The Planted Matching Problem

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Joint work with Mehrdad Moharrami (Michigan) and Jiaming Xu (Duke)

December 8, 2019

Planted problems: good solution + noise

Constraint satisfaction, optimization

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Generative models, Bayesian inference

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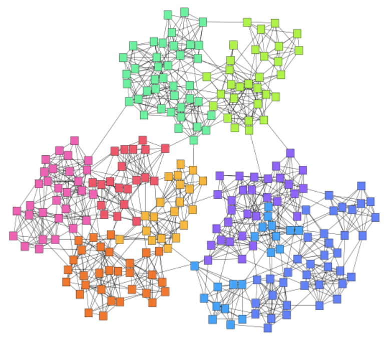

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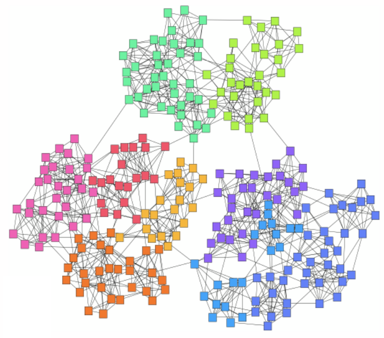

Information-theoretic (a.k.a. statistical) and computational barriers

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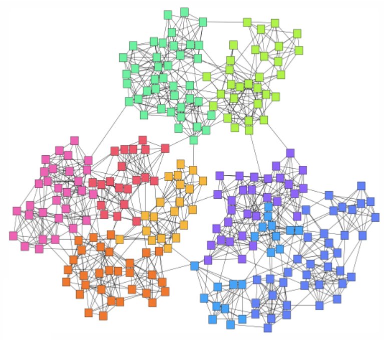
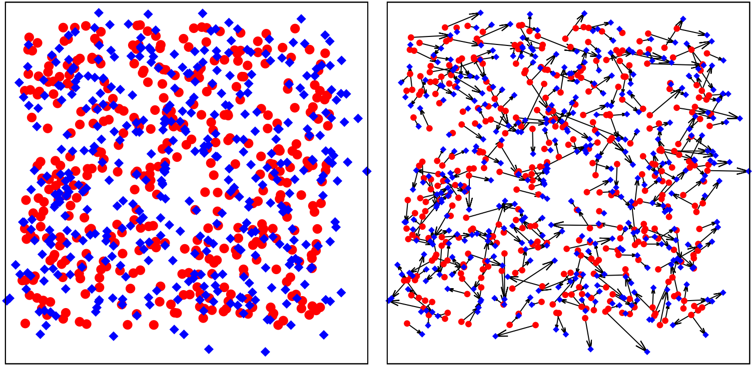


Information-theoretic (a.k.a. statistical) and computational barriers
Statistical physics $\Rightarrow$ conjectures, proofs, and algorithms

Planted matchings: particle tracking

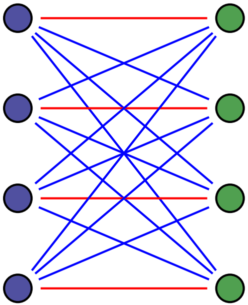
Tracking particles advected by turbulent fluid flow



[Chertkov-Kroc-Krzakala-Vergassola-Zdeborová PNAS'10]

Goal: recover the underlying true one-to-one mapping of the particles
Flocks of birds, swimming microbes, ...

The planted assignment model



- A complete bipartite graph
- A hidden perfect matching M
- Edge weight

$$W_{ij} \overset{\text{ind.}}{\sim} \begin{cases} P & e \in M \\ Q & e \notin M \end{cases}$$

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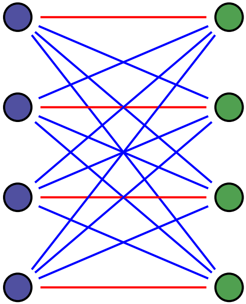

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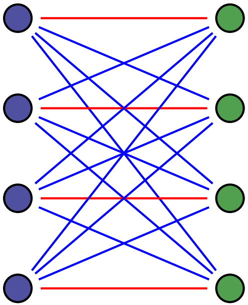

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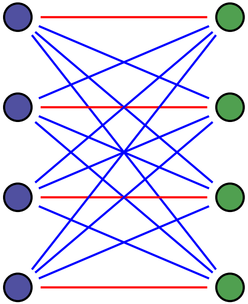

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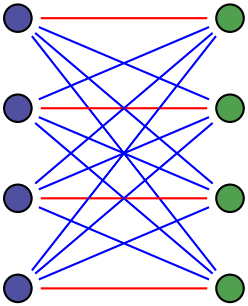

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- A phase transition in λ, and exact results

Main result: phase transition at $\lambda = 4$

Theorem (Moharrami-M.-Xu '19)

$$\textbf{overlap:} \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\left|\widehat{M} \cap M\right|\right] = \begin{cases} 1 & \text{if } \lambda \geq 4 \\ \alpha(\lambda) & \text{if } 0 < \lambda < 4 \end{cases}$$

where $\alpha(\lambda) = 1 - 2\int_0^\infty (1 - F(x))\,(1 - G(x))\,V(x)W(x)\,dx < 1\,,$

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and F, G, V, W is the unique solution to a system of ODEs:

$$\begin{aligned} \dot{F} &= (1-F)(1-G)V \\ \dot{G} &= -(1-F)(1-G)W \\ \dot{V} &= \lambda(V - F) \\ \dot{W} &= -\lambda(W - G) \end{aligned}$$

Boundary conditions: $F(x), V(x), G(-x), W(-x) \to \begin{cases} 1 & x \to +\infty \\ 0 & x \to -\infty \end{cases}$

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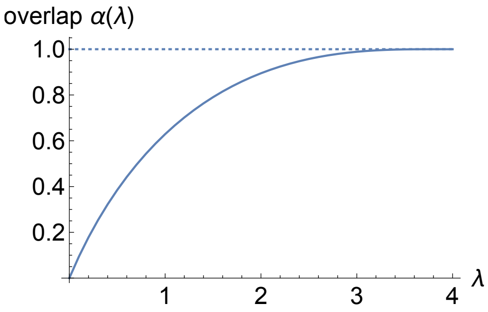

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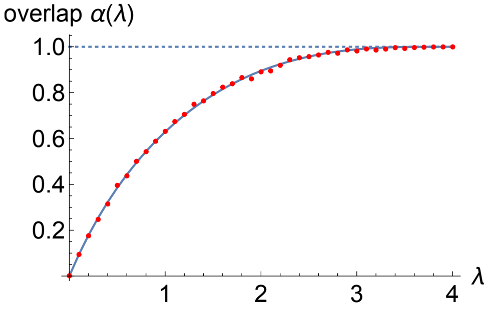
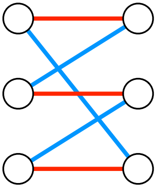


When $\lambda \geq 4$: count augmenting cycles



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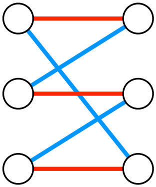

- Probability that M' has lower total weight than M is

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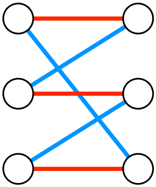

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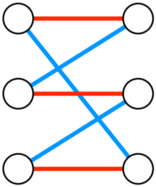

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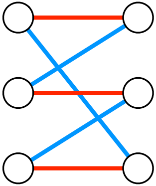

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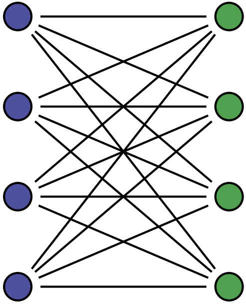
$\Rightarrow$ Sum over ℓ: total probability a planted edge is in augmenting cycle is $o(1)$ if $\lambda \geq 4$

Warmup: the (un-planted) random assignment problem



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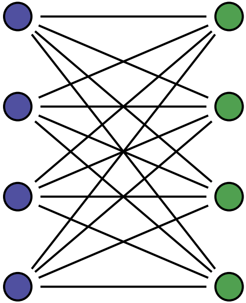

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- [Walkup '79, Mézard-Parisi '87, Aldous '92, Steele '97, Aldous '01, ...]

$$\lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}\left[\min_{\pi} \sum_{i=1}^{n} W_{i\pi(i)}\right] = \frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$$

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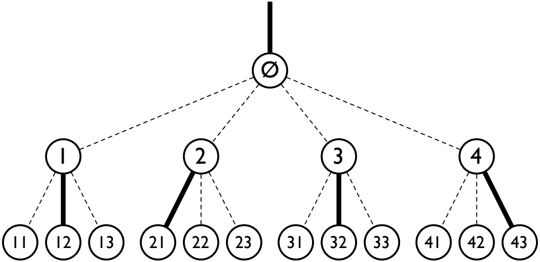

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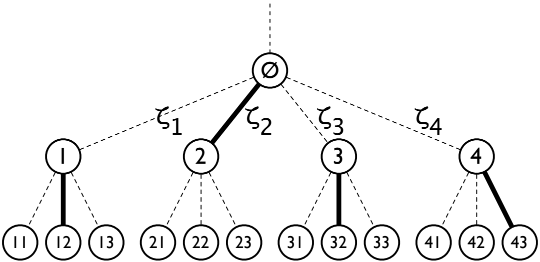

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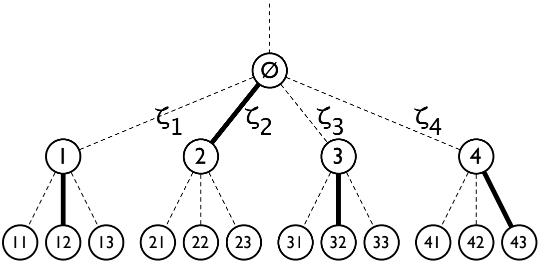

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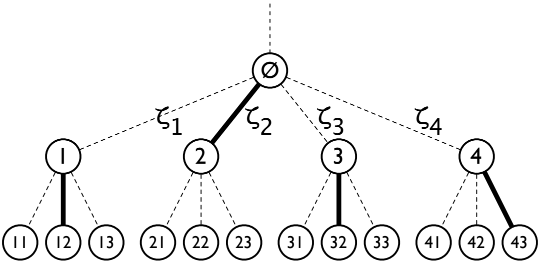

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arrivals $\zeta_1, \zeta_2, \ldots$ of a Poisson process with rate 1

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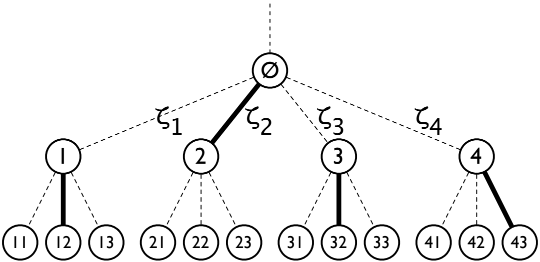

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$$X \stackrel{d}{=} \min_{i \geq 1} \{\zeta_i - X_i\}$$

From distributional to differential equations

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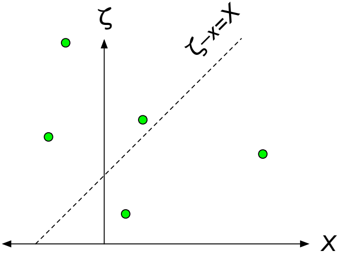
Generate pairs (ζ, x): two-dimensional Poisson process with density $f(x)$



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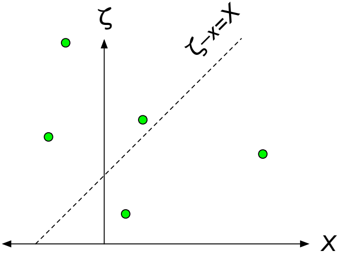

Define the cdf $\bar{F}(x) = 1 - F(x) = \mathbb{P}[X > x] = \mathbb{P}[\forall i : \zeta_i - x > X_i]$

$$\bar{F}(x) = \exp\left(-\int_{-x}^{\infty} \bar{F}(t)\,dt\right) \quad \Rightarrow \quad \frac{dF(x)}{dx} = F(x)F(-x)$$

From distributional to differential equations, cont'd

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From distributional to differential equations, cont'd

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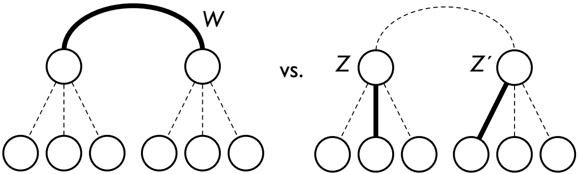

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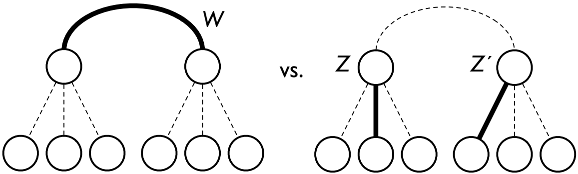
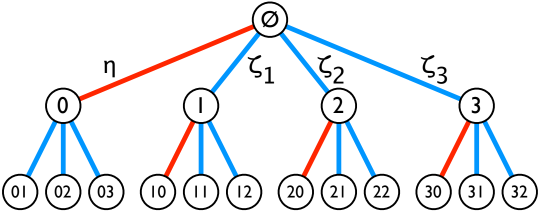


Contribution of a single edge: $\mathbb{E}\left[W\,\mathbf{1}[W < X + X']\right]$

$$= \frac{1}{4}\,\mathrm{Var}[X+X'] = \frac{1}{2}\,\mathrm{Var}[X] = \frac{\pi^2}{6}$$

Now with planted edges

Partner in planted matching is either parent or child 0, other children sorted $1, 2, 3, \ldots$



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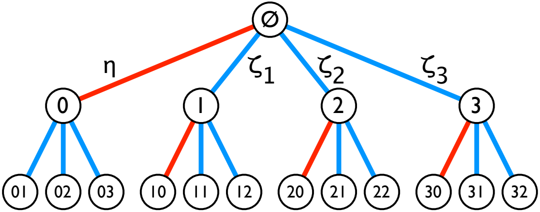

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Recursion:

$$X_\varnothing = \min \left\{ W_{\varnothing,0} - X_0, \; \min_{i \geq 1} \{ W_{\varnothing,i} - X_i \} \right\}$$

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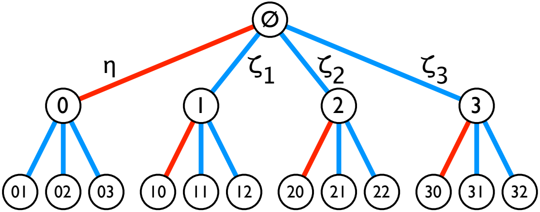

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$$Y \overset{d}{=} \min \{\eta - Z, Z'\}$$

$$Z \overset{d}{=} \min_i \{\zeta_i - Y_i\}$$

From distributional to differential equations, redux

$$\boxed{\begin{aligned} Y &\stackrel{d}{=} \min\left\{\eta - Z, Z'\right\} \\ Z &\stackrel{d}{=} \min\left\{\zeta_i - Y_i\right\}_{i=1}^{\infty} \end{aligned}}$$

where $\eta \sim \mathrm{Exp}(\lambda)$ and ζ_i are Poisson arrivals

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$\dot{V}$ and $\dot{W}$ from $\eta \sim \mathrm{Exp}(\lambda)$, integration by parts

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where $\eta \sim \mathrm{Exp}(\lambda)$ and ζ_i are Poisson arrivals

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$$\boxed{\begin{aligned} \dot{F} &= (1-F)(1-G)V \\ \dot{G} &= -(1-F)(1-G)W \\ \dot{V} &= \lambda(V-F) \\ \dot{W} &= -\lambda(W-G) \end{aligned}}$$

$\dot{V}$ and $\dot{W}$ from $\eta \sim \mathrm{Exp}(\lambda)$, integration by parts

Boundary conditions: $F(+\infty) = V(+\infty) = 1$, $F(-\infty) = V(-\infty) = 0$

No solution for $\lambda \geq 4$

At least no sensible one. . .

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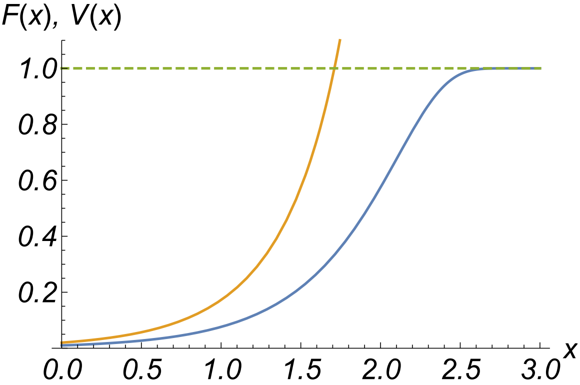
At least no sensible one...
Want $F(+\infty) = V(+\infty) = 1$. But...



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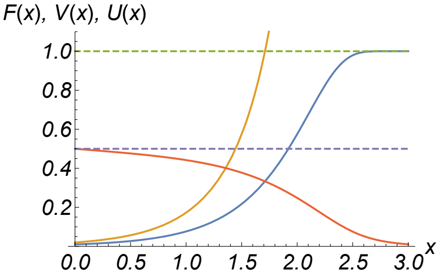

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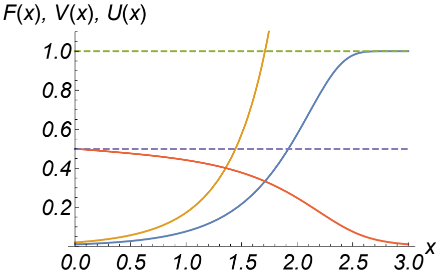

$$\dot{U} = -\lambda U(1-U) + (1-F)(1-G) \leq -\lambda U(1-U) + 1$$

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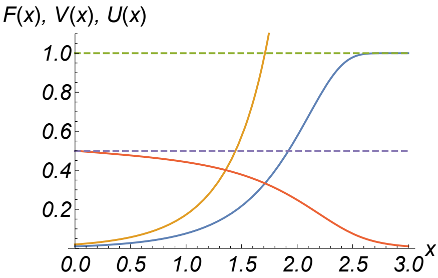

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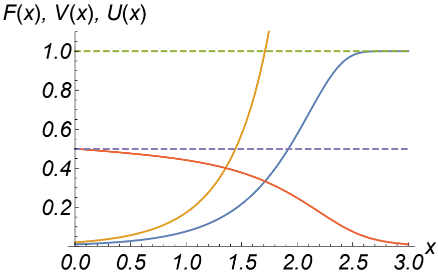

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No fixed distribution on finite values: cost of un-planted edge is $+\infty$

$\Rightarrow$ almost perfect recovery

A unique solution when $\lambda < 4$

$(F, G, V, W) \Longleftrightarrow (U, V, W)$: three-dimensional dynamical system

$$\boxed{\begin{aligned}
&\dot{U} = -\lambda U(1-U) + (1-UV)\,(1-(1-U)W) \\
&\dot{V} = \lambda V(1-U) \\
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&\text{Initial conditions: } U(0) = \frac{1}{2}, V(0) = W(0) = \epsilon
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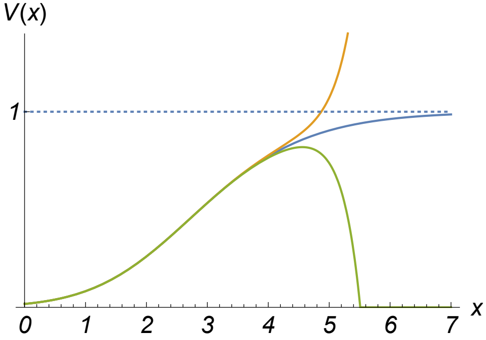
Lemma

If $\lambda < 4$ there is a unique $\epsilon_0 \in (0,1)$ such that

- *If $\epsilon \in [0, \epsilon_0)$, $U(x) \to +\infty$*
- *If $\epsilon = \epsilon_0$, $U(x) \to 1$ and $V(x) \to 1$*
- *If $\epsilon \in (\epsilon_0, 1]$, $V(x) \to +\infty$*

A unique solution when $\lambda < 4$

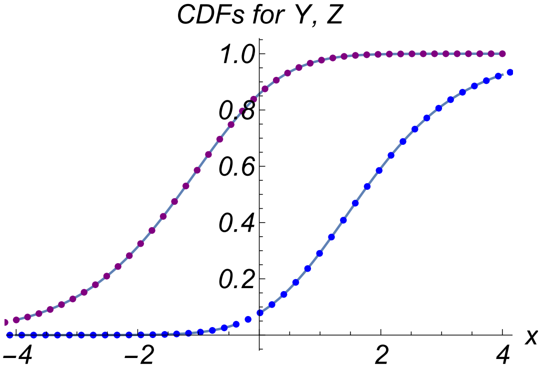
Geometric intepretation: $(U = 1, V = 1, W = 0)$ is a saddle point
If $V(0) = W(0) = \epsilon_0$ we approach the saddle along its unstable manifold



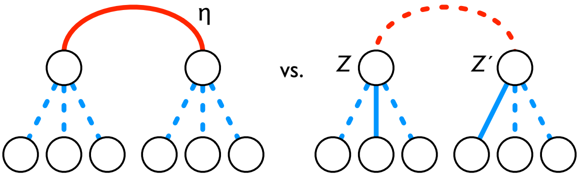
This gives cdfs $F, V \to 1$ of the unique fixed point distribution

A numerical experiment

$\lambda = 2.5$, population dynamics with $N = 10^6$



Finally, computing the overlap for $\lambda < 4$



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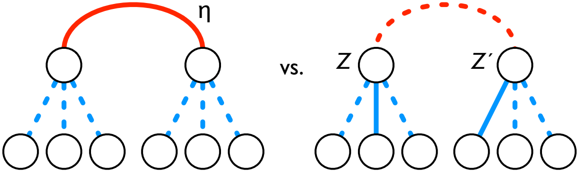


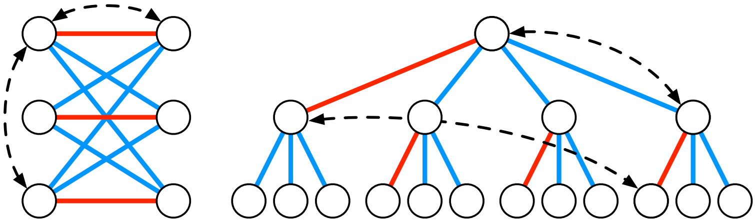
$$\alpha(\lambda) = \mathbb{P}\left[\eta < Z + Z'\right] = 1 - \mathbb{E}_\eta \int_{-\infty}^{+\infty} f(x) F(\eta - x)\, dx$$

$$= 1 - \int_{-\infty}^{+\infty} f(x)\, \mathbb{E}_\eta F(\eta - x)\, dx$$

$$= 1 - \int_{-\infty}^{+\infty} (1 - F(x))(1 - G(x))\, V(x) W(x)\, dx$$

$$= 1 - 2 \int_{0}^{+\infty} (1 - F(x))(1 - G(x))\, V(x) W(x)\, dx$$

Proving it: Local weak convergence (Aldous 1992, 2001)



- $\widehat{M}$ only depends on weights $\Rightarrow$ symmetry in the joint distribution of weights and matching
- Vertex-transitive involutions on $K_{n,n}$ or infinite tree T_∞
- A random matching is *involution invariant* if it has these symmetries
- We have constructed an involution invariant M_{opt} on T_∞ and computed its cost and overlap

Proving it: Local weak convergence (Aldous 1992, 2001)

- Easy: any invariant sequence $\{M_n\}$ of matchings on $K_{n,n}$ has a subsequence $\{M_{n_j}\}$ that converges to a (possibly random) invariant matching on T_∞

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- Together these imply $\lim_{n\to\infty} \mathrm{overlap}(\widehat{M}_n) = \mathrm{overlap}(\widehat{M}_\infty)$

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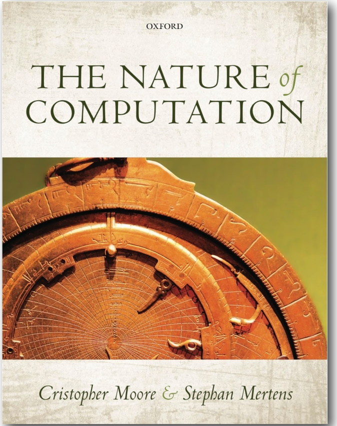
5. Spatial structure (particle tracking)?
6. Other planted structures: spanning trees, traveling salespeople?

Shameless Plug



www.nature-of-computation.org

To put it bluntly: this book rocks! It somehow manages to combine the fun of a popular book with the intellectual heft of a textbook.

Scott Aaronson, UT Austin

This is, simply put, the best-written book on the theory of computation I have ever read; one of the best-written mathematical books I have ever read, period.

Cosma Shalizi, Carnegie Mellon

A creative, insightful, and accessible introduction to the theory of computing, written with a keen eye toward the frontiers of the field and a vivid enthusiasm for the subject matter.

Jon Kleinberg, Cornell