CS500 Homework Zero

Prof. Cris Moore

This is a sort of pretest to see how well-versed you are in the math you'll need for this course. It will not be graded! But it is intended as a taste of the kind of math we'll need. As always, collaboration is allowed if you get stuck, but please use this to assess your readiness for the course.

1. How many subsets does a set of size n have? You have two choices for each element, either to include it or not, so the number of subsets is 2^n.

2. If I add two n-digit numbers, how many digits does their sum have? At most $n+1$. What about their product? At most $2n$.

3. Roughly speaking, what is $\log_2 1{,}000{,}000{,}000$? About 30, since $2^{10} \approx 1,000$.

4. True or false: $a^{\log b} = b^{\log a}$. True: both are $\mathrm{e}^{\log a \log b}$.

5. Name a function f such that $f(x) = 2f(x-1)$. $f(x) = 2^x$

6. Name a function f such that $f(x) = f(x/2) + 1$. $f(x) = \log_2 x$

7. Name a function f such that $f(x) = 2f(x/4)$. $f(x) = \sqrt{x}$

8. There are 10 popular vacation spots, and three people each choose one randomly without talking with each other. What is the probability that a) they all choose the same place? 1/100 b) two of them choose the same place and the third is elsewhere? 27/100 c) they choose three different places? 72/100 (Note that these probabilities should sum to 1, and they do.)

9. Approximately, what is $\sum_{i=1}^{i} 1/i = 1 + 1/2 + 1/3 + \cdots + 1/n$? This the harmonic series; it converges to $\ln n$ plus a constant.

10. Give a good approximation to e^x when x is close to zero. (Don't just say 1, please.) The Taylor series: $1 + x + x^2/2 + \cdots$

11. True or false: if $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$. Give a proof or a counterexample. False: if $f = 2\log_2 n$ and $g = \log_2 n$, then $2^{f(n)} = n^2$ but $2^{g(n)} = n$.

12. Suppose I have a balanced binary tree of depth d. How many leaves does it have? 2^d (assuming that a tree of depth zero consists just of its root) How many internal nodes does it have? $2^d - 1$ When d is large, what fraction of the nodes are leaves? 1/2

13. NoC Problem 1.2 (the Pigeonhole Principle) The pigeons are the vertices, and the holes are the degrees. The degree ranges from 0 (connected to nobody) to $n-1$ (connected to everyone else), and there are n vertices. It sounds like there are enough pigeons for all the holes—but if there is a vertex of degree zero, then there can't be one of degree $n-1$, and vice versa. Thus there are only $n-1$ holes, ranging from 0 to $n-2$ or from 1 to $n-1$. Two pigeons must be in the same hole, so two vertices must have the same degree.

14. NoC Problem 1.7 (Hamiltonian grids) Color the vertices of the grid black and white like a checkerboard. The color alternates each time we take a step, so if there is a Hamiltonian cycle, the total number of vertices must be even. Thus at least one of m and n must be even. Conversely, if there are both even, there is a Hamiltonian cycle which weaves back and forth across the grid and then back along one edge.

15. NoC Exercise 2.4 (doubling the speed of the computer). In four years, our computer will be four times as fast as it is now. In each case, we want to find the value of n' such that $t(n') = 4t(n)$. If $t(n) = \log_2 n$, this gives $n' = n^4$. If $t(n) = \sqrt{n}$, then $n' = 16n$. If $t(n) = n$, then $n' = 4n$. If $t(n) = n^2$, then $n' = 2n$. If $t(n) = 2^n$, then $n' = n + 2$. If $t(n) = 4^n$, then $n' = n + 1$.