , and second-stage equilibrium strategy profiles are no longer linear in x , except for special cases. (They are, however, still described by theorem 1.) This is the selection problem. It is not just a statistical issue. It affects the basic structure of equilibrium, because it affects inference not only of the econometrician but of individuals constructing the network.

How can one proceed? From the perspective of the reduced form model (3) coefficients B , the only effect that endogeneity can have under the information regime we have described is through $E(\varepsilon|x) = E(\mu(x, z)|x)$.¹⁷ This expression is in fact nothing more than Heckman's classic control function (e.g. Heckman (1979), Heckman and Robb (1986)), where basic idea is to use economic theory to model the violation of orthogonality between the x and ε that is induced by the dependence of z on x . In the present context, so long as $\mu(x, z)$ does not depend linearly on x , identification will still hold. To be clear, the robustness of identification to endogenous network formation exploits the quadratic game structure that leads to linear equilibrium strategy profiles. But this is true for general control function approaches; they break down when $E(\varepsilon|x)$ is linear in x . Hence Heckman's fundamental idea that self-selection can be addressed by incorporating self-selection into the analysis, rather than using instrumental variables, applies to social interactions contexts.¹⁸

Where would instrumental variables approaches come into play in this setting? Suppose that the researcher has available a vector of observable individual attributes v . From the vantage

¹⁷This formally demonstrates and generalizes the interpretation given by Goldsmith-Pinkham and Imbens (2013) of social network endogeneity as an omitted variable problem.

¹⁸The idea that selection on unobservables can aid in identification of social effects via control functions was first shown in Brock and Durlauf (2001). Brock and Durlauf (2006) provide a more general treatment when agents select into non-overlapping groups and the sociomatrix weights are required to be equal as occurs for the linear-in-means model. Our current discussion makes two important extensions of this earlier work. First, an explicit game for the sequential formation of social networks and the subsequent choices of actors in the network are described. Second, the analysis indicates that the control function approach applies to a much wider class of environments than had previously been established.

point of this two-stage game, the critical question involves the timing by which this information is revealed. If agents observe v by the outset of the second stage, then endogenous network formation means that one needs to analyze $E(z_i|x, v)$. But this means that v no longer constitutes an instrument, since it is correlated with the errors in the regressions that emerge in the second stage of the game. In this sense, the pro forma use of instruments on the grounds that they are associated with the payoffs of network formation and not behaviors conditional on the network is invalid. Once one introduces instruments to account for network heterogeneity, one needs to account for their implications for the second stage regression errors, which will, outside of special cases, be present even if the payoff in the second stage is independent of the instrument.

7. Conclusion

In this paper, we have provided a theoretical and econometric characterization of linear social interactions models. Our analysis provides both a clear description of the behavioral assumptions needed to employ these models as well as the conditions under which the primitive utility parameters that characterize social influences may be recovered. Our results demonstrate the possibilities and limits to identification as determined by the degree of prior information on the sociomatrices that determine how the characteristics and behaviors of others affect each individual's utility. The absence of any *a priori* knowledge on these matrices unsurprisingly means that identification fails. We show that for the most common case in the empirical literature, namely when these matrices are known *a priori*, identification holds generically. Variants of the workhorse linear-in-means model for which identification fails are in fact knife edge cases.

We further explore a range of possible forms of *a priori* knowledge that represent intermediate cases compared to these two extreme information assumptions. These intermediate cases correspond to plausible sources of *a priori* information as derived from economic theory or empirical social structure measurement. We also address the identification question when a researcher is limited to aggregated data of various types. Finally, we argue that endogenous network formation does not constitute an unbridgeable impediment to identification.

In terms of future research, we see a number of important directions. First, our findings may be understood as fleshing out parts of the “assumptions/possibilities” frontier in terms of the edges between different types of *a priori* information on social structure and identification. There is no reason to believe that the cases we have examined span the possible types of information that may be available to a researcher, so there is certainly more work to be done in fully characterizing the environments in which identification does or does not hold.

Second, the operationalization of the control function approach to addressing network endogeneity needs to be developed. Third, we have not addressed issues of estimation. This suggests a necessary complementary paper to this one if one wishes to make our results operational. Fourth, while we have addressed the question of how our identification results are affected by endogenous social structure, we have not addressed how this endogeneity can, when explicitly modeled, facilitate identification. For example, if group memberships are associated with prices, then prices can help to uncover social effects, as demonstrated in recent advances in the econometrics of hedonic models. (See Ekeland, Heckman, and Nesheim (2004)). As discussed above, the control functions associated with the changes in conditional error distributions conditional on group membership may be able to facilitate identification.

Finally, information on social interactions may be encoded in the composition of the groups themselves. Becker’s (1957) model of taste-based discrimination implies that information on the presence of discriminatory preferences is embodied both in any black/white wage gap and in the degree of segregation of workers across firms. All these directions emphasize the importance of extending the theoretical and econometric arguments developed here in directions that fully exploit the codetermination of social structure and associated behavioral outcomes.

Appendix

A1. Networks and Sociomatrices

Our networks can be represented by a vertex set V and an adjacency relation \sim , which is symmetric. If $i \sim j$, i then i and j influence each other, but the influence may be unequal. Degree of influence is represented by a *weighted sociomatrix* M , where m_{ij} is the degree or weight of influence that j has on i . By virtue of E.2 and E.3, \sim is symmetric, but it need not be the case that $m_{ij} = m_{ji}$. However, $m_{ij} > 0$ iff $i \sim j$, so the location of zeros in the matrix is symmetric.

A pair of vertices i, j is connected by a sequence of length n if there is a sequence $i = k_0 \sim \dots \sim \dots k_n = j$. An i, j pair is connected by a path of length n if and only if $M_{ij}^n > 0$. A maximally connected set of vertices is called a *connected component*, or component for short. The vertices can be ordered so that M is block-diagonal, with each block corresponding to a single component. Any two individuals in a component are connected by a path of length equal to at most the number of component members less 1. Thus for large enough N the matrix $M + M^2 + \dots + M^N$ will be block diagonal with strictly positive blocks. If M is the weighted sociomatrix for a single component (the entire matrix or one block of a larger matrix), then it is irreducible, and we can rely on the consequences of the Perron-Frobenius Theorem. For sociomatrices with row-sums equal to 1, 1 is the Perron eigenvalue, the vector of 1s spans its right eigenspace, and its left eigenspace is spanned by a single strictly positive vector as well. If it has two cycles of lengths which are relative prime, M^n will converge as n grows to a matrix whose row vectors are the left Perrone eigenvector whose coefficients sum to 1.

A2. Proofs

We begin with the proof of theorem 1, the existence theorem. In fact we prove a more general theorem. Suppose that each individual i has his or her own ϕ_i . Define the matrices

$$\Phi_{ij} = \begin{cases} \frac{1}{1 + \phi_i} & \text{if } i = j, \\ 0 & \text{otherwise;} \end{cases} \quad \hat{\Phi}_{ij} = \begin{cases} \frac{\phi_i}{1 + \phi_i} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Assumption T.1 is modified appropriately:

T. 1'. For all i , $\phi_i \geq 0$. A and C are non-negative, for each $i \in V$, $\sum_j a_{ij}$ is either 0 or 1, and similarly for C . For all $i \in V$, $a_{ii} = 0$.

Theorem A1. *If the Bayesian game satisfies axioms T.1' and T.2, then the game has a unique Bayes-Nash equilibrium. The equilibrium strategy profile is*

$$f(x, z) = \Phi (I - \hat{\Phi}A)^{-1} (\gamma I + \delta C)x + g(x, z),$$

where $g(x, z)$ satisfies, for each i , the relation

$$g_i(x, z_i) = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} E (g_j(x, z_j) | x, z_i) .$$

If z is independent of x , then each $g(x, z_i)$ depends only on z_i . If the elements of z are pairwise independent, then $g_i(x, z_i) = (1 + \phi_i)^{-1} z_i + \mu_i(x)$.

This theorem breaks the strategy profile into two pieces. The first measures direct and contextual effects of the public type x , and the feedback through their peer effects. The second term measures the effects of each individual's private type and is estimate of the private types of others.

Proof of Theorem A1. Suppose that in the utility function the parameter ϕ is indexed by i . Give \mathcal{F} the L^2_ρ max norm; $\|f\| = \max_i \|f_i\|_2$. Let

$$\psi_i = \gamma x_i + \delta \sum_j c_{ij} x_j + z_i,$$

so that

$$u_i = \psi_i \omega_i - \frac{1}{2} \omega_i^2 - \frac{\phi_i}{2} (\omega_i - \sum_j a_{ij} \omega_j)^2 \quad (15)$$

Since the strategies are in L^2_ρ , the expected payoff to any i of any strategy profile f is finite, so preferences over strategies for the Bayesian game are well-defined.

The first-order conditions for expected utility maximization are that for each i , and given the strategy profile f_{-i} of the other individuals and type $(x, z_i) \in \mathcal{T}$,

$$\psi_i + \phi_i \sum_j a_{ij} \mathbb{E}(f_j(\psi_j, z_j) | x, z_i) - (1 + \phi_i) \omega_i = 0. \quad (16)$$

Since the problem is concave in ω_i , the first-order conditions are sufficient.

Define the operator $T : \mathcal{F} \rightarrow \mathcal{F}$ such that

$$(Tf)_i(\psi_i, z_i) = \frac{1}{1 + \phi_i} \psi_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} \mathbb{E}(f_j(\psi_j, z_j) | x, z_i).$$

A fixed point of T satisfies the first-order condition for all i , ψ_i and z_i ; thus it will be a Bayes-Nash equilibrium profile. Assumption T.1 and a computation shows that this map is a contraction in the norm topology with contraction constant $\phi = \max_i \phi_i / (1 + \phi_i)$, and so a fixed point exists, and is unique. The fixed-point strategy profile satisfies the sufficient first-order optimality conditions, and so it is a Bayes-Nash equilibrium.

Any strategy profile can be written in the form

$$f(x, z) = \Phi (I - \hat{\Phi} A)^{-1} (\gamma I + \delta C)x + g(x, z).$$

where $g_i(x, z)$ depends on z through z_i alone. Apply the operator T to see that f will be an equilibrium if and only if $g(x, z)$ satisfies, for each i ,

$$g_i(x, z) = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} E(g_j(x, z_j) | x, z_i). \quad (17)$$

Thus each g_i depends upon z only through z_i . From now on we take the arguments of each g_i to be x and z_i . Take

$$\mu(x, z)_i = \sum_j a_{ij} E(g_j(x, z_j) | x, z_i).$$

This proves the general characterization of equilibrium strategy profiles.

For the characterizations of the $g_i(x, z_i)$, define the operator T_g such that

$$(T_g h)_i = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} E(h_j(x, z_j) | x, z_i).$$

This operator too is a contraction on L^2_ρ , and so it has a unique fixed point, which is clearly g . The characterizations are proven by showing that the different assumptions imply that sets of g with given properties are invariant under T , and so the fixed point must be in this set.

To prove the second claim, suppose now that x and z are independent. Then for any function $h_j : z_j \mapsto \mathbf{R}$,

$$E(h_j(z_j) | x, z_i) = E(h_j(z_j) | z_i).$$

Consequently, the set of functions $h : (z) \mapsto \mathbf{R}^N$ is invariant under T_g . Thus each g_i depends only on z_i .

For the third claim, observe that if the private types are independent, then if $h_i(x, z_i)$ is of the form $(1 + \phi_i)^{-1} z_i + \mu_i(x)$,

$$T_g(h)_i(x, z) = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} \left(\frac{1}{1 + \phi_j} E(z_j | x) + \mu_j(x) \right),$$

since $E(z_j|x, z_i) = E(z_j|x)$. The sum over j is a function only of x , and so the set of all functions of this form is invariant under T_g . Thus the fixed point has this property too. This proves theorem A1.

To complete the proof of theorem 1, observe that if for all i and j , $\phi_i = \phi_j$, and if the z_i are independent of each other and of x , the fixed point of T_g can be computed directly, and gives equation (1a). \square

The remainder term $\mu(x, z)$ has to do with higher-order beliefs. Suppose, to simplify the exposition, that all the ϕ_i are identical. Equation (17) contains a recursion, and by iterating it, one sees that

$$\begin{aligned} \mu_i(x, z_i) = \frac{1}{1 + \phi} & \left(\frac{\phi}{1 + \phi} \sum_{i'} a_{ii'} E(z_{i'}|x, z_i) \right. \\ & \left. + \left(\frac{\phi}{1 + \phi} \right)^2 \sum_{i'} \sum_{i''} a_{ii'} a_{i'i''} E(E(z_{i''}|x, z_{i'}) |x, z_i) + \dots \right) \end{aligned}$$

The second term contains expressions whose meanings are, "i's expectation of j's expectation of $z_k \dots$ ".

Now we take up identification questions. We have assumed for convenience (E.5) that γ and δ are not both 0. Lemma 1 settles the question of identification when the true γ and δ are both 0.

Lemma 1. *Assume T.1–3 and E.1–E.5. The set of parameters $\{(\gamma, \delta, \phi) : \gamma = \delta = 0, \phi \geq 0\}$ is weakly identified from the conditional mean of ω . No parameter vector $(0, 0, \phi)$ is identified.*

Proof of Lemma 1. Let γ^* , δ^* and ϕ^* denote the true values of the un-starred parameters. If $\gamma^* = \delta^* = 0$, then $B(s) = 0$. If $B(s) = 0$, then since $B_\phi(s)$ is non-singular, $\gamma^*I + \delta^*C = 0$. The sociomatrix C has some positive off-diagonal element (E.3), so the unique solution to the equation $\gamma I + \delta C = 0$ is $\gamma = \gamma^*$ and $\delta = \delta^*$.

If $\gamma^* = \delta^* = 0$, then $B(s) = 0$, and ϕ affects ω only through its effect on ε . Since $E(\varepsilon|x) = 0$, $E(\omega|x)$ is independent of the parameter ϕ . \square

Proof of Theorem 2. Identification of μ follows from E.1 and equation (3). $E(\omega|x)$ is an affine function whose behavior on an open set is observed, so $E(\omega|x)$ is identified. Thus, the spanning assumption E.1 and the orthogonality assumption E.4 identify $B(s)$. Since C is stochastic, for the vector e of all 1s, $B(s)e = (\gamma + \delta)B_\phi(s)e = (\gamma + \delta)e$.

If $E(\omega|x)$ is independent of x , then $B(s) = 0$. Since $B_\phi(s)$ is always non-singular, $\gamma I + \delta C = 0$. E.3 implies that C is not a multiple of I , so $\gamma = \delta = 0$. Conversely, if $\gamma = \delta = 0$, then $B(s) = 0$ and ω is independent of x .

Suppose that for all i , $E(\omega_i|x) = E(\omega_i|x_i)$ and that some ω_i is not independent of x_i . Then $B(s) = \alpha I$, and α is identified and equal to $\gamma + \delta$. Since we are not in the previous case, $\alpha \neq 0$. There are two cases to consider. First, suppose $A \neq C$. Then $\alpha(1 + \phi)I - \alpha\phi A = \gamma I + \delta C$. Since both sociomatrices have 0 diagonals, $\gamma = (1 + \phi)\alpha$ and $-\alpha\phi A = \delta C$. Since both sociomatrices are stochastic, $-\alpha\phi = \gamma$. Since $A \neq C$ they are independent, so $\delta = 0$ and $\phi\alpha = 0$. So $\gamma = \alpha$, and since $\alpha \neq 0$, $\phi = 0$.

Suppose next that $A = C$. Then $(1 + \phi)\alpha I - \phi A = \gamma I + \delta A$. Since for some $i \neq j$, $a_{ij} \neq 0$, the matrices I and A are independent, so $\gamma = (1 + \phi)\alpha$ and $-\phi = \delta$. Thus

$$\gamma + \alpha\delta = \alpha$$

$$\gamma + \delta = \alpha.$$

If $\gamma + \delta \neq 1$, this equation system has the unique solution $\gamma = \alpha$ and $\delta = 0$.

The converse is obvious. □

The proof of theorem 3 requires that I , A , C and AC be distinct sociomatrices.

Lemma 2. *Suppose that I , A , C and AC are linearly dependent and the peer- and contextual-effects networks overlap. If $A \neq C$, then I , A , C and AC are distinct.*

Proof. Neither A nor C equals I since they have only 0s on the diagonal. If $AC = I$, then there must be an even number of individuals, divided into pairs, such that each i links in the peer-effects

network only to his corresponding j , and each j links in the contextual-effects network only to her corresponding i . Since i is influenced by j if and only if j is influenced by i , j links only to i in the peer-effects network and i links only to j in the contextual-effects network. Since the matrices are stochastic, each weight has to be 1, and so $A = C$.

For matrix $M = A, C$, $\tilde{M} = (1/N) \sum_{n=1}^{N-1} M^n$. Suppose $AC = C$. Then $\tilde{A}C = C$ and $\tilde{A}\tilde{C} = \tilde{C}$. Indices can be arranged so that the matrix \tilde{C} is block diagonal with strictly positive blocks, each block corresponding to a component of the contextual-effects network. For i in one contextual-effects component and j in another, $0 = \tilde{c}_{ij} = \sum_k \tilde{a}_{ik}\tilde{c}_{kj}$. Since the $\tilde{c}_{kj} > 0$ for all k in the same component as j , $\tilde{a}_{ik} = 0$ for all such k , and so the components of the peer-effects network are subsets of those of the contextual effects network.

Without loss of generality assume that the contextual effects network has only one component, and arrange the indices so that A is in block diagonal form. Then the i th block of \tilde{A} is strictly positive. Partition C into corresponding blocks. For the k th block of A , $A_k C_{kk} = C_{kk}$. If any column of C_{kk} has a non-zero element, then $\tilde{A}C = C$ implies that the corresponding diagonal element of C_{kk} exceeds 0, which is a contradiction. Thus $C_{kk} = 0$. The contextual-effects network does not overlap the peer-effects network.

If $AC = A$, arguing as before shows that the contextual-effects components are subsets of the peer-effects components. The preceding argument works again, and we conclude that the peer-effects network does not overlap the contextual-effects network. \square

Proof of theorem 3. First observe that if ϕ is identified, so are the remaining utility parameters.

For if (ϕ, γ', δ') and $(\phi, \gamma'', \delta'')$ give rise to the same reduced form B , then

$$\frac{\gamma' - \gamma''}{1 + \phi} \left(I - \frac{\phi}{1 + \phi} A \right)^{-1} = \frac{\delta'' - \delta'}{1 + \phi} \left(I - \frac{\phi}{1 + \phi} A \right)^{-1} C,$$

and so $(\gamma' - \gamma'')I = (\delta' - \delta'')C$. Since C has 0s on its diagonal, $\gamma' = \gamma''$ and since $C \neq 0$, $\delta' = \delta''$.

Suppose that distinct $(\phi', \gamma', \delta')$ and $(\phi'', \gamma'', \delta'')$ give rise to the same reduced form B with row sum b . Then $\phi' \neq \phi''$, $\gamma' + \delta' = \gamma'' + \delta'' = b$, and $B_\phi(s')(\gamma'I + \delta'C) = B_\phi(s'')(\gamma''I + \delta''C)$. Multiply this last condition out. Since $B_\phi(s)^{-1}$ is a power series in A and all such series commute, we derive that $B(s') = B(s'')$ if and only if $B_\phi(s'')(\gamma'I + \delta'C) = B_\phi(s')(\gamma''I + \delta''C)$ if and only if

$$\begin{aligned} & ((1 + \phi'')\gamma' - (1 + \phi')\gamma'')I + ((1 + \phi'')\delta' - (1 + \phi')\delta'')C \\ & \quad + (\phi'\gamma'' - \phi''\gamma')A + (\phi'\delta'' - \phi''\delta')AC = 0. \quad (18) \end{aligned}$$

In words, a linear combination of the four matrices which equals 0, $\alpha I + \beta C + \theta A + \tau AC = 0$, is given by $(1 + \phi'')\gamma' - (1 + \phi')\gamma'' = \alpha$, $(1 + \phi'')\delta' - (1 + \phi')\delta'' = \beta$, $\phi'\gamma'' - \phi''\gamma' = \theta$ and $\phi'\delta'' - \phi''\delta' = \tau$. Algebraic manipulation shows that if the four matrices are linearly independent, that is, if $\alpha, \beta, \theta, \tau = 0$, then $(\phi, \gamma', \delta') = (\phi, \gamma'', \delta'')$.

Suppose now that the four matrices are linearly dependent. That is, there are $\alpha, \beta, \theta, \tau$ not all zero such that the linear combination is the 0 matrix. Taking account of the fact that the sum of γ and δ is identified, identification of the utility parameters fails if and only if there is a solution to the system of equations

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \gamma' \\ \gamma'' \\ b - \gamma' \\ b - \gamma'' \end{pmatrix} = \begin{pmatrix} \alpha \\ \theta \\ \beta \\ \tau \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} 1 + \phi'' & -1 - \phi' \\ -\phi'' & \phi' \end{pmatrix}$$

If identification is to fail, we must have $\phi' \neq \phi''$, and so the matrix M will be non-singular. The system will have a solution if and only if it is consistent, which is true if and only if

$$M \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix} = \begin{pmatrix} \alpha \\ \theta \end{pmatrix} \quad \text{and} \quad M \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix} = b(\phi'' - \phi') \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \beta \\ \tau \end{pmatrix}$$

so consistency is achieved if and only if

$$\begin{pmatrix} \alpha + \beta \\ \theta + \tau \end{pmatrix} = b(\phi'' - \phi') \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since I , C , A and AC all have row sums equal to 1, the coefficients of the trivial linear combination sum to 0. Algebra shows the necessary and sufficient consistency condition for this equation system to be consistent is to choose ϕ' and ϕ'' such that $\alpha + \beta = b(\phi'' - \phi')$. Since $b \neq 0$ by hypothesis, take $\phi' - \phi'' = (\alpha + \beta)/b$. If $\alpha + \beta \neq 0$, then $\phi'' - \phi' \neq 0$ and the system has a solution. If $\alpha + \beta = 0$, the system has no solution with $\phi'' - \phi'$, and so the utility parameters are identified.

Next, observe that we cannot have $\alpha + \beta = 0$. If so, we also have $\theta + \tau = 0$. Furthermore, if the four matrices are linearly dependent and $A \neq C$, $\tau \neq 0$. Without loss of generality, choose $\tau = -1$. Then $AC = \alpha(I - C) + A$. Stochasticity implies $\alpha = 1$, and this conclusion contradicts lemma 2. \square

Proof of Theorem 4.i. This result is a corollary of theorem 3. If $A \neq C$ and C is the linear-in-means sociomatrix, then I , A , C and AC are distinct matrices. We will show that they are dependent. Since C has only one component the overlap condition (see Definition 3) is satisfied. The diagonal elements of AC are $1/(N - 1)$. For $i \neq j$, $AC_{ij} = (N - 1)^{-1} \sum_{k \neq i} a_{jk} = (N - 1)^{-1}(1 - a_{ij})$. (The second equality makes use of the assumption that C has only one block.) Thus $AC = (N - 1)^{-1}I - (N - 1)^{-1}A + C$, and the claim follows from theorem 3. \square

Proof of Theorem 4.ii. The argument is the same. The diagonal elements of AC are $1/(N - 1)$. For $i \neq j$, $AC_{ij} = (N - 1)^{-1} \sum_{k \neq i} c_{kj} = (N - 1)^{-1}(1 - c_{ij})$. Both statements hold because the column sums of C are 1. \square

Proof of Theorem 4.iii. First we show that linear independence is a sufficient condition for identification. Then we show that linear-in-means is necessary and sufficient for linear dependence.

Finally, we check directly that if C is the linear-in-means matrix, the utility parameters are not identified.

Lemma 3. *If $A = C$ and $\gamma + \delta \neq 0$, then linear independence of I , A and A^2 is sufficient for identification.*

Proof of lemma 3. Let $r = \phi / (1 + \phi)$. Suppose $(\gamma', \delta', \phi')$ and (γ, δ, ϕ) have the same reduced form with row sum b . In this case, $B(s) = (1 + \phi)^{-1} (I - (\phi / (1 + \phi))A)^{-1} (\gamma I + \delta A)$ and so we derive as in the proof of theorem 3 that if the three matrices are independent,

$$\begin{aligned} (1 - r)\gamma - (1 - r')\gamma' &= 0 \\ (1 - r)\delta - (1 - r)r'\gamma - (1 - r')\delta' + (1 - r')r\gamma' &= 0 \\ (1 - r)r'\delta - (1 - r')r\delta' &= 0. \end{aligned}$$

Since $r, r' < 1$, either γ and γ' both equal 0 or neither do. If they both do, then $\delta = \delta' = b$ and $r / (1 - r) = r' / (1 - r')$ so $\phi = \phi'$. If not, we derive $(r - r')(1 - r)(\gamma - b) = (r - r')(1 - r')(\gamma' - b) = 0$. If $r = r'$ then $\gamma = \gamma'$ and so $\delta = \delta'$. If $r \neq r'$ then $\gamma = \gamma' = b$ and $\delta = \delta' = 0$. Then $\gamma + \delta = b \neq 0$ implies that $r = r'$, a contradiction. \square

Next is the result relating dependence of I , A and A^2 to the linear-in-means matrices. We state this result in some generality because it applies as well to other econometric social interaction models.

Theorem A2. *If A is any stochastic sociomatrix (satisfying T.1) such that I , A and A^2 are linearly dependent, then A is block-diagonal with each block corresponding to a component of the peer-effects network, each block is a linear-in-means matrix for that block, and all components contain the same number of individuals.*

Proof of Theorem A2. Suppose that $\rho_I I + \rho_A A + \rho_{AA} A^2 = 0$ with not all of the scalars 0. It cannot be the case that $\rho_{AA} = 0$ because A cannot be a scalar multiple of I . Thus, recalling that

the row sums of the three matrices are all 1 and that A^2 is non-negative, we have

$$A^2 = \rho I + (1 - \rho)A \text{ for some } 0 \leq \rho \leq 1.$$

The following formula can be verified by induction:

$$A^n = \psi(n)I + (1 - \psi(n))A, \quad \psi(n) = \sum_{k=1}^{n-1} (-1)^{k-1} \rho^k, \quad (19)$$

with $\psi(n) > 0$ for all n if $\rho > 0$, and $\lim_n \psi(n) = \psi(\infty) \equiv \rho / (1 + \rho)$.

If $\rho = 1$, then $A^2 = I$. In this case the peer-effects network is the union of strongly connected components of size 2. To see this, observe first that for each i there is a j such that $a_{ij}, a_{ji} > 0$. Second, for each i there is only one j such that $a_{ij} > 0$. If instead $a_{ij}, a_{ji} > 0$ and $a_{ik} > 0$ for some $k \neq j$, then $A_{jk}^2 > 0$, a contradiction.

If $\rho = 0$, then $A^2 = A$. The peer-effects network contains no cycles. Clearly there can be no two-cycles, else for some i , $a_{ii} = A_{ii}^2 > 0$. Suppose, to the contrary, that there is a path $1 \rightsquigarrow 2 \rightsquigarrow \dots \rightsquigarrow k \rightsquigarrow 1$ for $k \geq 3$. Then $a_{12}, a_{23} > 0$ implies $A_{13}^2 > 0$, and so $a_{13} > 0$ and $1 \rightsquigarrow 3$. Continuing in this fashion, $1 \rightsquigarrow j$ for any j in the cycle. Call this the ‘‘argument about cycles’’. In particular, it holds for $j = k$. Thus $a_{1k} > 0$, and since $a_{k1} > 0$, we have found a two-cycle.

Since there are no cycles, starting from any individual and following any directed edge, we ultimately reach an individual who is connected to no one, that is $a_{ij} = 0$ for all j . Thus A is nilpotent; there is a power $k < n$ such that $A^k = 0$. $A = A^2$ implies $A = A^k = 0$. No one is connected to anyone, and the peer-effects network is the union of components of size 1.

The remaining case has $0 < \rho < 1$. When these inequalities hold, $A_{ii}^2 > 0$ for all i , and so every individual is in a connected component of size at least 2. Since for any i and j in the same component there is a cycle containing them both, the preceding argument about cycles shows that $a_{ij}, a_{ji} > 0$, and so the component is a clique. (Alternatively, there is a path of some length n between them, and equation (19) shows that $a_{ij} > 0$.)

If i is in one component and j is in another, suppose it were the case that $i \rightsquigarrow j$. (i) There could be no k in the first component and l in the second component such that $l \rightsquigarrow k$; otherwise the argument about cycles implies that i and j would be in the same component. (ii) Every individual in i 's component would be influenced by j , and therefore every individual in i 's component would be influenced by everyone in j 's component. (iii) If someone in j 's component were influenced by k in yet another component, then everyone in i 's component would be influenced by everyone in k 's component. In other words, if we suppose that individuals in one component are influenced by individuals in another component, then we could order the components $V_1 \succ V_2 \cdots V_k$ where $V_i \succ V_j$ means that (someone, and therefore) everyone in V_i is influenced by (someone, and therefore) everyone in V_j . This order would be transitive. Consequently, individuals could be enumerated in such a way to make A block upper-triangular. The rows corresponding to a given component V would have non-zero entries in their corresponding columns, except for the diagonal elements. Columns corresponding another component V' would have 0 elements in these rows unless $V \succ V'$. In this case, all elements would be positive.

We will now show that none of this can happen; that each member of any component links only to other members of her component. There can be no links between distinct components, and consequently individuals can be enumerated so that A is block-diagonal, with each block indecomposable. To see this, consider a maximal component of the order that is not also minimal; that is, the component influences no other component but is influenced by some other component. Suppose our component has diagonal block B in A . The corresponding block in A^2 is B^2 , and so $B^2 = \rho I + (1 - \rho)B$. The row-sums of A are 1, and each row in the block comprising B has positive elements outside of the block. Therefore each row sum of B is less than 1. Let $0 < \alpha < 1$ denote the maximal row-sum of the block B . Then the maximal row sum of B^n is no greater than α^n , which converges to 0. On the other hand, $B^n = \psi(n)I + (1 - \psi(n))B$. Thus the maximal row-sum of B^n converges to $(\rho + \alpha)/(1 + \rho) > 0$, establishing a contradiction.

The sociomatrix A is block-diagonal, and each block is itself indecomposable. We claim that if the social network is not the union of components of size 2, connected pairs, then A is aperiodic.

First observe that A has no blocks of size 2. For suppose that A contains a block B of size 2. Then $B^2 = I$ and hence $\rho = 1$, a case we have already dispensed with. Thus each block has size at least three. Since B^2 and B^3 are both strictly positive linear combinations of B and I , the network contains cycles of lengths 2 and 3. The greatest common divisor of all cycle-lengths belonging to i is thus 1, and therefore B is aperiodic.

We have now established that A is block-diagonal and that each block is indecomposable and aperiodic. Thus A^n converges to a block-diagonal matrix A^∞ wherein each block is at least 3×3 and has rank 1. The i th row vector has in its non-zero columns the left Perron eigenvector of the block to which i belongs. Since $A_{ii}^\infty = \psi(\infty)I$, every coefficient in the left Perron eigenvector of every block is $\psi(\infty)$. It follows that $a_{ij} = \rho$ for all $i \neq j$. Since the sum of the coefficients of each row is 1, it follows that all components are the same size n , $\rho = 1/(n - 1)$, and so A is the linear-in-means sociomatrix with identically-sized components. \square

Finally, we establish the reflection principle for linear in means matrices.

Lemma 4. *If $A = C$ and is a linear-in-means matrix in which all blocks are of equal size, then the utility parameters are not identified.*

Proof of Lemma 4. If all blocks are the same size, a calculation shows that the reduced form has identical diagonal elements and identical off-diagonal elements. The matrix B_ϕ^{-1} has this form because it is the discounted sum of powers of C , and an induction shows that the powers have this form. Its product with $\gamma I + \delta C$ also then has this form. Thus the reduced form is described by two numbers that are algebraic functions of the parameters. The inverse image of the map from parameters to reduced forms therefore has dimension of at most 2, and so the three parameters cannot be uniquely determined. \square

This concludes the proof of theorem 4. \square

Proof of Corollary 2. We will verify the condition of theorem 3. If the hypothesis of the corollary holds, then $A \neq C$. Let i and j be two individuals connected only by paths containing both

peer-effect and contextual-effect links. Choose a path of minimal length connecting them. Then since according to E.2 and E.3 we can traverse the path in both directions, there will be a triple $k \rightsquigarrow_A l \rightsquigarrow_C m$ along the path. There can be no $k \rightsquigarrow m$ link in either network, and $k \neq m$, or it would be possible to find a shorter path. Suppose that I, A, C and AC are linearly dependent. Since $A \neq C$, we can write $AC = \rho_I I + \rho_A A + \rho_C C$. Since $AC_{km} > 0$, at least one of ρ_A and ρ_C is not zero. We have $AC_{km} = \rho_A A_{km} + \rho_C C_{km}$, however, and so one of these matrix elements must be non-zero. That is, we have at least one of $k \rightsquigarrow_A m$ and $k \rightsquigarrow_C m$, a contradiction. \square

The proof of theorem 5 is long, tedious, and without merit beyond its existence. It will be useful to rewrite the social interaction effects with the parameter $r = \phi / (1 + \phi)$, with $r \in [0, 1)$.

Define $a(r)^T = (1 - r)e^T(I - rA)^{-1}$, where e is a vector of suitable length and T denotes transpose. The effect of r is isolated in the column-sum vector a . We need these facts:

Lemma 5. (a) For every sociomatrix A which is not bistochastic, the map $r \mapsto a(r)$ is an injection.

(b) For all A and $r \in [0, 1)$, $a(r) \gg 0$.

(c) $\sum_x a_x(r) = N$.

Proof. We use the relationship $a^T = (1 - r)e^T + ra^T A$ which is easily derived from the definition of a .

(a) If $a(r') = a(r'') = a$ for $r' \neq r''$, then

$$(r'' - r')e^T + (r' - r'')a^T A = 0,$$

so $a^T A = e^T$ and therefore $a = e$, and A is bistochastic.

(b) Without loss of generality, suppose that A is irreducible. (Otherwise consider each component of the peer-effects network \mathcal{A} separately.) $(1 - r)e^T(I - rA)^{-1}$ equals $(1 - r)(e + reA + r^2eA^2 + \dots)$. This is the sum of nonnegative vectors, and some eA^T is strictly positive.

(c) The row sums of A are 1. \square

Another technical lemma we need is this: Fix N , and let S denote the set of all triples (α, r, C) which solve the equation system

$$\alpha(a_i(r) - a_j(r)) - \beta \sum_{x \in V} a_x(r)(c_{xi} - c_{xj}) = 0 \text{ for } i \neq j,$$

and let $S_C \subset \mathcal{M}_C$ denote its projection onto the set of all contextual-effects sociomatrices.

Lemma 6. *The set S_C is closed and has dimension at most $2 + (N - 1)^2$, which is less than $\dim \mathcal{M}_C$ for $N \geq 3$.*

The set of stochastic matrices had dimension $N(N - 1)$, which exceeds $2 + (N - 1)^2$ when $N \geq 3$. The proof involves facts about semi-algebraic sets — sets defined by finite numbers of polynomial inequalities, which can be found, for instance, in Bochnak, Coste, and Roy (1987).

Proof of lemma 6. This system contains $N - 1$ equations. According to lemma 5, $a(r) \gg 0$, and so the derivative with respect to C of the left-hand side is surjective onto \mathbf{R}^{N-1} . Consequently the solution set is a semi-algebraic set of co-dimension $N - 1$, which is to say, of dimension $2 + (N - 1)^2$. It is also compact. The projection of S onto S_C is compact and has dimension at most $2 + (N - 1)^2$ since semi-algebraic functions (projection, in this case) cannot increase the dimension of their domains. □

Proof of Theorem 5. Since \bar{x}^g is observed, σ_x^2 is identified. It is convenient to define, for the g th group,

$$f^g(\gamma, \phi, \beta, A^g, C^g) = \frac{1}{n^g} e \cdot B_\phi^g (\gamma I + (\beta - \gamma) C^g) (\gamma I + (\beta - \gamma) C^{gT}) B_\phi^{gT} e^T.$$

Then $v^g = f^g(\gamma, \phi, \beta, A^g, C^g) - f^0(\gamma, \phi, \beta, A^0, C^0)$.

Define $F = F^1, \dots, F^g$ such that

$$\begin{aligned} F^g(\gamma', r', \gamma'', r'', \beta, A^g, C^g) &= f^g(\gamma', r', \beta, A^g, C^g) \\ &\quad - f^g(\gamma', r', \beta, A^0, C^0) - f^g(\gamma'', r'', \beta, A^g, C^g) \\ &\quad + f^g(\gamma'', r'', \beta, A^0, C^0) \end{aligned}$$

The domain of F is taken to be $\mathbf{R}^4/\Delta^2 \times \mathbf{R} \times M_{\mathcal{A}} \times M_{\mathcal{C}}$ where \mathcal{A} and \mathcal{C} denote the peer- and contextual-effects networks, respectively and the first set is that of all quadruples $(\gamma', r', \gamma'', r'')$ such that not both $\gamma' = \gamma''$ and $r' = r''$.

Fix the A^g, C^g and s , and consider the equation

$$F(\gamma', r', \gamma'', r'', \beta, A^0, \dots, A^g, C^0, \dots, C^g) = 0.$$

Since β is already identified, we need only to identify γ and r . The statistic (v^1, \dots, v^g) does not distinguish γ', r' from γ'', r'' (given β) if and only if $(\gamma', r', \gamma'', r'')$ solves the equation, and $(\gamma', r') \neq (\gamma'', r'')$. Thus we must show that for generic C^1, \dots, C^g ,

$$F(\gamma', r', \gamma'', r'', \beta, A^0, \dots, A^g, C^0, \dots, C^g) = 0$$

has no solution in \mathbf{R}^4/Δ^2 .

We will show that if $F^g = 0$, then $D_{C^g}F^g$ is surjective onto \mathbf{R} . If so, it follows that DF_{C^1, \dots, C^g} is surjective onto \mathbf{R}^g . Consequently, 0 is a regular value of F , and we conclude from the transversality theorem that for almost all C^1, \dots, C^g , 0 is a regular value of the map

$$F(\cdot, \beta, A^1, \dots, A^g, C^1, \dots, C^g) : \mathbf{R}^4/\Delta^2 \rightarrow \mathbf{R}^g.$$

Because F is semi-algebraic, the set of critical C^1, \dots, C^g for which this may fail is closed and lower-dimensional in $\mathcal{M}'_{\mathcal{C}}$. When the map has 0 as a regular value, the inverse image of 0 is a manifold of co-dimension G . For $G \geq 5$, this implies that the solution set in \mathbf{R}^4/Δ^2 has negative dimension, that is, it is empty.

It remains only to show that if $F^g = 0$, then $D_{C^g}F^g$ is surjective onto \mathbf{R} . Observe first that

$$D_{C^g}F^g = D_{C^g}f^g(\gamma', r', \beta, A^g, C^g) - D_{C^g}f^g(\gamma'', r'', \beta, A^g, C^g).$$

The derivative $D_{C^g}f^g$ is a linear map from the tangent space of M_C to \mathbf{R} . That tangent space is spanned by the set of all matrices H_{kij} whose vw th entry is 1 if $v = k$ and $w = i$, -1 if $v = k$ and $w = j$, and 0 otherwise. In words, H_{kij} shifts a little bit of j 's influence on k to i .

A calculation shows that

$$\begin{aligned} D_{C^g}f^g H_{kij} &= a_k(r)(s - \gamma) \left(\gamma(a_i(r) - a_j(r)) \right. \\ &\quad \left. + (s - \gamma) \sum_x a_x(r)(c_{ix} - c_{jx}) \right) \\ &\equiv a_k(r) \varrho_{ij}(\gamma, r) \end{aligned}$$

where $a = eB_\phi$. Thus if $D_{C^g}F^g$ is not surjective at $(\gamma', r', \gamma'', r'')$, then

$$a_k(r') \varrho_{ij}(\gamma', r') = a_k(r'') \varrho_{ij}(\gamma'', r''). \quad (20)$$

First we show that for generic C and all γ and r , there is a pair $i \neq j$ such that $\varrho_{ij}(\gamma, r) \neq 0$. Suppose not. Since $\gamma \neq s$, for all i, j pairs

$$\gamma(a_i(r) - a_j(r)) = (\gamma - s) \sum_x a_x(r)(c_{xi} - c_{xj}).$$

Fix j . Then for all $i \neq j$, the equation system

$$\alpha(a_i(r) - a_j(r)) = \sum_x a_x(r)(c_{xi} - c_{xj})$$

has a solution. From lemma 6, the set of matrices S_C for which this system has a solution is a closed and lower-dimensional subset of \mathcal{M}'_C . So for $C \in S_C^c$, the equation system has no solution, and hence some $\varrho_{ij} \neq 0$.

Now suppose that $D_{C^g}F^g$ is not surjective at $(\gamma', r', \gamma'', r'')$, so that equation (20) holds. Lemma 5 states that the sum over u of $a_k(r)$ is N , independent of r . Consequently, summing

over u in equation (20), we see that for all i and j , $q_{ij}(\gamma', r') = q_{ij}(\gamma'', r'')$. Since for at least one i, j pair, $q_{ij}(\gamma', r') \neq 0$, it follows that for all u , $a(r') = a(r'')$. Conclude from lemma 5 that $r' = r''$. Thus $D_{C^g} F^g$ can only fail to be surjective at points $(\gamma', r', \gamma'', r')$.

A calculation now shows that for all $i \neq j$,

$$(s - \gamma' - \gamma'')(a_i(r') - a_j(r')) = (2s - \gamma' - \gamma'') \sum_x a_x(r') (c_{xi} - c_{xj}),$$

and by hypothesis $\gamma' \neq \gamma''$. If $s \neq 0$, then at least one of $s - \gamma' - \gamma''$ and $2s - \gamma' - \gamma''$ must not be 0. That is, the equation system

$$\alpha(a_i(r') - a_j(r')) = \beta \sum_x a_x(r') (c_{xi} - c_{xj})$$

has a solution (α, β, r', C) has a solution with not both $\alpha = \beta = 0$. It cannot be the case that $\beta = 0$, for if so, then $\alpha \neq 0$, and $a_i(r') = a_j(r')$ for all i and j . But if this were the case, then $a(r') = e$ and it follows that $r' = 0$ or that A is bistochastic. We have ruled out both cases by assumption. Since $\beta \neq 0$, it follows that if $D_{C^g} F^g$ is not surjective, then

$$\alpha(a_i(r') - a_j(r')) = \sum_x a_x(r') (c_{xi} - c_{xj}),$$

and again from lemma 6, this can only happen for $C^g \in S_C$. □

If $-\gamma^*/\delta^*$ is not an eigenvalue of C , then B will be non-singular, and γ^* , δ^* and ϕ^* solve the equation

$$(1 + \phi)I - \phi A = \gamma B^{-1} + \delta C B^{-1}. \quad (21)$$

We will use this fact in the proof of theorem 6.

Proof of Theorem 6. Axiom E.2 and the hypothesis imply that both a_{ij} and a_{ji} are 0, so we have two locations with 0's in the peer-effects sociomatrix. Let γ^* , δ^* and ϕ^* denote the true parameter values, and recall that $B = B(s)$ is identified (by theorem 2). Since it is nonsingular, they solve equation (21). We also know that $b = \delta^* + \gamma^*$ is identified (by theorem 2). Thus the following

two-by-two equation system in γ and δ has as one solution $\delta = \delta^*$ and $\gamma = \gamma^*$.

$$\begin{aligned} \gamma B_{ij}^{-1} + \delta \sum_k c_{ik} B_{kj}^{-1} &= 0 \\ \gamma + \delta &= b. \end{aligned} \tag{22}$$

The system is degenerate if and only if $B_{ij}^{-1} - \sum_k c_{ik} B_{kj}^{-1} = 0$. If it is not degenerate, equation (22) can be solved for γ^* , and since $b = \delta^* + \gamma^*$, this gives δ^* . Finally, identify ϕ from the diagonal of equation (21) — the equation

$$1 + \phi = \gamma^* B_{jj}^{-1} + \delta^* c_{jk} B_{kj}^{-1}$$

has ϕ^* as its unique solution. □

Proof of Theorem 7. The proof for this theorem applies the classical rank condition for linear simultaneous equation systems to the system

$$(1 - r)^{-1}(I - rA)\omega - (\gamma I + \delta C)x = \eta, \quad \text{where } r = \phi/(1 + \phi).$$

The strategy of the proof is to identify one equation, say the equation for individual 1, and use its coefficients to identify the utility parameters. The last part is straightforward: Normalize the equation system so that the sum of the coefficients corresponding to the ω , the endogenous variables, is 1. Then the ω_1 coefficient gives ϕ since $a_{11} = 0$; ϕ equals the coefficient value less 1. The x_1 coefficient identifies γ , and the sum of the x_j coefficients for $j \neq 1$ identifies δ .

Now we develop the rank condition. Let M denote the $(N - 1) \times (K + L)$ matrix of the form

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

where $K = \#\{j : j \notin A\}$, $L = \#\{j : j \notin C\}$, u_1 is the $K \times K$ matrix whose rows and columns correspond to the $j \in \{j : j \notin A\}$, u_2 has rows corresponding to the remaining individuals except individual 1, and v_1 and v_2 have corresponding to $\{j : j \notin C\}$. The order condition for identification of the equation corresponding to individual 1 is that $K + L \geq N - 1$, and the

rank condition is that this matrix have full (row) rank. Notice that in the case where the excluded individuals sum to exactly $N - 1$, while each of the first K rows and first K columns correspond to the same individual, this is not necessarily true for the last L rows and columns: The columns correspond to individuals not contextually connected to individual 1, while the columns correspond to individuals peer-connected to 1. We will show that if the order condition is satisfied, the rank condition holds for generic A and C .

To establish the rank condition we must show that if x and y are vectors such that $u_1x + v_1y = 0$ and $u_2x + v_2y = 0$, then $x = 0$ and $y = 0$. Assume without loss of generality that the individuals not peer-connected to 1 are individuals 2 through $K + 1$. The matrix u_1 is of the form $(1 - r)(I - \tilde{A})$ where the ij element of \tilde{A} is just a_{ij} . The Perron eigenvalue of $\tilde{A} < 1$, so u_1 is invertible. If the first equation is satisfied, $x = u_1^{-1}v_1y$, and so the set of all pairs satisfying the first equation is an L -dimensional subspace of \mathbf{R}^{N-1} , a space of codimension K .

Now consider the second equation. If $L \geq K$, for every δ and γ it will be true that for generic C the matrix v_2 is non-singular and that the set of all (x, y) pairs solving the second equations intersects the set of solutions to the first equations transversally. This solution set is of codimension L , so the intersection is a vector subspace of codimension $K + L = N - 1$; that is their intersection is a set of dimension 0, a point, which must be the zero vector. If $L < N - 1$ then any given C can be perturbed to meet the conditions without violating the summability constraint on rows or the constraints that any diagonal terms in the submatrix are 0.

If $L \leq K < N - 1$, then for generic A corresponding to those i, j pairs where i is peer-connected to individual 1 and j is not, there will be an $L \times L$ submatrix of A of full rank, whose entries correspond to pairs in this set, and again the matrix is generically such that the two solution sets intersect transversally, and hence have only 0 in the intersection. Finally, if $K = N - 1$ then the matrix M is only $[u_1 \ v_1]$, and we have already argued that u_1 has full row rank. \square

Proof of Theorem 8. Let \hat{x}^g denote the mean of the x_j for $j \in g$, including individual i . Equations (14) become

$$\mathbb{E}(\bar{\omega}|x_i, \bar{x}^g) = \mu_g + b_g \hat{x}^g + b_{gi} x_i \quad (23)$$

$$\mathbb{E}(\omega_i|x_i, \bar{x}^g) = \mu_i + b_{-i} \hat{x}^g + b_{ii} x_i$$

The four coefficients are identified, and the problem is to determine the values of the utility parameters from these four values without knowing A . Observe that $b_{-i} + b_{ii} = \gamma + \delta$, a row-sum of B . Thus $\gamma + \delta$ is identified. Furthermore, $b_g + b_{gi} = (N - 1)(\gamma + \delta)$. Consequently there are only three independent values among these four coefficients. For fixed C , let F_C denote the map that takes quadruples $(\gamma, \delta, \phi, A)$ to triples (b_{ii}, b_{-i}, b_{gi}) with the given C matrix. This map is smooth, and so the implicit function theorem can be used to study solutions of the equation

$$F_C(\gamma, \delta, \phi, A) = (b_{ii}, b_{-i}, b_{gi}).$$

It will be convenient to take $i = 1$ for the calculations, to reparametrize with $r = \phi/(1 + \phi)$, and to work with the map $G_V : (\gamma, \delta, r, \epsilon) \mapsto (b_{11}, \sum_k b_{k1}, \sum_i b_{1i})$. This function is a non-singular linear transformation of F_C , and so we can identify utility parameters from b_{ii} , b_{gi} and b_{-i} if and only if we can identify them from these sums of elements of B as well.

We will show that there is a particular direction H for a perturbation of peer-effects matrices A such that for generic A and any $\phi > 0$, the derivative of the map $(\gamma, \delta, \phi, \epsilon) \mapsto G_C(\gamma, \delta, \phi, A + \epsilon H)$ has full-rank at $(\gamma, \delta, \phi, 0)$ and the partial derivative $\partial_\epsilon \neq 0$. Choose now γ^* , δ^* and ϕ^* , and an A for which the preceding statement holds, and denote the corresponding statistics $(s_1, s_2, s_3) \equiv (b_{11}^*, \sum_k b_{k1}^*, \sum_i b_{1i}^*)$. The derivative map is surjective in a neighborhood $J \times I$ of $(\gamma^*, \delta^*, \phi^*, 0)$ where I is an open interval around $\epsilon = 0$ and J is an open rectangle in \mathbf{R}^3 containing $(\gamma^*, \delta^*, r^*)$, and so the intersection of the inverse image of $(b_{11}^*, \sum_k b_{k1}^*, \sum_i b_{1i}^*)$ with $J \times I$ is a manifold of dimension 1. In fact, we show that $J \times I$, the partial derivative $\partial_{\gamma\delta\epsilon} G_C : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is surjective. This immediately implies that r is not identified. Suppose we parametrize the manifold locally in a neighborhood $(\gamma^*, \delta^*, \phi^*, 0)$ as $\psi(\lambda) = (\gamma(\lambda), \delta(\lambda), r(\lambda), \epsilon(\lambda))$ where

$\psi(0) = (\gamma^*, \delta^*, \phi^*, 0)$. Suppose that $D\psi(0) = (x_\gamma, x_\delta, x_r, x_\epsilon)$. Then $D\psi(0) \neq 0$, and

$$0 = \partial_r G_C(\psi(0)) \cdot x_r + \partial_{\gamma\delta\epsilon} G_C(\psi(0)) \cdot (x_\gamma, x_\delta, x_\epsilon).$$

To show that r is not identified, it suffices to show that $x_r \neq 0$. Suppose $x_r = 0$. Since $D\psi(0) \neq 0$, $x_r = 0$ requires that $(x_\gamma, x_\delta, x_\epsilon) \neq 0$. If so, then $DG_C(\psi(0)) \cdot D\psi(0) \neq 0$, which is a contradiction.

Now we calculate. First observe that for generic A , there will exist i and j such that $b_{1i} \neq b_{1j}$, and $a_{1i}, a_{1j} > 0$. Next, observe that $D_A B \cdot H = -r(1 - rA)^{-1}HB$, where $r = \phi/(1 + \phi)$. Observe, too, that $s_2 = \gamma + \delta$. To show that $DG_C(\gamma^*, \delta^*, \phi^*, A)$ is surjective, it suffices to show that $\partial_{\gamma\delta\epsilon} G_C(\gamma^*, \delta^*, \phi^*, A + \epsilon H)|_{\epsilon=0}$ has rank 3. Choose H to be the matrix H_{kij} where i and j are as above, $u \neq 1$, $h_{ki} = -h_{kj} = 1$, and all other elements of H are 0. Computing,

$$\partial_{\gamma\delta\epsilon} G_C(\gamma^*, \delta^*, \phi^*, A + 0H) = \begin{pmatrix} (1 - r^*)d_{1,1} & (1 - r^*)\sum_u d_{1u}c_{u1} & -r^*(b_{i1} - b_{j1})d_{1k} \\ 1 & 1 & 0 \\ (1 - r^*)\sum_v d_{v1} & (1 - r^*)\sum_v \sum_u d_{vk}c_{u1} & -r^*(b_{i1} - b_{j1})\sum_u d_{uk} \end{pmatrix}$$

where d_{ul} is the ul th element of $(I - r^*A)^{-1}$. Since $b_{i1} - b_{j1} \neq 0$ and $0 < r^* < 1$, this matrix is non-singular if and only if the following matrix is non-singular:

$$\begin{pmatrix} d_{1,1} & \sum_u d_{1u}c_{u1} & d_{1k} \\ 1 + \phi^* & 1 + \phi^* & 0 \\ \sum_v d_{v1} & \sum_v \sum_u d_{vk}c_{u1} & \sum_u d_{uk} \end{pmatrix}$$

Notice that δ^* and γ^* have disappeared. For fixed r^* and C it is generic in A that this matrix is non-singular. \square

References

- Badev, A. 2013. "Discrete Games with Endogenous Networks: Theory and Policy." Working Paper. University of Pennsylvania.
- Bassani, C. 2008. "Parent Classroom Involvement and the Development of Social Capital: A Reading Program in East Vancouver." *Journal of Educational Enquiry* 8 (2):51–70.
- Bayer, P., S. Ross, and G. Topa. 2008. "Place of Work and Place of Residence: Informal Hiring Networks and Labor Market Outcomes." *Journal of Political Economy* 116 (6):1150–1196.
- Becker, G. 1974. "A Theory of Social Interactions." *Journal of Political Economy* 82 (6):1063–1093.
- Benhabib, J., A. Bisin, and M. Jackson. 2011a. *Handbook of Social Interaction*, vol. 1. Amsterdam: North Holland.
- . 2011b. *Handbook of Social Interaction*, vol. 2. Amsterdam: North Holland.
- Bifulco, R., J. Fletcher, and S. Ross. 2011. "The Effect of Classmate Characteristics on Post-secondary Outcomes: Evidence from the Add Health." *American Economic Journal: Economic Policy* 3 (1):25–53.
- Blume, L., W. Brock, S. Durlauf, and Y. Ioannides. 2011. "Identification of Social Interactions." In *Handbook of Social Economics*, vol. 1B, edited by J. Benhabib, A. Bisin, and M. Jackson. Amsterdam: North Holland, 853–964.
- Bochnak, J., M. Coste, and M-F. Roy. 1987. *Géométrie Algébrique Réelle*. Berlin: Springer-Verlag.
- Bramoullé, Y., H. Djebbari, and B. Fortin. 2009. "Identification of Peer Effects Through Social Networks." *Journal of Econometrics* 150:41–55.
- Brock, W. and S. Durlauf. 2001. "Interaction-Based Models." In *Handbook of econometrics*, vol. 5, edited by J. Heckman and E. Leamer. Amsterdam: North-Holland, 3297–3380.
- . 2006. "Multinomial Choice with Social Interactions." In *The Economy as an Evolving Complex System, Vol. 3*, edited by S. Durlauf and L. Blume. New York: Oxford University Press, 175–206.
- Calabrese, S., D. Epple, T. Romer, and H. Sieg. 2006. "Local Public Good Provision: Voting, Peer Effects, and Mobility." *Journal of Public Economics* 90 (6):959–981.

- Calvó-Armengol, A. and R. Ilkiliç. 2009. "Pairwise-Stability and Nash Equilibria in Network Formation." *International Journal of Game Theory* 38 (1):51–79.
- Calvó-Armengol, A., E. Patacchini, and Y. Zenou. 2009. "Peer Effects and Social Networks in Education." *Review of Economic Studies* 76 (4):1239–1267.
- Campbell, M. E., R. Haveman, and B. Wolfe. 2011. "Economic Inequality and the Distribution of Offspring Human Capital." Unpublished paper.
- Cipollone, P. and A. Rosolia. 2007. "Social Interactions in High School: Lessons from an Earthquake." *The American Economic Review* 97 (3):948–965.
- Conley, T. and C. Udry. 2010. "Learning About a New Technology: Pineapple in Ghana." *American Economic Review* 100 (1):35–69.
- Corcoran, M., R. Gordon, D. Laren, and G. Solon. 1992. "The Association between Men's Economic Status and their Family and Community Origins." *Journal of Human Resources* 27 (4):575–601.
- Datcher, L. 1982. "Effects of Community and Family Background on Achievement." *Review of Economics and Statistics* 64 (1):32–41.
- Davezies, L., X. d'Haultfoeuille, and D. Fougère. 2009. "Identification of Peer Effects Using Group Size Variation." *The Econometrics Journal* 12 (3):397–413.
- De Giorgi, G., M. Pellizari, and S. Redaelli. 2010. "Identification of Social Interactions through Partially Overlapping Peer Groups." *American Economic Journal: Applied Economics* 2 (2):241–275.
- Dhillon, A. and J.-F. Mertens. 1996. "Perfect Correlated Equilibria." *Journal of Economic Theory* 68 (2):279–302.
- Ekeland, I., J. Heckman, and L. Nesheim. 2004. "Identification and Estimation of Hedonic Models." *Journal of Political Economy* 112 (1, pt.2):S60–S109.
- Epple, D. and R. Romano. 1998. "Competition Between Private and Public Schools, Vouchers, and Peer-Group Effects." *The American Economic Review* 88 (1):33–62.

- . 2011. “Peer Effects in Education: A Survey of the Theory and Evidence.” In *Handbook of Social Economics*, vol. 1a, edited by J. Benhabib, A. Bisin, and M. Jackson, chap. 11. Amsterdam: North Holland, 1053–1163.
- Fisher, F. 1966. *The Identification Problem in Econometrics*. New York: McGraw-Hill.
- Gale, D. and L. Shapley. 1962. “College Admissions and the Stability of Marriage.” *American Mathematical Monthly* 69 (1):9–15.
- Glaeser, E., B. Sacerdote, and J. Scheinkman. 1996. “Crime and Social Interactions.” *Quarterly Journal of Economics* 111 (2):507–48.
- . 2003. “The Social Multiplier.” *Journal of the European Economic Association* 1 (2-3):345–3The Social Multiplier53.
- Goldsmith-Pinkham, P. and G. W. Imbens. 2013. “Social Networks and the Identification of Peer Effects.” *Journal of Business & Economic Statistics* 31 (3):253–64.
- Graham, B. 2008. “Identifying Social Interactions Through Conditional Variance Restrictions.” *Econometrica* 76 (3):643–60.
- Granovetter, M. S. 1973. “The Strength of Weak Ties.” *American Journal of Sociology* 78 (6):1360–80.
- Heckman, J. 1979. “Sample Selection Bias as a Specification Error.” *Econometrica* 47 (1):153–161.
- Heckman, J. and R. Robb. 1986. “Alternative Methods for Solving the Problem of Selection Bias in Evaluating the Impact of Treatments on Outcomes.” In *Drawing inferences from self-selected samples*, edited by H. Wainer. New York: Springer-Verlag, 63–113.
- Henderson, V., P. Mieszkowski, and Y. Sauvageau. 1978. “Peer Group Effects and Educational Production Functions.” *Journal of Public Economics* 10 (1):97–106.
- Hsieh, C.-S. and L.-F. Lee. 2012. “A Structural Modeling Approach for Network Formation and Social Interactions with Applications to Students’ Friendship Choices and Selectivity on Unobservables.” Working Paper. Ohio State University.
- Jackson, M. and A. Wolinsky. 1996. “A Strategic Model of Social and Economic Networks.” *Journal of Economic Theory* 71 (1):44–74.

- Koopmans, T. 1953. "Identification Problems in Economic Model Construction." In *Studies in Econometric Method*, edited by W. C. Hood and T. Koopmans. New York: Wiley and Sons, 27–48.
- Lee, L.-F. 2007. "Identification and Estimation of Econometric Models with Group Interactions, Contextual Factors and Fixed Effects." *Journal of Econometrics* 140 (2):333–74.
- Lee, L.-F., X. Liu, and X. Lin. 2010. "Specification and Estimation of Social Interaction Models with Network Structures." *Econometrics Journal* 13 (2):145–176.
- Lin, N. 2002. *Social Capital: A Theory of Social Structure and Action*, vol. 19. Cambridge: Cambridge University Press.
- Manski, C. 1993. "Identification of Endogenous Social Effects: The Reflection Problem." *Review of Economic Studies* 60 (3):531–42.
- Manski, C. and J. Mayshar. 2003. "Private Incentives and Social Interactions: Fertility Puzzles in Israel." *Journal of the European Economic Association* 1 (1):181–211.
- McManus, D. 1992. "How Common is Identification in Parametric Models?" *Journal of Econometrics* 53 (1):5–23.
- Moffitt, R. 2001. "Policy Interventions, Low-Level Equilibria, and Social Interactions." In *Social dynamics*, edited by S. Durlauf and H. P. Young. Cambridge MA: MIT Press, 45–82.
- Montgomery, J. 1994. "Weak Ties, Employment, and Inequality: An Equilibrium Analysis." *American Journal of Sociology* 99 (5):1212–36.
- Myerson, R. 1991. *Game Theory: Analysis of Conflict*. Cambridge MA: Harvard University Press.
- Nakajima, R. 2007. "Measuring Peer Effects on Youth Smoking Behaviour." *Review of Economic Studies* 74 (3):897–935.
- Patacchini, E., E. Rainone, and Y. Zenou. 2012. "Student Networks and Long-Run Educational Outcomes: The Strength of Strong Ties." CEPR Discussion Paper No. DP9149.
- Plotnick, R. D. and S. D. Hoffman. 1999. "The Effect of Neighborhood Characteristics on Young Adult Outcomes: Alternative Estimates." *Social Science Quarterly* 80 (1):1–18.
- Rege, M., K. Telle, and M. Votruba. 2012. "Social Interaction Effects in Disability Pension Participation: Evidence from Plant Downsizing." *The Scandinavian Journal of Economics*

114 (4):1208–1289.

Sacerdote, B. 2001. “Peer Effects with Random Assignment: Results for Dartmouth Roommates.” *Quarterly Journal of Economics* 116 (2):681–704.

———. 2011. “Peer Effects in Education: How Might They Work, How Big are They and How Much Do We Know Thus Far?” In *Handbook of the Economics of Education*, vol. 3, edited by E. Hanushek and L. Woessmann. Elsevier, 249–277.

Sharkey, P. and F. Elwert. 2011. “The Legacy of Disadvantage: Multigenerational Neighborhood Effects on Cognitive Ability.” *American Journal of Sociology* 116 (6):1934–1981.

Sheng, S. 2012. “Identification and Estimation of Network Formation Games.” Working Paper. University of Southern California.

Sirakaya, S. 2006. “Recidivism and Social Interactions.” *Journal of the American Statistical Association* 101 (475):863–877.

Sui-Chu, E. and J. Willms. 1996. “Effects of Parental Involvement on Eighth-Grade Achievement.” *Sociology of Education* 69 (2):126–141.

Topa, G. 2001. “Social Interactions, Local Spillovers and Unemployment.” *The Review of Economic Studies* 68 (2):261.