Wealth Dynamics in Competitive Markets

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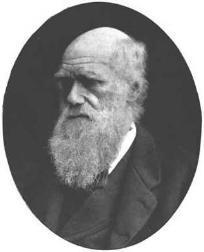
Lectures at Scuola Superiore Sant’Anna

21-22 June, 2011

Biology and Economics

In order to pass the B.A. examination, it was, also, necessary to get up Paley's *Evidences of Christianity*, and his *Moral Philosophy* . . . The logic of this book and as I may add of his *Natural Theology* gave me as much delight as did Euclid. The careful study of these works, without attempting to learn any part by rote, was the only part of the Academical Course which, as I then felt and as I still believe, was of the least use to me in the education of my mind. I did not at that time trouble myself about Paley's premises; and taking these on trust I was charmed and convinced of the long line of argumentation.

Charles Darwin



. . . when we come to inspect the watch, we perceive. . . that its several parts are framed and put together for a purpose, e.g. that they are so formed and adjusted as to produce motion, and that motion so regulated as to point out the hour of the day; that if the different parts had been differently shaped from what they are, or placed after any other manner or in any other order than that in which they are placed, either no motion at all would have been carried on in the machine, or none which would have answered the use that is now served by it. . . the inference we think is inevitable, that the watch must have had a maker – that there must have existed, at some time and at some place or other, an artificer or artificers who formed it for the purpose which we find it actually to answer, who comprehended its construction and designed its use.

William Paley, *Natural Theology*

Biology and Economics

In October 1838, fifteen months after I had begun my systematic inquiry, I happened to read for amusement Malthus on Population, and being prepared to appreciate the struggle for existence which everywhere goes on, from long-continued observation of the habits of animals and plants, it at once struck me that under these circumstances favourable variations would tend to be preserved, and unfavourable ones to be destroyed. The result would be the formation of a new species.

Charles Darwin

. . . the most important book I read was Malthus' *Principle Of Population* . . . It was the first work I had yet read treating any of the problems of philosophical biology, and its main principles remained with me as a permanent possession, and twenty years later gave me the sought-after clue to the effective agent in the evolution of organic species.

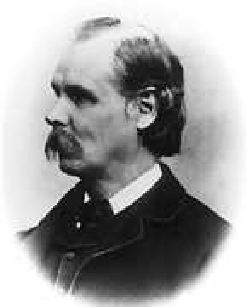
Alfred Wallace

Why Biology?

The Mecca of the economist is economic biology rather than economic dynamics.

–Alfred Marshall (1898)

In economics every event causes permanent alterations in the conditions under which future events can occur . . . When any causal disturbance has caused a great increase of any commodity, and has thereby led to the introduction of extensive economies, these economies are not readily lost. Developments of mechanical appliances, of division of labour, and of means of transport, and of improved organisation of all kinds, when they have been once obtained are not readily abandoned.



Why Biology?

- Competition for resources, Robbins (1935).
- Invisible hand explanations, Nozick (1994), *AER* 84. A pattern or institutional structure that apparently only could arise by conscious design instead can originate or be maintained through the interactions of agents having no such overall pattern in mind.
- multiplicity of scales

Uses of Biological Thinking

- A source and testbed for economic hypotheses
  - Profit maximization
  - Behavioral economics
- Intertemporal market models with sufficient heterogeneity exhibit a population dynamic.

Features of Selection

- Path dependence
- Frozen “accidents”
- Discontinuity in phenotypic change
- The systemic nature of the genome is important
  - Evolution of evolvability
  - Self-organization
- Group selection

Why Not Cultural Evolution?

- Intentionality — gaming the system. This is not Penrose's (1954) critique of biological analogy in economics.
- Cultural evolution involves the proliferation of packets of information, *codices* in G. C. Williams' terminology. Biological analogies for evolution, he suggests, are more likely to be found in epidemiology than in population genetics, which he defines as 'that branch of epidemiology that deals with infectious elements transmitted exclusively from parent to offspring.'
- Unenforceable contracts in biological systems. Institutional infrastructure to trade in economic systems that we often ignore.
- Biological emphasis on interaction processes which are often ignored in market analysis.

On Exactitude in Science **. . . In that Empire, the Art of Cartography attained such Perfection that the map of a single Province occupied the entirety of a City, and the map of the Empire, the entirety of a Province. In time, those Unconscionable Maps no longer satisfied, and the Cartographers Guilds struck a Map of the Empire whose size was that of the Empire, and which coincided point for point with it. The following Generations, who were not so fond of the Study of Cartography as their Forebears had been, saw that that vast Map was Useless, and not without some Pitilessness was it, that they delivered it up to the Inclemencies of Sun and Winters. In the Deserts of the West, still today, there are Tattered Ruins of that Map, inhabited by Animals and Beggars; in all the Land there is no other Relic of the Disciplines of Geography.**

Suarez Miranda,Viajes de varones prudentes, Libro IV,Cap. XLV, Lerida, 1658

Profit Maximization Why Not?

- Do firms maximize profits?

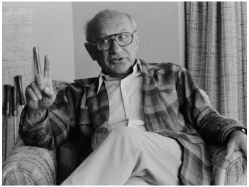
  Hall and Hitch (1939), “Price Theory and Business Behaviour”, *Oxford Economic Papers* 2, 12–45.

  Lester (1946), “Shortcomings of Marginal Analysis for Wage-Unemployment Problems”, *American Economic Review* 36, 63–82.

- Machlup, F. (1946), “Marginal Analysis and Empirical Research”, *American Economic Review* 35, 519–54.
- Machlup, F. (1955), “The Problem of Verification in Economics”, *Southern Economic Journal* 22, 1-21.
- Friedman, M. (1953), “The Methodology of Positive Economics”, in *Essays in Positive Economics*.

The propositions of economic theory, like all scientific theory, are obviously deductions from a series of postulates. And the chief of these postulate are all assumptions involving in some way simple and *indisputable* facts of experience relating to the way in which the scarcity of goods which is the subject of our science actually shows itself in the world of reality. . . . The main postulate of the theory of production is the fact that there are more than one factor of production. . . . *These are not postulates the existence of whose counterpart in reality admits of extensive dispute once their nature is fully realized. We do not need controlled experiments to establish their validity: they are so much the stuff of our everyday experience that they have only to be stated to be recognized as obvious.* . . . And it is from the existence of the conditions they assume that the general applicability of the broader propositions of economic science is derived.

An Essay on the Nature and Significance of Economic Science, 1932 (1935)



To put this point less paradoxically, the relevant question to ask about the "assumptions" of a theory is not whether they are descriptively "realistic," for they never are, but whether they are sufficiently good approximations for the purpose in hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions. The two supposedly independent tests thus reduce to one test.

The Methodology of Positive Economics, p. 15

Biological Responses

Adaptation of he individual firm:

- Harrod (1939), “Price and Cost in Entrepreneurs’ Policy”, *Oxford Economic Papers*.

Selection across firms:

- Alchian (1950), “Uncertainty, Evolution, and Economic Theory”, *Journal of Political Economy* 58, 211–21.

Biological Responses

Whenever this determinant (of business behavior) happens to lead to behavior consistent with rational and informed maximization of returns, the business will prosper and acquire resources with which to expand; whenever it does not the business will tend to lose resources and can be kept in existence only by the addition of resources from the outside. The process of natural selection thus helps to validate the hypothesis (of profit maximization) or, rather, given natural selection, acceptance of the hypothesis can be based largely on the judgment that it summarizes appropriately the conditions for survival.

Friedman (1953), pg. 22

Biological Responses

- Winter (1971), “Satisficing, Selection, and the Innovating Remnant”, *Quarterly Journal of Economics* 85, 237–61.
- Nelson and Winter (1982), *An Evolutionary Theory of Economic Change*.

Criticisms

- Koopmans (1957), *Three Essays on the State of Economic Science*

Model

Capitalists own firms and employ workers.

- J commodities.
- Each worker i has a time-stationary endowment $e_i \in \mathbf{R}^{\mathbf{J}}_{+}/\{0\}$. Each capitalist h owns firm h and an initial output bundle ω_0^{h+}.
- Individual k has utility function $U_k(c) = \sum_{t=1}^{\infty} \beta_k u_k(c_t)$ on consumption plans, where $u_k : \mathbf{R}^{\mathbf{J}}_{+} \to \mathbf{R}$ and $0 < \beta_k < 1$. The u_i are C^2, strictly concave, strictly monotonic and satisfy an Inada condition on the boundary.
- Firms are described by their technology sets $T^h \subset \mathbf{R}^{\mathbf{J}}$. For $\omega \in T^h$, $\omega^+ = \omega \vee 0 \in \mathbf{R}^{\mathbf{J}}_{+}$ are outputs, and $\omega^- = \omega \wedge 0 \in \mathbf{R}^{\mathbf{J}}_{-}$ are inputs. For each firm, inputs and outputs are fixed.

Blume and Easley, "Optimality and natural selection in markets," *JET* 107, 2002.

Decision Rules

Firms buy input at date t and sell output at date $t + 1$.

Definition. A decision rule for firm h is a function

$$d^h : (p, q, y) \hookrightarrow (\omega^-, \omega^+)$$

mapping input prices p, output prices q and input expenditures y into production plans such that

1. $d^h(p, q, y) \in T^h$,
2. $p \cdot d^{h-}(p, q, y) = y$,
3. d is u.h.c.
4. d is homogeneous; for all positive scalars α, β,

$$d(\alpha p, \beta q, \alpha y) = d(p, q, y).$$

Decision Rules

Example: Constrained profit maximization.

$$d(p,q,y) = \arg\max_{\omega} q\,\omega^{+} + p\,\omega^{-}$$

$$\text{s.t.} \qquad \omega \in T^h$$

$$p\,\omega^{-} = y$$

This will be homogeneous if T^h is a cone.

Constrained Equilibrium

Definition. A constrained equilibrium is a sequence $\left(p_t^*, (c_t^{i*})_{i=1}^I, (c_t^{h*}, \omega_t^{h*})_{h=1}^H\right)_{t=1}^\infty$ with $p_t^* \in \mathbf{R}_+^{\mathbf{J}} / \{0\}$ such that

1. For all workers i, $(c_t^{i*})_{t=1}^\infty$ solves

$$\max_c U(c)$$

$$\text{s.t. for all } t,\ p_t^* \cdot (c_t - e^i) \leq 0$$

$$c_t \in \mathbf{R}_+^{\mathbf{J}}.$$

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2. For all capitalists h, $(c_t^{h*}, \omega_t^{h*})_{t=1}^{\infty}$ solves

$$\max_{c,\omega} U(c)$$

$$\text{s.t. for all } t,\ p_t^*(c_t - \omega_t) \leq 0,$$

$$(\omega_t^{h*-}, \omega_{t+1}^{h*+} \in d(p_t^*, p_{t+1}^*, p_t^* \cdot \omega_t^{h*-}),$$

$$c \in \mathbf{R}_+^{\mathbf{J}}.$$

and $(\omega_0^{h*+})_{h=1}^{H}$ is given.

Constrained Equilibrium

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3. At every date t,

$$\sum_i c_t^{i*} + \sum_h c_t^{h*} - \sum_h \omega_t^{h*} - \sum_i e_i = 0.$$

Constrained Equilibria

- Prices can be rescaled period by period.
- Constrained equilibrium is recursive. Starting a constrained equilibrium at date t gives a constrained equilibrium.

Competitive Equilibrium

Definition. A competitive equilibrium is a sequence $\left(q_t^*, (c_t^{i*})_{i=1}^I, (c_t^{h*}, \omega_t^{h*})_{h=1}^H\right)_{t=1}^\infty$ with $q_t^* \in \mathbf{R}_+^\mathbf{J} / \{0\}$ such that

1. For all workers i, $(c_t^{i*})_{t=1}^\infty$ solves

$$\max_c U(c)$$

$$\sum_{t=1}^\infty p_t^* \cdot (c_t - e^i) \leq 0$$

$$c_t \in \mathbf{R}_+^\mathbf{J}.$$

Competitive Equilibrium

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$$\max_{c,\omega} U(c)$$

$$\sum_{t=1}^{\infty} p_t^* (c_t - \omega_t) \leq 0,$$

$$(\omega_t^{h*-}, \omega_{t+1}^{h*+}) \in T^h,$$

$$c_t \in \mathbf{R}_+^{\mathbf{J}}.$$

and $(\omega_0^{h*+})_{h=1}^H$ is given.

Competitive Equilibrium

Definition. A competitive equilibrium is a sequence $\left(q_t^*, (c_t^{i*})_{i=1}^I, (c_t^{h*}, \omega_t^{h*})_{h=1}^H\right)_{t=1}^\infty$ with $q_t^* \in \mathbf{R}_+^{\mathbf{J}} / \{0\}$ such that

3. At every date t,

$$\sum_i c_t^{i*} + \sum_h c_t^{h*} - \sum_h \omega_t^{h*} - \sum_i e_i = 0.$$

In a competitive equilibrium, every firm maximizes profits, and equilibrium profits are 0.

Selection

Let $\rho^h(p^{h-}, p^{h+}, y)$ denote firm h's revenue function.

Assume that on an equilibrium path, ρ_y exists, and that for all t, $\liminf_{y \to 0} \rho_y(p_t^*, p_{t+1}^*, y) \neq 0$.

Theorem. In any constrained equilibrium, and for any two capitalists h and k with discount factors β_h and β_k,

$$\prod_{\tau=1}^{t-1} \frac{\beta_k \rho_{y\tau}^k}{\beta_h \rho_{y\tau}^h} \to 0 \text{ implies } \lim_t \frac{p_t \omega^{k-}}{\sum_i p_t \omega_t^{i-}} \text{ and } c_t^k \to 0.$$

Proof of Theorem

The capitalist's solution solves

$$\max \sum_t \beta^t v^h(p_t, z_t)$$

$$\text{s.t. for all } t,\ z_t + y_t = m_t$$

$$m_{t+1} = \rho^h(p_t, p_{t+1}, y_t)$$

where m_t is revenue at the beginning of t, z_t is consumption expenditure, y_t is input expenditure and v^h is an indirect utility function.

Proof of Theorem

F.O.C.

$$v_z^h(p_t, z_t^h) = \beta_h \rho_{yt}^h v_z^h(p_{t+1}, z_{t+1}^h)$$

so

$$\frac{v_z^h(p_{t+1}, z_{t+1}^h)}{v_z^k(p_{t+1}, z_{t+1}^k)} = \frac{\beta_k \rho_{yt}^k}{\beta_h \rho_{yt}^h} \frac{v_z^h(p_t, z_t^h)}{v_z^k(p_t, z_t^k)}$$

$$= \prod_{\tau=1}^{t-1} \frac{\beta_k \rho_{y\tau}^k}{\beta_h \rho_{y\tau}^h} \frac{v_z^h(p_1, z_1^h)}{v_z^k(p_1, z_1^k)}$$

and so for each good l,

$$\frac{D_l u^h(c_{t+1}^h)}{D_l u^k(c_{t+1}^k)} \propto \prod_{\tau=1}^{t-1} \frac{\beta_k \rho_{y\tau}^k}{\beta_h \rho_{y\tau}^h}$$

If the RHS goes to 0, so does the LHS, and so $c_t^k \to 0$.

Selection

Let $r_t^h = \rho(p_t, p_{t+1}, p_t\omega_t^{h-}) / p_t\omega_t^{h-}$ denote the date t average return on investment.

Assume. For every firm h and prices (p, q), the revenue function is concave in y.

Corollary. Suppose capitalist h is a constrained profit-maximizer, and suppose that for capitalist k, ρ_y^h is non-increasing in y, and

$$\prod_{\tau=1}^{t-1} \frac{\beta_k r_\tau^k}{\beta_h r_\tau^h} \to 0.$$

Then the theorem's conclusions still hold.

Selection

Corollary. If h and k are profit maximizers, if

$$\limsup_t \prod_{\tau=1}^{t-1} \frac{r_\tau^k}{r_\tau^h} < \infty \text{ and if } \frac{\beta_k}{\beta_h} < 1,$$

then $\lim_t c_t^k = 0$ and $\lim y_t^k / \sum_h y_t^h \to 0$.

Selection

Assume. Common discount factor, all inputs are endowed and all consumer goods are produced.

Theorem. Every competitive equilibrium consumption plan is stationary from date 2 on.

Theorem. A stationary competitive equilibrium is a constrained equilibrium for some assignment of initial outputs $(\omega_0^h)_{h=1}^H$.

Selection

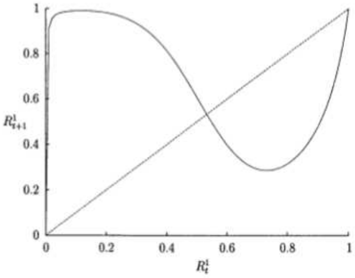
Theorem. The allocation resulting from any stationary and interior constrained equilibrium is a competitive allocation.

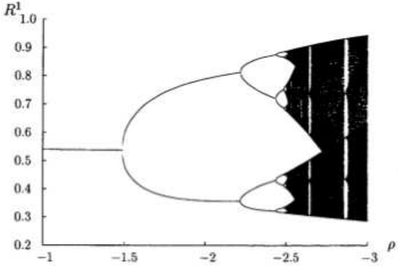
Theorem. If a constrained equilibrium is locally stable, then the limit allocation is competitive.

Dynamics

Two produced consumption goods, x and y, one factor z. Linear production with unit outputs $(1, 0.1)$ and $(0.001, 1)$.

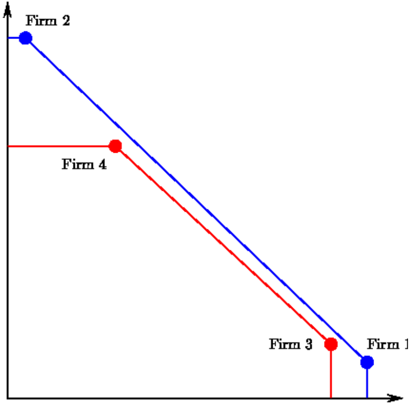
$$u(x, y) = \sum_{t=1}^{\infty} \beta^t \log \left(x_t^\rho + y_t^\rho\right)^{1/\rho}$$

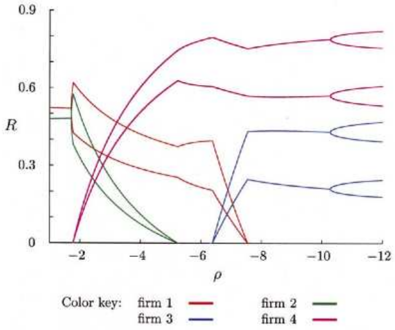




Dynamics

Four firms. with unit output vectors $(1, 0.1)$, $(0.05, 1)$, $(0.9, 0.15)$, $(0.3, 0.7,)$.





Evolution in Finance

“The trading floor is a jungle,” he went on, “and the guy you end up working for is your jungle leader. Whether you succeed here or not depends on knowing how to survive in the jungle.”

Michael Lewis, *Liar’s Poker*

Kelly's Betting System

You are offered a gamble with probability $2/3$ of winning and $1/3$ of losing. You may bet up to your current wealth. The amount you bet is either doubled or lost. You will bet for 50 days.

$$x_0 \text{ given.}$$

$$0 \le b_k \le X_{k-1}$$

$$x_k = \begin{cases} X_{k-1} + b_k & \text{with prob. } 2/3, \\ X_{k-1} - b_k & \text{with prob. } 1/3. \end{cases}$$

Kelly's Betting System

Maximize end date wealth:

$$\begin{aligned} \mathrm{E}\{X_{50}|X_{49}\} &= X_{49} + \frac{2}{3}b_{49} - \frac{1}{3}b_{49} \\ &= X_{49} + \frac{1}{3}b_{49} \end{aligned}$$

so take $b_{49} = X_{49}$.

$$\begin{aligned} \mathrm{E}\{X_{50}|X_{49}\} &= \frac{4}{3}X_{49} \\ \mathrm{E}\{X_{50}|X_{48}\} &= \frac{4}{3}\,\mathrm{E}\{X_{49}|X_{48}\} = \frac{16}{9}X_{48} \\ &\vdots \\ \mathrm{E}\{X_{50}|X_0\} &= \left(\frac{4}{3}\right)^{50} X_0 \end{aligned}$$

Kelly's Betting System

$$\mathrm{E}\{X_{50}|X_0\} = \left(\frac{4}{3}\right)^{50} X_0 = 1.77 \times 10^6 X_0$$

BUT

$$X_{50} = \begin{cases} 2^{50} X_0 & \text{with probability } 0.00000000157, \\ 0 & \text{with probability } 0.99999999843. \end{cases}$$

Kelly's Betting System

A Proportional Betting System: Invest fraction π in the bet.

$$X_n = (1+\pi)^{Z_n}(1-\pi)^{n-Z_n}X_0 \qquad Z_n \text{ is \# wins}$$

$$= \exp\{Z_n \log(1+\pi) + (n - Z_n)\log(1-\pi)\}X_0$$

$$= \exp n \left\{\frac{Z_n}{n}\log(1+\pi) + \left(1 - \frac{Z_n}{n}\right)\log(1-\pi)\right\} X_0$$

$$\simeq \exp n \left\{\frac{2}{3}\log(1+\pi) + \frac{1}{3}\log(1-\pi)\right\} X_0$$

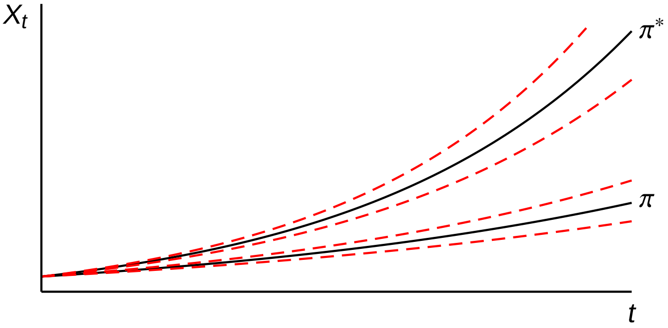
Maximize the rate: $\pi = 1/3$.

$$\text{growth rate is} \quad \frac{2}{3}\log\frac{4}{3} + \frac{1}{3}\log\frac{2}{3} \approx 0.056633$$

$$X_{50} \approx 16.97 X_0$$

Kelly's Betting System

Let $\pi \neq 1/3$ be another proportional betting rule. The SLLN implies that for any exponential confidence bands around the mean growth path, X_t will almost surely be inside the bands for all large t.



Breiman on Kelly

"Assume that we are hardened and unscrupulous types with an infinitely wealthy friend. ..."

- X takes values in $\mathcal{S} = \{1, \ldots, S\}$. $\Pr\{X = s\} = p_s$.
- There are J assets. Asset j pays off $A(s, j)$ in state s.
- A bet is a vector $\{\alpha_1, \ldots, \alpha_J\}$, where if wealth share b_j is bet on asset j, and if the event $\{X = s\}$ is realized, we receive $\sum_j \alpha_j A(s, j)$. W.l.o.g. we bet everything.
- W^n is our fortune after n plays.

Definition. A game is favorable if there is a strategy such that $W^n \to \infty$ a.s.

Optimal Gambling for Favorable Games

α_k^n The fraction of W_{n-1} invested in asset k.

$\mathcal{B}^n$ The set of date n feasible portfolios.

V^n The return on portfolio α^n,

$$V^n = \sum_j \alpha_j^n A(s, j) \quad \text{if } X_n = s;$$

$$W^{n+1} = V^{n+1} W^n.$$

G^n Growth rate, $G^n = \log V^n$. Depends on α^n.

A Single Stage

The expected growth rate is

$$G(\alpha, p) \equiv \sum_s p_s \log \sum_j \alpha_j A(s, j).$$

The maximal expected growth rate is

$$G(p) = \max_{\alpha \in \mathcal{B}} G(\alpha, p)$$

Properties of $G(\alpha, p)$

Lemma. $W(\alpha, p)$ is concave in α and linear in p. $G(p)$ is convex in p.

Lemma. If $\mathcal{B}$ is convex, then set of log-optimal portfolios is convex.

Lemma. Any two portfolios α' and α'' that maximize the expected growth rate G have the same rate of return in each state i; for all i,

$$\sum_s \alpha_j' A(s,j) = \sum_s \alpha_j'' A(s,j) \qquad \text{for all } s.$$

Optimal Portfolios

Suppose that each $\mathcal{B}^n$ is the simplex.

Theorem. A portfolio α^* is log-optimal iff

$$\mathsf{E}\left\{\frac{A(s,j)}{\sum_j \alpha_j^* A(s,j)}\right\} \begin{array}{c} = \\ \leq \end{array} 1 \quad \begin{array}{l} \text{if } \alpha_j^* > 0, \\ \text{if } \alpha_j^* = 0. \end{array}$$

Proof

Let α^* be optimal, and consider a deviation in direction of feasible portfolio α. Take $\alpha_\lambda = (1 - \lambda)\alpha^* + \lambda\alpha$. Then

$$\frac{d}{d\lambda} G(\alpha_\lambda)\bigg|_{\lambda \downarrow 0} \leq 0$$

for all feasible portfolios α. Computing

$$\begin{aligned}
\frac{d}{d\lambda} G(\alpha_\lambda)\bigg|_{\lambda \downarrow 0} &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \mathrm{E}\left\{ \log \frac{(1-\lambda)\sum_j \alpha_j^* A(s,j) + \lambda \sum_j \alpha_j A(s,j)}{\sum_j \alpha_j^* A(s,j)} \right\} \\
&= \lim_{\lambda \downarrow 0} \mathrm{E}\left\{ \frac{1}{\lambda} \log\left(1 + \lambda \left(\frac{\sum_j \alpha_j A(s,j)}{\sum_j \alpha_j^* A(s,j)} - 1 \right)\right)\right\} \\
&= \mathrm{E}\left\{ \frac{\sum_j \alpha_j A(s,j)}{\sum_j \alpha_j^* A(s,j)} - 1 \right\}
\end{aligned}$$

Proof (cont.)

Thus for all $\alpha \in \mathcal{B}$,

$$\mathrm{E}\left\{\frac{\sum_j \alpha_j A(s,j)}{\sum_j \alpha_j^* A(s,j)}\right\} \leq 1.$$

This expression is linear in α, so it suffices to check it for the extreme points of $\mathcal{B}^n$. Thus

$$\mathrm{E}\left\{\frac{A(s,j)}{\sum_j \alpha_j^* A(s,j)}\right\} \leq 1.$$

If $\alpha *_j > 0$, the line from asset j extends below 0, so the inequality is two-sided, that is, it holds with equality. If $\alpha_j^* = 0$, it cannot be extended, and we have the weak inequality.

More on Optimal Portfolios

Let $V_\alpha = \sum_j \alpha_j A(s,j)$ denote the random return from an arbitrary portfolio.

Theorem.

$$\mathrm{E}\left\{\frac{V_\alpha}{V_{\alpha'}}\right\} \leq 1 \text{ for all } \alpha \in \mathcal{B} \text{ iff } \mathrm{E}\left\{\log \frac{V_\alpha}{V_{\alpha'}}\right\} \leq 1 \text{ for all } \alpha \in \mathcal{B}.$$

Proof.

The second condition is log-optimality, so Only If follows from the first-order condition. If follows from Jensen's inequality:

$$\mathsf{E}\left\{\log \frac{V_\alpha}{V_{\alpha'}}\right\} \leq \log \mathsf{E}\left\{\frac{V_\alpha}{V_{\alpha'}}\right\} \leq \log 1 = 0.$$

Asymptotic Optimality

Lemma. Let W_*^n be the wealth of an investor after n rounds of using a log-optimal portfolio, and let W^n denote the wealth from any other portolio rule. Then

$$\mathrm{E}\{\log W_*^n\} = nW(p) \geq \mathrm{E}\{\log W^n\}.$$

The proof is obvious.

Asymptotic Optimality

Theorem.

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{W^n}{W_*^n} \leq 0 \text{ with pr. } 1.$$

Proof. From the Kuhn-Tucker conditions,

$$\mathrm{E}\left\{\frac{W^n}{W_*^n}\right\} \leq 1.$$

From Markov's inequality

$$\Pr\{W^n > t_n W_*^n\} = \Pr\left\{\frac{W^n}{W_*^n} > t_n\right\} < \frac{1}{t_n},$$

so,

Asymptotic Optimality

$$\Pr\left\{\frac{1}{n}\log\frac{W^n}{W_*^n} > \frac{1}{n}\log t_n\right\} < \frac{1}{t_n}.$$

Take $t_n = n^2$, and summing,

$$\sum_{n=1}^{\infty}\Pr\left\{\frac{1}{n}\log\frac{W^n}{W_*^n} > \frac{2\log n}{n}\right\} < \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Borel-Cantelli says

$$\Pr\left\{\frac{1}{n}\log\frac{W^n}{W_*^n} > \frac{2\log n}{n} \text{ infinitely often}\right\} = 0,$$

Asymptotic Optimality

Thus for almost every sequence $(x_1, \ldots)$ there is an N such that for $n \geq N$,

$$\frac{1}{n} \log \frac{W^n}{W_*^n} \leq \frac{2 \log n}{n},$$

from which the theorem follows.

The Cost of Being Wrong

Suppose you believe the true distribution is q, when the truth is p. How badly off will you be?

Theorem. If α^q is log-optimal for probability distribution q, then

$$\Delta W = W(\alpha^p, p) - W(\alpha^q, p) \leq \sum_s p_s \log \frac{p_s}{q_s}$$

Proof

$$\Delta W = \sum_s p_s \log\left(\sum_j \alpha_j^p A(s,j)\right) - \sum_s p_s \log\left(\sum_j \alpha_j^q A(s,j)\right)$$

$$= \sum_s p_s \log\left(\frac{\sum_j \alpha_j^p A(s,j)}{\sum_j \alpha_j^q A(s,j)}\right)$$

$$= \sum_s p_s \log\left(\frac{\sum_j \alpha_j^p A(s,j)}{\sum_j \alpha_j^q A(s,j)} \frac{q_s}{p_s} \frac{p_s}{q_s}\right)$$

$$= \sum_s p_s \log\left(\frac{\sum_j \alpha_j^p A(s,j)}{\sum_j \alpha_j^q A(s,j)} \frac{q_s}{p_s}\right) - \sum_s p_s \log\left(\frac{p_s}{q_s}\right)$$

Proof

$$\sum_s p_s \log\left(\frac{\sum_j \alpha_j^p A(s,j)}{\sum_j \alpha_j^q A(s,j)} \frac{q_s}{p_s}\right) \leq \log \sum_s p_s \frac{\sum_j \alpha_j^p A(s,j)}{\sum_j \alpha_j^q A(s,j)} \frac{q_s}{p_s}$$

$$\leq \log \sum_s q_s \frac{\sum_j \alpha_j^p A(s,j)}{\sum_j \alpha_j^q A(s,j)}$$

$$\leq \log 1$$

$$= 0$$

This result gives an upper bound on the value of additional information for the optimal portfolio.

WHY WE SHOULD NOT MAKE MEAN LOG OF WEALTH BIG THOUGH YEARS TO ACT ARE LONG

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> He who acts in *N* plays to make his mean log of wealth as big as it can be made will, with odds that go to one as *N* soars, beat me who acts to meet my own tastes for risk.

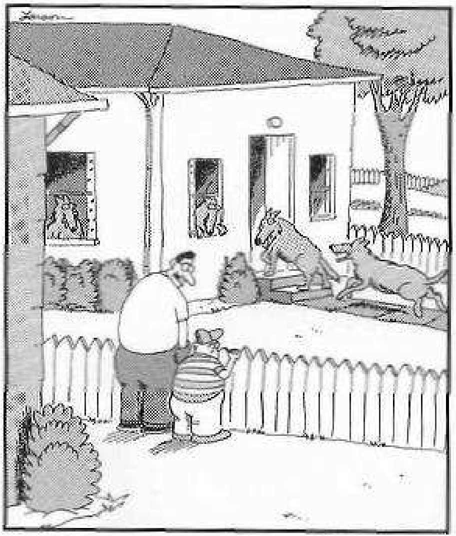
Who doubts *that*? What we do doubt[1] is that *it* should make us change our views on gains and losses – should taint our tastes for risk.

To be clear is to be found out. Know that life is not a game with net stake of one when you beat your twin, and with net stake of nought when you do not. A win of ten is not the same as a win of two. Nor is a loss of two the same as a loss of three. *How much* you win by counts. *How much* you lose by counts.

As soon as we see *this* clear truth, we are back to our own tastes for risk. Mean log of wealth then bores those of us with tastes for risk not real near to one odd (thin!) point on the line of *all* the tastes for risk – and this holds for each *N*, *with N as big as you like.*

No need to say more.[2] I've made my point.[3] And, save for the last word, have done so in prose of but one syllable.

Evolutionary Arguments in Financial Markets



"I know you miss the Wainrights, Bobby, but they were weak and stupid people—and that's why we have wolves and other large predators."

Given the uncertainty of the real world, the many actual and virtual traders will have many, perhaps equally many, forecasts. . . If any group of traders was consistently better than average in forecasting stock prices, they would accumulate wealth and give their forecasts greater and greater weight. In this process, they would bring the present price closer to the true value.

Robert Cootner

. . . dependence in the noise generating process would tend to produce ‘bubbles’ in the price series. . . If there are many sophisticated traders in the market, however, they will be able to recognize situations where the price of a common stock is beginning to run up above its intrinsic value. If there are enough of these sophisticated traders, they may tend to prevent these ‘bubbles’ from ever occurring.

Eugene Fama

Arrow Securities

- There are S securities, one for each state. Each security pays of \$1 in its state, and \$0 otherwise. The price of security s is p_s.
- Investor i invests fraction $\alpha_s^i \geq 0$ of his wealth w^i in asset s. His budget constraint is $\sum_s \alpha_s^i = 1$.
- Assets are in 0 net supply, so the total amount returned by the asset that pays off in state s is the total amount invested in all assets, $w = \sum_i w^i$. Market clearing implies

$$\sum_i \frac{\alpha_s^i w^i}{p_s} = w \qquad p_s = \sum_i \alpha_s^i \frac{w^i}{w}$$

Dynamics

Suppose income is reinvested repeatedly.

$$w_t^i = \prod_{\tau=0}^{t} \prod_{s=1}^{S} \left(\frac{\alpha_{s\tau}^i}{p_{s\tau}} \right)^{1_s(\omega_\tau)} w_0^i$$

Suppose that states are iid, that $\Pr\{\omega_t = s\} = q_s$, and that investors use constant share rules.

$$w_t^i = \prod_{\tau=0}^{t} \prod_{s=1}^{S} \left(\frac{\alpha_s^i}{p_{s\tau}} \right)^{1_s(\omega_\tau)} w_0^i$$

$$\log w_t^i = \sum_{\tau=0}^{t} \sum_{s} 1_s(\omega_\tau) \left(\log \alpha^i - \log p_{s\tau} \right) + \log w_0^i$$

Comparing Investors

Suppose that *i* and *j* both use constant share rules.

$$\log \frac{w_t^i}{w_t^j} = \sum_{\tau=0}^{t} \sum_s 1_s(\omega_\tau) \left(\log \alpha^i - \log \alpha^j\right) + \log \frac{w_0^i}{w_0^j}$$

From the law of large numbers,

$$\begin{aligned}
\frac{1}{t+1} \log \frac{w_t^i}{w_t^j} &\to \sum_s q_s \left(\log \alpha^i - \log \alpha^j\right) \\
&= \sum_s q_s \left(\log q_s - \log \alpha^j\right) - \sum_s q_s \left(\log q_s - \log \alpha^i\right) \\
&= I(q;\alpha_j) - I(q;\alpha_i)
\end{aligned}$$

Selection

The SLLN give a simple market selection result:

Theorem. If for trader i, $I(q, \alpha_i) > \min_j I(q, \alpha_j)$, then $\dfrac{w_t^i}{\sum_j w_t^j} \to 0$.

- So being relative-entropy minimal is necessary for survival.
- Is it sufficient?

Generalizations I

Suppose that $S = R \times S$ where R has public states and S has state components that cannot be traded on. Suppose that there are $|R|$ assets, and asset r pays off $a(s, r)$ in state (s, r). Then

$$w_t^i = \prod_{\tau=0}^{t} \prod_{s=1}^{S} \prod_{r=1}^{R} \alpha_{r\tau}^i \left(\frac{a(s, r)}{p_{r\tau}} \right)^{1_{sr}(\omega_\tau)} w_0^i$$

$$\log w_t^i = \sum_{\tau=0}^{t} \sum_{sr} 1_{sr}(\omega_\tau) \log \alpha_{r\tau}^i + 1_{sr}(\omega_\tau) \log \frac{a(s, r)}{p_{r\tau}} + \log w_0^i$$

- The log-optimal portfolio is independent of prices.
- Markets are incomplete.
- Selection will depend upon the marginal distribution of r.

Generalizations II

For an arbitrary asset structure, the log-optimal rule will be price dependent, and thus time-dependent. Nonetheless,

Theorem. If $\liminf_t \frac{1}{t} \sum_{\tau=1}^{t} W(\alpha_\tau^i, q) - W(\alpha_\tau^j, q) > 0$ then $\frac{w_t^j}{w_t^i} \to 0$.

Generalizations III

- If traders are constrained in their portfolios, log-optimality still dominates.
- Adjustment for savings rates.

If trader *i* saves at rate δ_i, define the survival rate

$$\kappa_i = \log \delta_i - I(q, \alpha_i)$$

A maximal survival rate is necessary for long-run survival.

Long Run Asset Pricing

Suppose some trader uses the log-optimal portfolio.

- First order conditions imply

$$\mathrm{E}\left\{\frac{a_k(s)/p_k}{\sum_j \alpha_j a_j(s)/p_j}\right\} = 1 \text{ if } \alpha_k > 0.$$

- The long run equilibrium condition is that for all s

$$\sum_j \frac{\alpha_j a_j(s)}{p_j} = 1.$$

Thus

$$p_k = \mathrm{E}\left\{a_k(s)\right\}.$$

Survival: Sufficient Conditions

Suppose traders 1 and i both have maximal survival index.

$$\begin{aligned}\log \frac{w_t^i}{w_t^1} &= \sum_{\tau=0}^{t} \sum_s 1_s(\omega_\tau)(\log \delta_i + \log \alpha_s^i) - \dots \\ &= \sum_s \Big(n_s(\omega_t) - tp_s\Big) \log \alpha_s^i + t\{\log \delta_i + p_s \log \alpha_s^i)\} - \dots \\ &= \sum_s \Big(n_s(\omega_t) - tp_s\Big)(\log \alpha_s^i - \log \alpha_s^1)\end{aligned}$$

Survival: Sufficient Conditions

$$\begin{pmatrix} \log \frac{w_t^2}{w_t^1} \\ \vdots \\ \log \frac{w_t^I}{w_t^1} \end{pmatrix} = \begin{pmatrix} \log \alpha_1^2 - \log \alpha_1^1 & \cdots & \log \alpha_S^2 - \log \alpha_S^1 \\ \vdots & & \vdots \\ \log \alpha_1^I - \log \alpha_1^1 & \cdots & \log \alpha_S^I - \log \alpha_S^1 \end{pmatrix} \cdot \begin{pmatrix} n_1(\omega_t) - tp_1 \\ \vdots \\ n_S(\omega_t) - tp_S \end{pmatrix}$$

$$= \begin{pmatrix} \log \frac{\alpha_1^2}{\alpha_S^2} - \log \frac{\alpha_1^1}{\alpha_S^1} & \cdots & \log \frac{\alpha_{S-1}^2}{\alpha_S^2} - \log \frac{\alpha_{S-1}^1}{\alpha_S^1} \\ \vdots & & \vdots \\ \log \frac{\alpha_1^I}{\alpha_S^I} - \log \frac{\alpha_1^1}{\alpha_S^1} & \cdots & \log \frac{\alpha_{S-1}^I}{\alpha_S^I} - \log \frac{\alpha_{S-1}^1}{\alpha_S^1} \end{pmatrix} \cdot \begin{pmatrix} n_1(\omega_t) - tp_1 \\ \vdots \\ n_{S-1}(\omega_t) - tp_{S-1} \end{pmatrix}$$

$$\begin{pmatrix} \log \frac{w_t^2}{w_t^1} \\ \vdots \\ \log \frac{w_t^I}{w_t^1} \end{pmatrix} = A \cdot \begin{pmatrix} n_1(\omega_t) - tp_1 \\ \vdots \\ n_{S-1}(\omega_t) - tp_{S-1} \end{pmatrix}$$

- If there is an x such that $A \cdot x \ll 0$, then there is an open cone C such that $A \cdot y \ll 0$ for all $y \in C$. In this case, for all $\epsilon > 0$, $w_t^1 > 1 - \epsilon$ infinitely often.
- If for all x there is a row of A such that $a \cdot x > 0$, then there is always some trader $i \neq 1$ who "beats" trader 1. If $S > 3$, then for all $\epsilon > 0$ the event $w_t^1 > \epsilon$ is transient.

There is an x such that $A \cdot x \ll 0$ iff there is no non-trivial, non-negative linear combination λ of the rows of A such that $\lambda \cdot A = 0$.

That is, there is no non-negative vector of weights λ summing to 1 such that

$$\sum_{i \geq 2} \lambda_i \left(\log \frac{\alpha_s^i}{\alpha_S^i} \right)_{s=1}^{S-1} = \left(\log \frac{\alpha_s^1}{\alpha_S^1} \right)_{s=1}^{S-1}.$$

That is, $\limsup_t w_t^1 = 1$ iff $\left(\log \frac{\alpha_s^1}{\alpha_S^1} \right)_{s=1}^{S-1}$ is not in the convex hull of the remaining vectors of log-odds ratios.

If $S > 3$,

- **Interior traders vanish.**
- **For extremal traders, $\limsup_t w_{it} \big/ \sum_j w_{jt} = 1$.**

Survival requires much patience and extreme beliefs.

A Simple Exchange Economy I

- One non-storable good.
- Uncertain endowments
- Long-lived consumers maximize the expectation of the present discounted value of utility streams.
- Consumers are heterogeneous — they may differ in beliefs, discount factors and payoff functions.
- Markets are complete.

A Simple Exchange Economy II

States: S, a finite set.

A **path** is denoted $\sigma = (\sigma_1, \sigma_2, \ldots)$, $\sigma_t \in S$.
There is a **"true" probability** p on $(\Sigma, \mathcal{F})$. For any probability distribution q,

$$q_t(\sigma) \equiv q\Big(\{\sigma_1, \ldots, \sigma_t\} \times S \times S \ldots\Big).$$

Consumers: **Beliefs** p^i on $(\Sigma, \mathcal{F})$.
Discount factors $0 < \beta_i < 1$.
Consumption plans $c = \big(c_1, \ldots\big)$, c_t t-meas.
Endowments e_i, strictly positive consumption plans.
Payoff functions $u_i : \mathbf{R}_+ \to \mathbf{R}$.
Preferences

$$U^i(c) = E_{p^i}\left\{\sum_{t=1}^{\infty} \beta_i^{t-1} u_i\Big(c_t(\sigma)\Big)\right\}$$

Assumptions

A.1. (Payoff Functions) The payoff functions u_i are C^1, strictly concave and monotonic, and satisfy an Inada condition at 0.

A.2. (Endowments)

$$0 < f = \inf_{t,\sigma} \sum_i e_t^i(\sigma) \leq \sup_{t,\sigma} \sum_i e_t^i(\sigma) = F < \infty.$$

A.3. (Truth) For all i, t and σ, if $p_t(\sigma) > 0$ then

$$p_t^i(\sigma) > 0.$$

We will assume A.1-3 throughout.

Optimal Allocations

If $c^* = (c^{1*}, \ldots, c^{I*})$ is an interior **Pareto optimal allocation** of resources, then there is a vector of welfare weights $(\lambda^1, \ldots, \lambda^I) >> \mathbf{0}$ such that c^* solves the problem

$$\max_{(c^1,\ldots,c^I)} \quad \sum_i \lambda^i U^i(c)$$

such that for all t and σ,

$$\sum_i c_t^i(\sigma) - \sum_i e_t^i(\sigma) = 0,$$

$$\text{and } \forall i \quad c_t^i(\sigma) \geq 0.$$

First Order Conditions

For all σ and t:

There are numbers $\lambda_i > 0$ and $\eta_t(\sigma) > 0$ s.t., for all i:

$$\lambda^i \beta_i^{t-1} u_i'\big(c_t^i(\sigma)\big) p_t^i(\sigma) - \eta_t(\sigma) = 0.$$

So for any i and j and for all σ and t:

$$\frac{u_i'\big(c_t^i(\sigma)\big)}{u_j'\big(c_t^j(\sigma)\big)} = \frac{\lambda^j}{\lambda^i} \frac{\beta_j^{t-1}}{\beta_i^{t-1}} \frac{p_t^j(\sigma)}{p_t^i(\sigma)}.$$

Analysis of the IID Case

$p^i_t(\sigma)$ generated by iid draws with distribution $p^i = (p^i_1, \ldots, p^i_S)$.

$$p^i_t(\sigma) = \prod_{s=1}^{S} (p^i_s)^{n^s_t(\sigma)}$$

Taking logs and dividing by t gives

$$\frac{1}{t} \log \frac{u^{i\prime}(c^i_t(\sigma))}{u^{j\prime}(c^j_t(\sigma))} = \frac{1}{t} \log \frac{\lambda_j}{\lambda_i} + \log \frac{\beta_j}{\beta_i} + \frac{1}{t} \sum_s n^s_t(\sigma)(\log p^j_s - \log p^i_s)$$

From the SLLN, the lhs converges a.s. to

$$\Big(\log \beta_j - I_p(p^j)\Big) - \Big(\log \beta_i - I_p(p^i)\Big),$$

where $I_p(q) = E_p \log \frac{p(s)}{q(s)}$ is the relative entropy of p with respect to q.

Long-Run Fates in the IID Case

Definition. Consumer i survives iff $\limsup c_t^i > 0$. She vanishes iff $\lim_t c_t^i = 0$.

Trader i's survival index is $\kappa_i = \log \beta_i - I_p(p^i)$.

$$\frac{1}{t} \log \frac{u^{i\prime}(c_t^i(\sigma))}{u^{j\prime}(c_t^j(\sigma))} \to \kappa_j - \kappa_i$$

Theorem. If $\kappa_i \neq \max_j \kappa_j$, then trader i vanishes almost surely.

Bayesian Learning

- States are iid with distribution p on S.
- Consumer i considers a model set Θ_i and has prior beliefs μ_i on Θ_i.
- Each $\theta \in \Theta_i$ describes a distribution p_θ on Σ.
- The collection of p_θ's and μ_i induce a joint distribution on $\Theta_i \times \Sigma$.
- p_t^i is the marginal distribution on the t-fold product $S \times \cdots \times S$.

Bayesian Learning

Theorem. Suppose traders i and j are Bayesians with model sets Θ_i that are manifolds. Suppose for each $k = i, j$ that:

1. $\beta_i = \beta_j$;
2. The parameter $\theta \in \Theta_k$, is identified;
3. that each Θ_k is a manifold of dimension d_k;
4. and that trader k's prior belief has strictly positive density on Θ_k with respect to Lebesgue measure.

Suppose $d_i > d_j$ and $\Theta_j \subset \Theta_i$. Then for all $\theta \in \Theta_j$, $c_t^i \to 0$ in probability. If $\beta_i \neq \beta_j$, the consumer with the lower discount factor vanishes.

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Other work: See the *Handbook of Financial Markets* for related surveys.

The Survival of Noise Traders?

DeLong, Shleifer, Summers and Waldmann, “The survival of noise traders in financial markets,” *Journal of Business*, 1991.

- In our exchange model, traders with better information drive out those with worse information, *ceteris paribus*. DSSL claim otherwise.
- This paper is cited as a justification for much current research.
- It illustrates the misuse of “continuum-of-agent models”.
- Similar modelling issues arise in other continuum models, e.g. Bewley Models.

The Survival of Noise Traders?

- There are two continuua of traders: Group I_p and I_q.
- Initial endowments are 1 for each group, uniformly distributed.
- Traders bet on the flip of identical i.i.d. coins, one for each trader. The true H probability is p.
- For every dollar i bets correctly, he earns $2.
- Traders have identical CRRA utility functions with parameter $\gamma < 1$ and a common discount factor $0 < \beta < 1$.
- How does the wealth of each group evolve?

Let α^i denote the share of wealth invested in H. For the trader with belief r,

$$\alpha = \frac{r^{1/1-\gamma}}{r^{1/1-\gamma} + (1 - r)^{1/1-\gamma}}$$

and

$$\delta = 2^{\gamma/1-\gamma}\beta^{1/1-\gamma}\left(r^{1/1-\gamma} + (1 - r)^{1/1-\gamma}\right)$$

The stochastic process of i's wealth is

$$w_t^i = (2\delta_i)^t \prod_{\tau=1}^{t} \alpha_i^{1_H(\omega_\tau^i)} (1 - \alpha_i)^{1 - 1_H(\omega_\tau^i)}$$

For traders i and j,

$$\log r_t \equiv \log \frac{w_t^i}{w_t^j} = t \log \frac{\delta_i}{\delta^j} + \sum_{\tau=1}^{t} 1_H(\omega_\tau^i) \log \alpha_i + 1_T(\omega_\tau^i) \log(1 - \alpha_i) - \sum_{\tau=1}^{t} 1_H(\omega_\tau^j) \log \alpha_j + 1_T(\omega_\tau^j) \log(1 - \alpha_j)$$

Suppose $i \in I_p$ and $j \in I_q$.

$$E_p\Big\{\log \delta_i + 1_H(s_t^i)\alpha_i + 1_T(s_t^i)\Big\} = \frac{1}{1-\gamma_i}(\gamma_i \log 2 + \log \beta_i - I_p(q^i) + C(p))$$

where $C(p) = p \log p + (1-p)\log(1-p)$. By the SLLN, almost surely:

$$\frac{1}{t}\log(r_t) \to \frac{1}{1-\gamma_i}\Big(\log \beta_i - I_p(q_i) + \gamma_i \log 2 + C(p)\Big) - \frac{1}{1-\gamma_j}\Big(\log \beta_j - I_p(q_j) + \gamma_j \log 2 + C(p)\Big)$$

If the r.h.s. > 0, then p-almost surely, $\frac{w_t^i}{w_t^j} \to \infty$.

Let w_t be the wealth of the rational group at time t.
Let x_t be the wealth of the irrational group at time t.
Let $\mathbf{r}_t = w_t / x_t$. Take $w_0 = x_0 = \mathbf{r}_0 = 1$.

$$\begin{aligned} w_{t+1} &= p\delta_{rat}(2pw_t) + (1-p)\delta_{rat}\Big(2(1-p)w_t\Big) \\ &= 2\delta_{rat}w_t\Big(p^2 + (1-p)^2\Big) \\ x_{t+1} &= p\delta_{irr}(2qx_t) + (1-p)\delta_{irr}\Big(2(1-q)x_t\Big) \\ &= 2\delta_{irr}x_t\Big(pq + (1-p)(1-q)\Big) \end{aligned}$$

so

$$\begin{aligned} \mathbf{r}_t &= \frac{w_t}{x_t} = \left(\frac{\delta_{rat}}{\delta_{irr}} \frac{p^2 + (1-p)^2}{pq + (1-p)(1-q)}\right)^t \\ &= \left(\frac{p^{1/1-\gamma} + (1-p)^{1/1-\gamma}}{q^{1/1-\gamma} + (1-q)^{1/1-\gamma}} \frac{p^2 + (1-p)^2}{pq + (1-p)(1-q)}\right)^t \end{aligned}$$

Results

Suppose $p > 1/2$.

Fact: *If $\gamma > 0$, then for all q sufficiently near to 1, $\mathbf{r}_t \to 0$ p-a.s.*

Fact: *If $\gamma < 0$, then for all p sufficiently near to $1/2$ and q sufficiently near to 1, $\mathbf{r}_t \to 0$ p-a.s. In this case, $\delta_{rat} > \delta_{irr}$.*

How can it be that $r_t \to \infty$ and $\mathbf{r}_t \to 0$?

In the DSSW analysis, wealth dynamics are derived by “applying” the SLLN to each group at each date.

- Analysis in the DSSW style computes

$$\lim_{t\to\infty} \lim_{N\to\infty} \frac{w_t}{x_t}.$$

- A correct analysis computes

$$\lim_{N\to\infty} \lim_{t\to\infty} \frac{w_t}{x_t}.$$

- These two limits are different:

$$\lim_{t\to\infty} \lim_{N\to\infty} \frac{w_t}{x_t} = 0$$

$$\lim_{N\to\infty} \lim_{t\to\infty} \frac{w_t}{x_t} = \infty$$