Robinson Crusoe

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1 The System

The point of economics is not individuals — firms, consumers — but the system: The interaction of many firms, many consumers, many markets. The simplest example of this is the *Robinson Crusoe economy*. In Robinson's world there is one person who produces coconut from his labor (and, presumably, a convenient tree). He spends his time eating coconuts, climbing the damn tree and laying around.

1.1 Technology

Labor can be turned into coconuts. The production function describes the tradeoffs. For our example,

$$f(x) = ax.$$

One needs to think about how the production function actually characterizes technology. One conventional way is that technology is fully characterized by a set of feasible *input-output* pairs, and the production function describes the relevant frontier.

1.2 Preferences

Preferences are described by a utility function.

$$u(c,l) = \frac{c^{1-\sigma} + l^{1-\sigma}}{1-\sigma}.$$

This is of the CES form. Labor is measured in units of weeks, and each week Robinson has exactly L weeks.

2 Optimality

For the moment, ignore markets. How well off can Robinson be? By foregoing leisure to make coconuts, what is the maximal level of utility he can achieve? Any solution to the following problem is an *optimal allocation of resources*.

$$\begin{aligned} \max \quad & u(c,l) \\ \text{s.t.} \quad & c = f(x) \\ & x + l \leq L \end{aligned}$$

The solution is described graphically below. It is characterized by a tangency between the production function and the consumer isoquants. The analytical characterization is in terms of marginal productivities and marginal rates of substitutions. The rate at which Rob is willing to give up leisure for coconuts is the MRS:

$$MRS_{lc} = \frac{MU_l}{MU_c} = \left(\frac{c}{l}\right)^{\sigma}$$

The marginal cost of production is $f'(x)$. So at an optimum,

$$\begin{aligned} f'(x) &= MRS_{lc}(c,l) \\ x + l &= L \end{aligned}$$

Where do these conditions come from? Substitute the constraints and maximize over the remaining free variable:

$$\max\{u\big(f(x), L-x\big) : x \geq 0\}.$$

In our problem,

$$l = \frac{L}{1 + a^{(\sigma-1)/\sigma}}, \qquad c = \frac{a^{1/\sigma}L}{1 + a^{(\sigma-1)/\sigma}}.$$

Figure 1: Optimality

3 Equilibrium

Although Robinson has no need for a market, this is an easy model in which to see how markets decentralize decisionmaking. Robinson has two roles on his island. He is an entrepreneur who owns a coconut firm, and a consumer who eats coconuts. In a market, the consumer will sell his labor to the firm. He will also own the firm, and so be entitled to it's profits. With this income he will buy coconuts. The market sets prices for coconuts and labor at which these exchanges will take place.

A competitive equilibrium involves identifying a demand curve for Robinson and a supply curve for the firm, and finding the price where they intersect. Alternatively, we can state a definition directly in terms of prices, utility maximization and profit maximization.

Let w denote the wage rate, and p (for the moment) the price of coconuts. Robinson the producer maximizes profits:

$$\begin{array}{rl} \max & pq - wx \\ \text{s.t.} & q \leq f(x) \end{array}$$

Rob the consumer maximizes utility:

$$\begin{array}{rl} \max & u(c,l) \\ \text{s.t.} & pc \leq w(L-l)+\pi \end{array}$$

Notice the budget constraint says directly that the value of coconut consumption must not exceed the value of labor income, $L-l$, plus profits. Rewrite this as

$$pc + wl \leq pL + \pi\,,$$

the value of coconut consumption plus leisure consumption does not exceed the value of his endowment plus profits.

Profits are homogeneous of degree 1 in prices, and so the budget constraint remains unchanged when prices are multiplied by a positive constant. Thus absolute price levels are not determined by market equilibrium. Only relative prices are determined. This is not deep. It says that we can measure prices in dollars, or in coconuts, or in hours, or in shekels; it does not matter. The choice of 'units of account' has no effect on the equilibrium. It will be convenient here to measure prices in terms of coconuts; that is, to take $p=1$. Coconuts are said to be the *numeraire good*.

Definition 1. *A* competitive equilibrium *is a vector (w,c,l,x,q) where p is a price of coconuts in terms of leisure, (c,l) is a consumption plan and (x,q) is a production plan, such that*

1. *$q=f(x)$ and $x\geq 0$; (technological feasibility)*
2. *$c,l\geq 0$; (the consumption plan (c,l) is in the consumption set)*
3. *(c,l) maximizes Robinson's utility on the budget set defined by the budget inequality $pc+l\leq pL+\pi$; (utility maximization)*
4. *(x,q) maximizes profits $\pi=pq-x$ on the set of technologically feasible production plans; (profit maximization)*
5. *$q=c$ and $l+x=L$. (feasibility)*

Computing the equilibrium is straightforward in our example. Consider profit maximization first. It must be true that $w\geq a$. If $w<a$, then profit maximization requires that entrepreneur Robinson hires an infinite amount of labor, and so the labor market cannot clear. It must also be true that $w\leq a$. If $w>a$, then profit maximization requires that Robinson's firm hires no one, and output

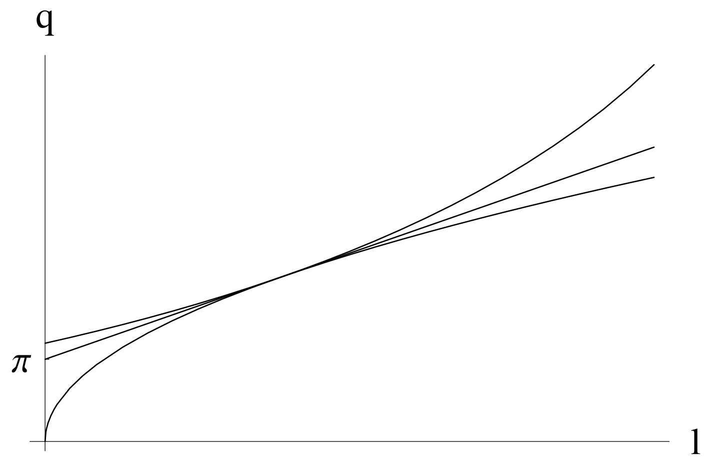

Figure 2: Equilibrium in General

will be 0. Market clearing requires that coconut consumption will be 0. Utility maximization requires that $(c/l)^{\sigma} = w$, and for no $\sigma > 0$ can $0^{\sigma} = w > a$. Thus in equilibrium, $w = a$.

The only fact about consumers we used in determining the equilibrium price was that, when the price of a good is 0, consumers will demand positive amounts. It rules out corners. Beyond this, prices are entirely determined by technology. This is property of *linear technologies*. In general, supply and demand interact. Actually they do here too, but the picture is trivial. Just require that demand touch neither axis, and then supply entirely determines equilibrium prices. On the other hand, at the equilibrium price, Robinson's firm is willing to produce any number of coconuts. Equilibrium quantity is thus determined entirely by supply.

The general case is illustrated in Figure 2. Isoprofit lines are upward-sloping straight lines parallel to the straight line in the graph. Profit maximization requires a production plan at the point of tangency of an isoprofit line and the technology set. The intersection of the isoprofit line with the vertical axis is the firm's profits measured in coconuts. (The intersection with the horizontal axis measures, from the left origin, profits measured in time units.) This isoprofit line thus also describes the consumer's budget set. Utility maximization requires a consumption plan at the tangency of the budget line and an indifference curve. Feasibility requires that the consumption and production plans are represented by the same point in the graph.

If the wage rate is w, the firm makes zero profits. Robinson's budget set is thus coincident with the production possibility set. The optimization problem Robinson the consumer faces is exactly

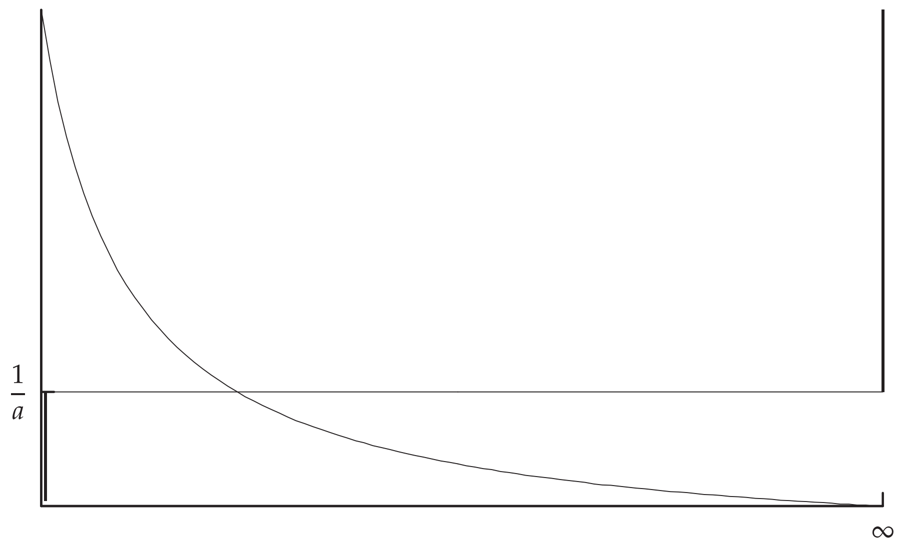

Figure 3: Supply and Demand with Linear Technologies

the "social" optimization problem. Hence the equilibrium quantities are the socially optimal quantities computed above. This connection between equilibrium and optimality is important. It is an instance of the **First Fundamental Theorem of Welfare Economics**.

4 Comparative Statics

How does equilibrium change if we change the following parameters: a, σ, and L? First,

$$\frac{\partial w}{\partial \sigma} = \frac{\partial w}{\partial L} = 0 \quad \text{and} \quad \frac{\partial w}{\partial a} = 1.$$

To get the change in consumption and production plans, we can differentiate the explicit formulae we have. Often, however, we define are equilibrium variables only implicitly, and so we will need other strategies.

5 A Taxation Problem

A typical problem is to ask after the effect of some government policy on market outcomes. So suppose Robinson is also a state, as well as a consumer and firm. As a state, he implements a sales tax on coconuts, and tax revenues are handed by the government back to the consumer in a *lump sum* fashion. This means that the consumer does not think about the effect of his purchases on his income.

Suppose that the market price charged by the firm is 1, and the price paid by the consumer is $1+t$. Total tax revenues are T. The firm's profit function remains unchanged, and so again $w=a$ and $\pi=0$. The consumer's budget constraint is now

$$(1+t)c + wl = wL + T\,.$$

The first-order conditions require, among other things, that

$$\frac{w}{1+t} = \frac{MU_l}{MU_c} = \left(\frac{c}{l}\right)^{\sigma}.$$

Already we see that $MU_l/MU_c \neq f'(x) = a$, so equilibrium will not be efficient.

To find the equilibrium, observe first that the wage rate once again has to be a, in units of (untaxed) coconuts. Thus on the demand side,

$$\frac{a}{1+t} = \left(\frac{c}{l}\right)^{\sigma}$$

Solve for c demand in terms of l:

$$c_d = \left(\frac{a}{1+t}\right)^{\frac{1}{\sigma}} l\,.$$

On the other hand, $x = L - l$, so if the labor market clears,

$$c_s = a(L-l).$$

Since the coconut market clears, supply equals demand, and so

$$l = \frac{(1+t)^{\frac{1}{\sigma}} L}{(1+t)^{\frac{1}{\sigma}} + a^{\frac{1-\sigma}{\sigma}}}$$

and now it's easy to compute the remainder of the equilibrium. Compute tax revenues.

6 Fish

Suppose now that Robinson owns time, but can produce coconuts or fish. The marginal product of time in fish is constant, and equals b. To keep things simple, suppose that Robinson is interested in coconuts and fish, but has no interest in lying around; the marginal utility of leisure is (identically) 0.

What is the PPF? What allocation of labor between fish and coconut production is optimal, and how much of each food is produced? What are the competitive equilibrium prices?

7 Applications

Surprising as it may seem, one consumer models have proven useful in a variety of economic applications:

1. Macroeconomics
2. Economic Growth Models: Solow, Koopmans
3. Finance: All asset pricing models are single agent exchange models!