Approximating the Permanent via Nonabelian Determinants

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Determinant and Permanent

- Let A be a $n \times n$ matrix with entries in $\{0,1\}$

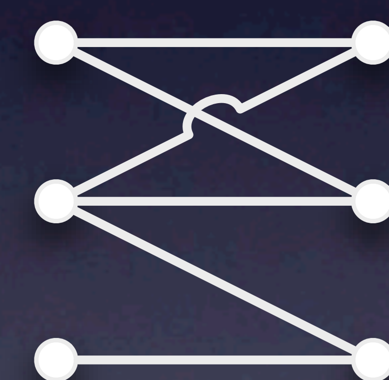
$$\det A = \sum_{\pi \in S_n} (-1)^{\pi} \prod_i A_{i,\pi i} \qquad \operatorname{perm} A = \sum_{\pi \in S_n} \prod_i A_{i,\pi i}$$

- $\det A$: geometric meaning, basis-independent, homomorphic, easy
- $\operatorname{perm} A$: combinatorial, basis-dependent, hard

A Combinatorial Picture

- Treat A as the adjacency matrix of a bipartite graph *G*:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



- Then $\mathrm{perm}\, A$ = # of *perfect matchings* in *G*

Complexity

- 0-1 PERMANENT is #P-complete [Valiant '79]
- However, it can be approximated using a rapidly-mixing Markov chain that samples random perfect matchings [Jerrum, Sinclair, Vigoda '04]
- Is there another approach, which is purely *algebraic?*

The Godsil-Gutman Estimator

- Let A be a $n \times n$ matrix with entries in $\{0, 1\}$
- Let $M_{ij} = \gamma_{ij} A_{ij}$ for uniformly random $\gamma_{ij} \in \{\pm 1\}$
- Then

$$\mathbb{E}\left[(\det M)^2\right] = \text{perm } A$$

What?!

- It's true and, better yet, simple:

$$\mathbb{E}\left[(\det M)^2\right] = \mathbb{E}\sum_{\pi,\sigma}(-1)^{\pi\sigma}\prod_i M_{i,\pi i}M_{i,\sigma i}$$

- only contributing terms appear when $\pi = \sigma$ and all $M_{i,\pi i}$ are nonzero:

$$= \sum_{\pi}(-1)^{\pi\pi}\left(\prod_i M_{i,\pi i}\right)^2 = \sum_{\pi}\prod_i A_{i,\pi i}$$

But What's the Variance?

- For matrix A, define $X = (\det M)^2$. Then

$$\mathbb{E}[X] = \text{perm } A$$

- How many samples do we need? Chebyshev:

$$t \approx \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2}$$

- Can we control this ratio?

Sadly...

- [KKLLL '93]: $\dfrac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} = 3^{n/2} \cdot \mathrm{poly}(n)$

- But, they show that if γ_{ij} is uniform on the unit circle, or even just the cube roots of 1,

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} = 2^{n/2} \cdot \mathrm{poly}(n)$$

- Can we do better?

Higher-dimensional algebras?

- Barvinok: what if the γ_{ij} are quaternions? Or higher-dimensional objects?

- [Chien, Rasmussen, Sinclair '03] In the *Clifford algebra* of dimension *d*:

$$\left(1 + O\left(\frac{1}{d}\right)\right)^{n/2}$$

- In particular, for the quaternions, $\left(\frac{3}{2}\right)^{n/2}$

How to define the determinant?

- In the nonabelian case, order matters
- The conventional determinant takes each product from top to bottom: no efficient algorithm is known!
- [Barvinok '00]: *symmetrize* each term:

$$\operatorname{sdet} M = \frac{1}{n!} \sum_{\pi,\alpha \in S_n} (-1)^\pi \prod_i M_{\alpha i, \pi \alpha i}$$

- $O(n^d)$ algorithm in a *d*-dimensional algebra

Algebraic Estimators

- Define $M_{ij} = \rho_{ij} A_{ij}$, where ρ_{ij} is drawn from some distribution on a nonabelian algebra A
- The Haar measure on unitary $d \times d$ matrices
- The Gaussian measure on $d \times d$ matrices (independent entries)
- $\det M$ takes values in A
- Define $X = \| \det M \|^2$ or $X = |\mathrm{tr}\, \det M|^2$

Our Results

- Two estimators, Haar or Gaussian measure:

$$X = |\mathbf{tr}\,\det M|^2 \qquad X_s = |\mathbf{tr}\,\mathrm{sdet}\ M|^2$$

- We establish the following ratios:

$$\frac{\mathbb{E}\left[X^2\right]}{\mathbb{E}\left[X\right]^2} = \left(1 + O\Big(\frac{1}{d}\Big)\right)^n \qquad \frac{\mathbb{E}[X_s^2]}{\mathbb{E}[X_s]^2} = \Omega\left(\frac{2^n}{n^d}\right)$$

- Ratios differ by $O(d^4)$ for $X = \|\det M\|^2$

Expansion

- Let's expand the symmetric estimator:

$$X_s = \sum_{\kappa,\lambda \vdash A} (-1)^{\kappa\lambda}\, \mathbb{E}_{\alpha,\beta} \left(\mathbf{tr} \prod_i \rho_{\alpha i,\kappa\alpha i} \right) \left(\mathbf{tr} \prod_i \rho^*_{\beta i,\lambda\beta i} \right)$$

where $\kappa \vdash A \Leftrightarrow \forall i : A_{i,\kappa i} = 1$

- Again, to contribute to the expectation, $\kappa = \lambda$ so each ρ_{ij} appears twice or not at all

Permuted Products

$$\begin{aligned}\mathbb{E}_{\{\rho_{ij}\}}[X_s] &= \sum_{\kappa \vdash A} \mathbb{E}_{\{\sigma_i\}}\, \mathbb{E}_{\alpha,\beta} \left(\mathbf{tr} \prod_i \sigma_{\alpha i} \right) \left(\mathbf{tr} \prod_i \sigma^*_{\beta i} \right) \\ &= a_d \cdot \operatorname{perm} A\end{aligned}$$

- where

$$a_d = \mathbb{E}_{\{\sigma_i\}}\, \mathbb{E}_{\alpha,\beta} \left(\mathbf{tr} \prod_i \sigma_{\alpha i} \right) \left(\mathbf{tr} \prod_i \sigma^*_{\beta i} \right)$$

- Covariance between a product of the same matrices σ_i taken in two different orders

The Cupcap Cometh

- What, for instance, is

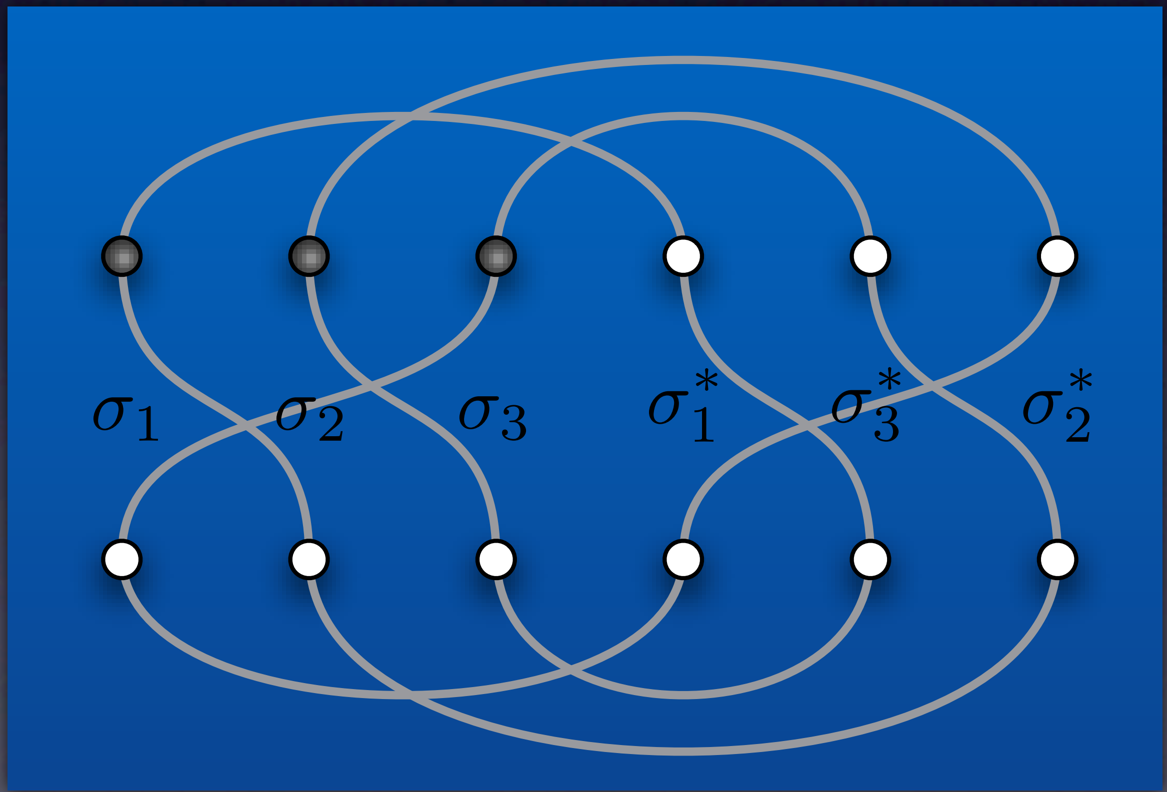
$$\mathbb{E}_{\sigma_1,\sigma_2,\sigma_3}\left(\mathbf{tr}\,\sigma_1\sigma_2\sigma_3\right)\left(\mathbf{tr}\,\sigma_1\sigma_3\sigma_2\right)^* \text{ ?}$$

- Both Haar and Gauss: $\mathbb{E}[\sigma^i_j(\sigma^k_\ell)^*] = \frac{1}{d}\delta^{ik}\delta_{j\ell}$
- Diagrammatically:

$$\mathbb{E}[\sigma \otimes \sigma^*] = \frac{1}{d}$$

Diagrams and Loops

- Form product by "weaving" matrices, and connect with cupcaps
- Tracing gives a factor of *d* for each loop



$$\mathbb{E}_{\sigma_1,\sigma_2,\sigma_3}\left(\mathbf{tr}\,\sigma_1\sigma_2\sigma_3\right)\left(\mathbf{tr}\,\sigma_1\sigma_3\sigma_2\right)^* = \frac{1}{d^2}$$

A Generating Function

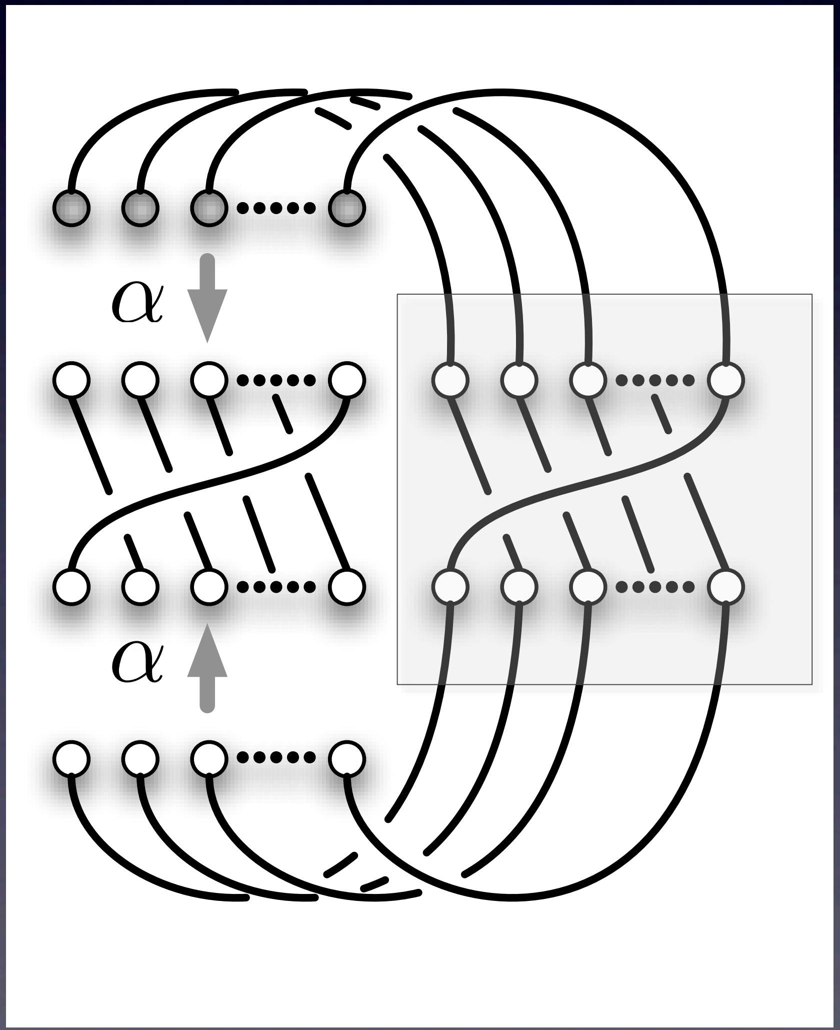
- Averaging over all permutations gives

$$a_d = \frac{1}{d^n} \mathbb{E}_\alpha \, d^{c([\alpha,r])}$$

- where $c(\pi)$ is the number of cycles and $r = (1\,2\,\cdots\,n)$ is a rotation

- Using Fourier analysis on S_n,

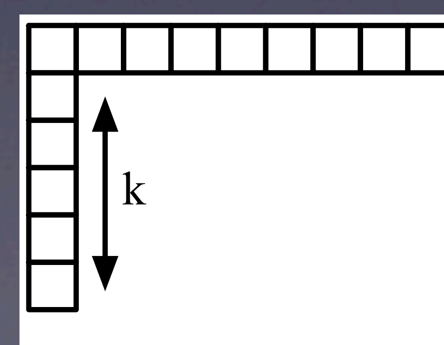
$$a_d = \frac{1}{d^n} \left(\binom{n+d}{n+1} - \binom{d}{n+1} \right)$$



Fourier Analysis

- First we write $a_d = \frac{1}{d^n}\mathbb{E}_{r,r'}d^{c(rr')}$
- Then a_d is an inner product $\left\langle d^{c(\cdot)}, P * P \right\rangle$

  where P is the uniform distribution on n-cycles
- $d^{c(\cdot)}$ is the trace of a *combinatorial representation:* action of S_n on strings of length n over $\{1,\ldots,d\}$
- Fourier coefficients are *Kostka numbers*
- P is supported on "hooks"



The k-hook

The Second Moment: Unsymmetrized

$$\mathbb{E}[X^2] = \sum_{C \vdash A} \sum_{(\kappa,\lambda,\mu,\nu) \vdash C} \mathbb{E}_{\{\rho_{ij}\}} \left(\mathbf{tr} \prod_i \rho_{i,\kappa i} \right) \left(\mathbf{tr} \prod_i \rho_{i,\lambda i} \right) \left(\mathbf{tr} \prod_i \rho^*_{i,\mu i} \right) \left(\mathbf{tr} \prod_i \rho^*_{i,\nu i} \right)$$

- The tuple $(\kappa, \lambda, \mu, \nu)$ determines a *double cycle cover*: each ρ_{ij} appears 2 or 4 times

$$(\operatorname{perm} A)^2 = \sum_{M_1 \vdash A} \sum_{M_2 \vdash A} = \sum_C 2^{|C|}$$

- $M_1 \oplus M_2 = C$, a cycle cover

Fourth-Order Operator

- In the Gaussian measure,

$$\mathbb{E}_\sigma\left[\sigma\otimes\sigma\otimes\sigma^*\otimes\sigma^*\right] = \frac{1}{d^2}\Big($$

$+$

$\Big)$

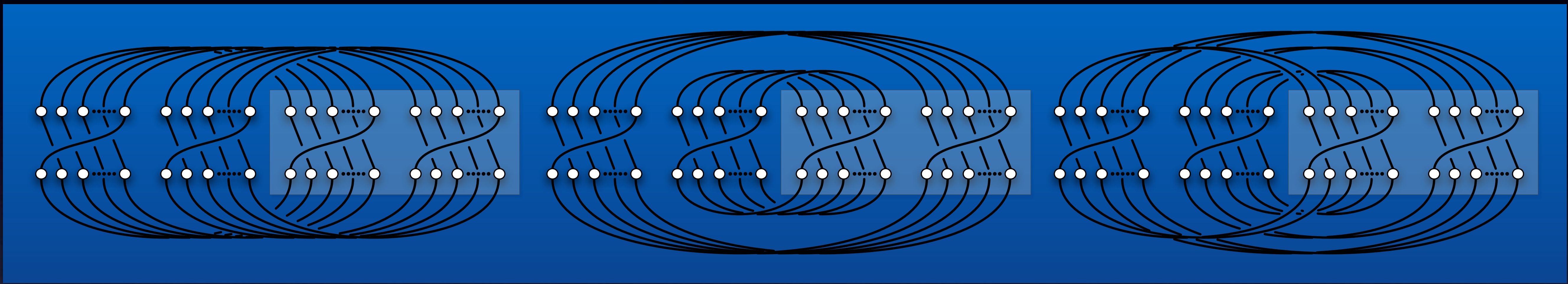
- In the Haar measure,

$$\mathbb{E}_\sigma\left[\sigma\otimes\sigma\otimes\sigma^*\otimes\sigma^*\right] \preceq \left(1+O\left(\frac{1}{d}\right)\right)\frac{1}{d^2}\Big($$

$+$

$\Big)$

The Second Moment: Unsymmetrized



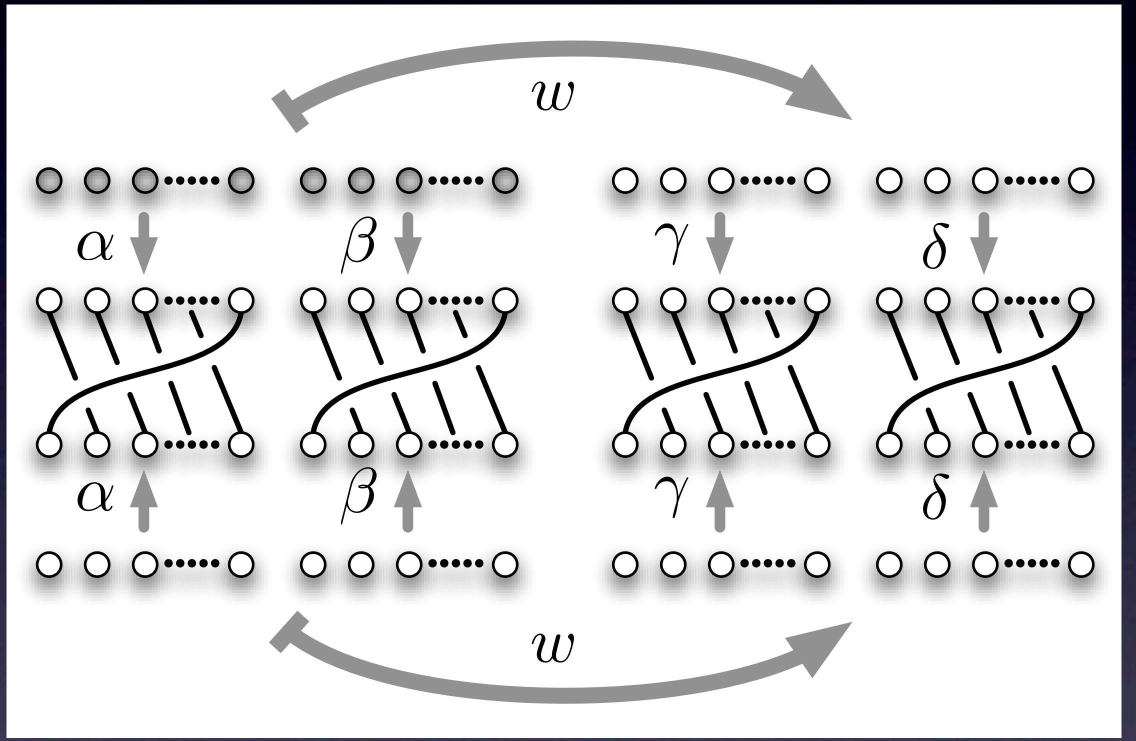
- Sum over all mixtures of (1,3),(2,4) and (1,4),(2,3) matchings
- Mixed matchings have fewer loops

$$2\sum_{i=0,2,4,\ldots}^{n}\binom{n}{i}d^{-i}=\left(\left(1+\frac{1}{d}\right)^{n}+\left(1-\frac{1}{d}\right)^{n}\right)$$

Second Moment, Symmetrized

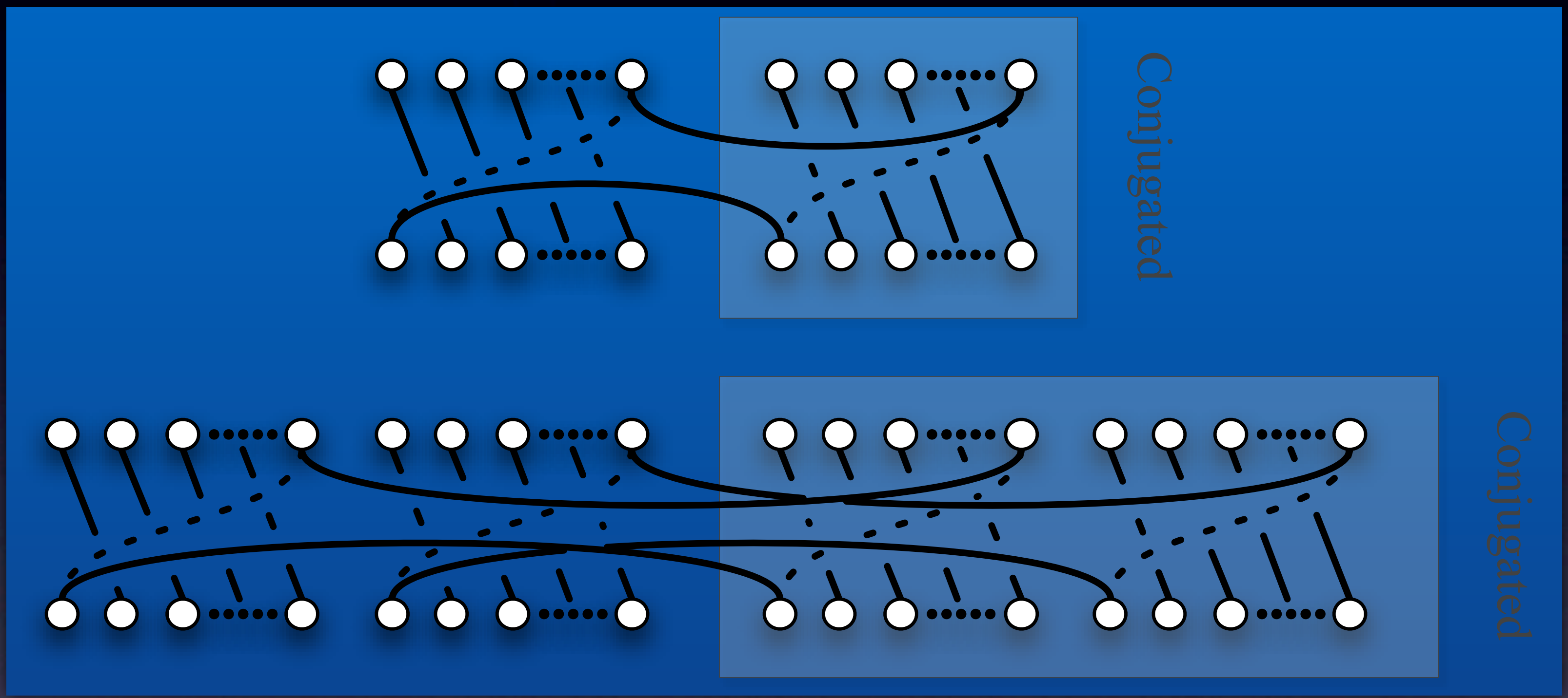
- Now the second moment can be bounded in terms of

$$\mathbb{E}_{\pi,\sigma\in S_{2n}} d^{c(\pi^{-1}(r,r)\pi\,\sigma^{-1}(r,r)\sigma)}$$



- Complicated distribution of pairs of n-cycles in S_{2n}
- Bound in terms of uniform distribution
- Littlewood-Richardson rule: restrictions of irreps of S_{2n} to the Young subgroup $S_n \times S_n$

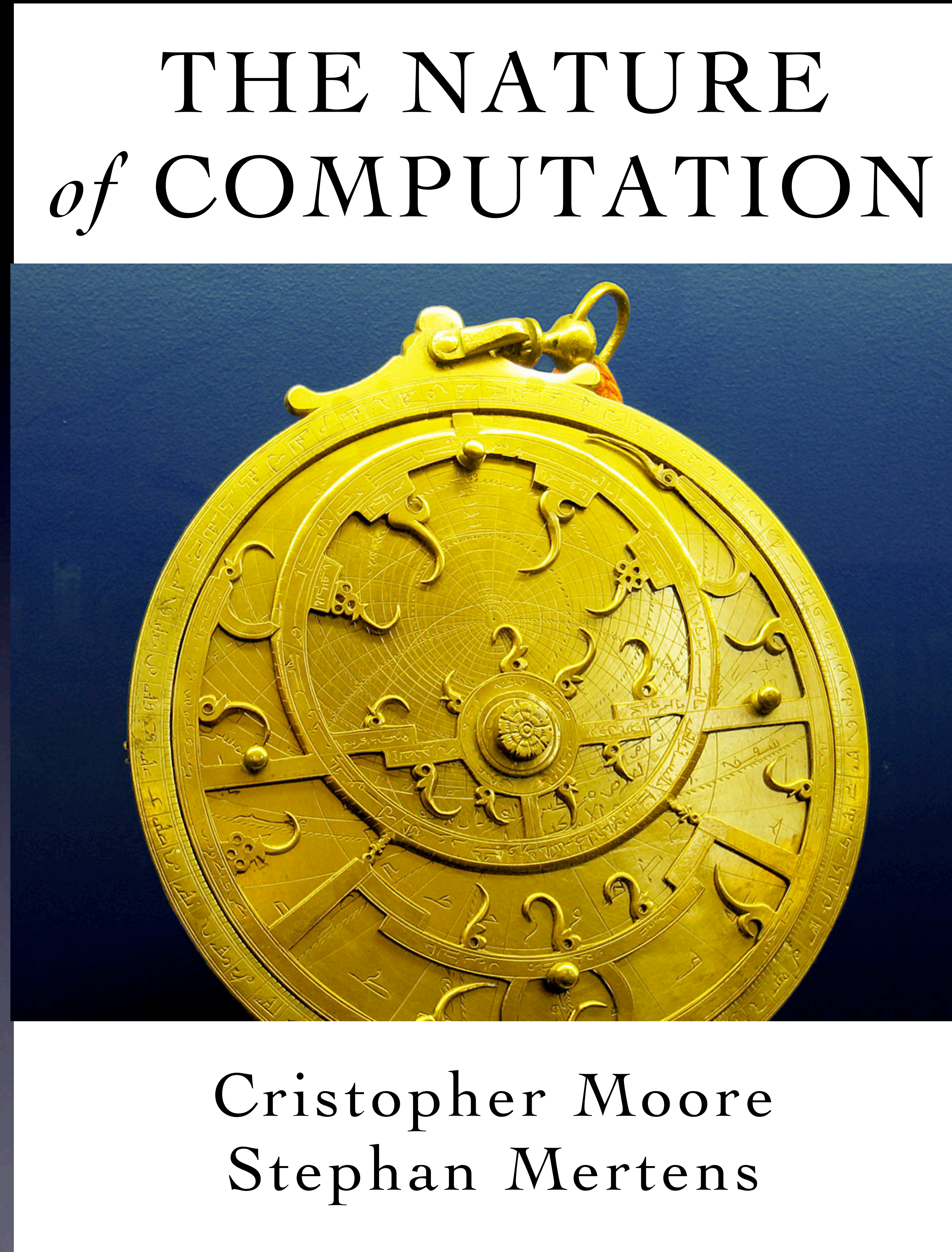
The Frobenius Estimators



- $\|M\|^2 = \mathbf{tr}\, M M^\dagger$ instead of $(\mathbf{tr}\, M)(\mathbf{tr}\, M^*)$
- Just rewire 2 or 4 edges

Shameless Plug

Oxford University Press, 2010



Acknowledgements

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