The Hunt for a Quantum Algorithm for Graph Isomorphism

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The Hidden Subgroup Problem

- Given a function *f(x)*, find the *y* such that

$$f(x+y) = f(x)$$

for all *x*.

- Given a function *f* on a group *G*, find the subgroup *H* consisting of *h* such that

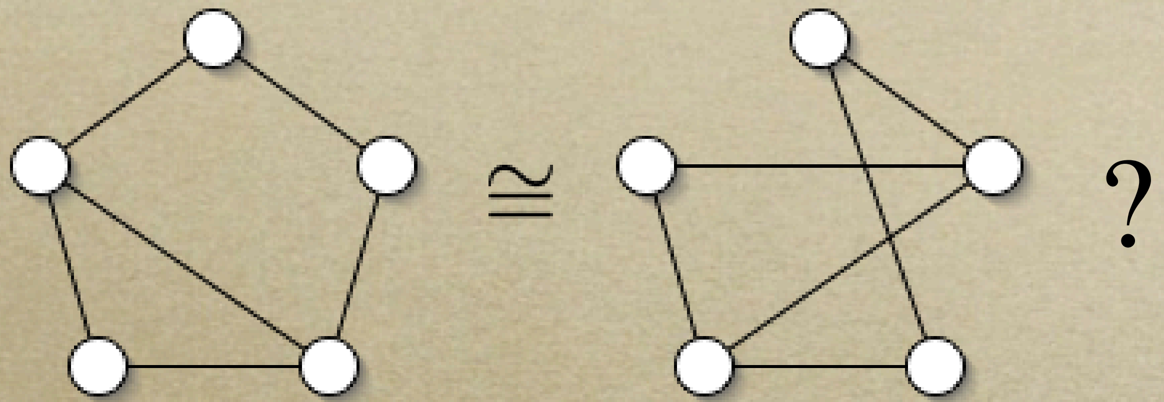
$$f(gh) = f(g)$$

for all *g*.

The Hidden Subgroup Problem

- This captures many quantum algorithms: indeed, most algorithms which give an exponential speedup.
  - $\mathbb{Z}_2^n$: *Simon's problem*
  - $\mathbb{Z}_n^*$: *factoring, discrete log (Shor)*
  - $\mathbb{Z}$: *Pell's equation (Hallgren)*
- What can the *non-Abelian* HSP do?

Graph Isomorphism



- Define a function *f* on S_{2n}. If both graphs are rigid, then either *f* is 1–1 and $H = \{1\}$, or *f* is 2–1 and $H = \{1, m\}$ for some involution *m* (of a particular type).

Standard Method: Coset States

- Start with a uniform superposition, $\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle$
- Measuring f gives a random coset of H:

$$|cH\rangle = \frac{1}{\sqrt{|H|}} \sum_{h \in H} |ch\rangle$$

or, if you prefer, a mixed state:

$$\rho = \frac{1}{|G|} \sum_{c \in G} |cH\rangle \langle cH|$$

The Fourier Transform

- We now perform a basis change. In $\mathbb{Z}_n$,

$$|k\rangle = \frac{1}{\sqrt{n}} \sum_x \mathrm{e}^{2\pi i k x/n}$$

and in $\mathbb{Z}_2^n$,

$$|k\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{k \cdot x}$$

- Why? Because these are homomorphisms from G to $\mathbb{C}$. These form a basis for $\mathbb{C}[G]$ with many properties (e.g. convolution)

Group Representations

- Homomorphisms from groups to matrices:

$$\sigma : G \to U(V)$$

- For instance, consider this three-dimensional representation of A_5.

- Any representation can be decomposed into a direct sum of *irreducible* representations.

Heartbreaking Beauty

- Given a “name” ρ and a row and column i, j,

$$|\sigma, i, j\rangle = \sqrt{\frac{d_\sigma}{|G|}} \sum_g \sigma(g)_{ij}$$

- Miraculously, these form an orthogonal basis for $\mathbb{C}[G]$:

$$\sum_{\sigma \in \widehat{G}} d_\sigma^2 = |G|$$

Group Actions

- Given a state in $\mathbb{C}[G]$ and a group element g, we can apply various *group actions:*

$$|x\rangle \to |xg\rangle \text{ or } \left|g^{-1}x\right\rangle \text{ or } \left|g^{-1}xg\right\rangle$$

- We can think of $\mathbb{C}[G]$ as a representation of G under any of these actions.
- Under (left or right) multiplication, the *regular representation* contains d_σ copies of each $\sigma \in \widehat{G}$.

Levels of Measurement

- For most group families, the QFT can be carried out efficiently, in polylog($|G|$) steps [Beals 1997; Høyer 1997; M., Rockmore, Russell 2004]
- *Weak sampling*: just the name σ
- *Strong sampling*: name, row and column σ, i, j in a basis of our choice (some bases may be much more informative than others)
- Intermediate: strong, but with a random basis

Fourier Sampling is Optimal

- The mixed state over (left) cosets

$$\rho = \frac{1}{|G|} \sum_{c \in G} |cH\rangle \langle cH|$$

is left G-invariant, hence block-diagonal.

- Measuring the irrep name (weak sampling) loses no coherence.
- Strong sampling is the only thing left to do!

Projections and Probabilities

- For each irrep σ, we have a projection operator

$$\pi_H^\sigma = \frac{1}{|H|} \sum_{h \in H} \sigma(h)$$

- The probability we observe σ is $\dfrac{d_\sigma |H| \operatorname{rk} \pi_H^\sigma}{|G|}$
- Compare with the *Plancherel distribution* $\dfrac{d_\sigma^2}{|G|}$ ($H = \{1\}$, the completely mixed state)

Weak Sampling Fails

- If $H = \{1, m\}$, we have $\quad (\chi_\sigma(g) = \operatorname{tr}\sigma(g))$

$$\operatorname{rk}\pi_H^\sigma = \frac{d_\sigma}{2}\left(1 + \frac{\chi_\sigma(m)}{d_\sigma}\right)$$

- In S_n, $\chi_\sigma(m)/d_\sigma$ is exponentially small, so the observed distribution is very close to Plancherel
- Weak sampling fails [Hallgren, Russell, Ta-Shma 2000]
- Random basis fails [Grigni, Schulman, Vazirani, Vazirani 2001]
- But, strong is stronger for some G... [MRRS 2004]

Now for Strong Sampling

- But what about a basis of our choice? Given σ, we observe a basis vector **b** with probability

$$\frac{\|\pi_H \mathbf{b}\|^2}{\operatorname{rk} \pi_H}$$

- Here we have $\|\pi_H \mathbf{b}\|^2 = \frac{1}{2}(1 + \langle \mathbf{b}, m\mathbf{b} \rangle)$
- How much does $\langle \mathbf{b}, m\mathbf{b} \rangle$ vary with m?

Controlling the Variance

- Expectation of an irrep σ over m's conjugates is

$$\mathrm{Exp}_m \sigma(m) = \frac{\chi_\sigma(m)}{d_\sigma} \mathbb{1}$$

so $\mathrm{Exp}_m \langle \mathbf{b}, m\mathbf{b} \rangle = \dfrac{\chi_\sigma(m)}{d_\sigma}$

- To turn the second moment into a first moment,

$$|\langle \mathbf{b}, m\mathbf{b} \rangle|^2 = \langle \mathbf{b} \otimes \mathbf{b}^*, m(\mathbf{b} \otimes \mathbf{b}^*) \rangle$$

Controlling the Variance

- Decompose $\sigma \otimes \sigma^*$ into irreducibles:

$$\sigma \otimes \sigma^* \cong \bigoplus_{\tau \in \widehat{G}} a_\tau \tau$$

Then

$$\mathrm{Var}_m \|\pi_H \mathbf{b}\|^2 \leq \frac{1}{4} \sum_{\tau \in \widehat{G}} \frac{\chi_\tau(m)}{d_\tau} \left\| \Pi_\tau^{\sigma \otimes \sigma^*} (\mathbf{b} \otimes \mathbf{b}^*) \right\|^2$$

- How much of $\mathbf{b} \otimes \mathbf{b}^*$ lies in low-dimensional τ?

Strong Sampling Fails

- Using simple counting arguments, we show that almost all of $\mathbf{b} \otimes \mathbf{b}^*$ lies in high-dimensional subspaces τ of $\sigma \otimes \sigma^*$.
- Since $\chi_\tau(m)/d_\tau$ is exponentially small, the observed distribution on $\mathbf{b}$ for *any* basis is exponentially close to uniform.
- No subexponential set of experiments on coset states can solve Graph Isomorphism.
  [M., Russell, Schulman 2005]

Entangled Measurements

- For any group, there exists a measurement on the tensor product of coset states

$$\underbrace{\rho \otimes \cdots \otimes \rho}_{k}$$

with $k = \mathrm{poly}(\log |G|)$ [Ettinger, Høyer, Knill 1999]

- What can we prove about entangled measurements?

Bounds on Multiregister Sampling

- Weak sample each register, observing

$$\boldsymbol{\sigma} = \sigma_1 \otimes \cdots \otimes \sigma_k$$

- Given a subset *I* of the *k* registers, decompose that part of the tensor product:

$$\bigotimes_{i \in I} \sigma_i \cong \bigoplus_{\tau \in \widehat{G}} a_\tau^I \tau$$

- This group action multiplies these registers by *g* and leaves the others fixed.

Bounds on Multiregister Sampling

- Second moment: analogous to one register, consider $\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}^*$. Given subsets I and J, define

$$E^{I,J}(\mathbf{b}) = \sum_{\tau \in \widehat{G}} \frac{\chi_\tau(m)}{d_\tau} \left\| \Pi_\tau^{I,J}(\mathbf{b} \otimes \mathbf{b}^*) \right\|^2$$

- For an arbitrary entangled basis, [M., Russell 2005]

$$\mathrm{Var}_m \left\| \Pi_H \mathbf{b} \right\|^2 \leq \frac{1}{4^k} \sum_{I,J \subseteq [k] : I,J \neq \emptyset} E^{I,J}(\mathbf{b})$$

Bounds on Multiregister Sampling

- With some additional work, this general bound can be used to show that $\Omega(n \log n)$ registers are necessary for S_n [Hallgren, Rötteler, Sen; M., Russell]
- But what form might this measurement take?
- Note that each subset of the registers contributes some information...

Subset Sum and the Dihedral Group

- The HSP in the dihedral group D_n reduces to random cases of Subset Sum [Regev 2002]
- Leads to a $2^{O(\sqrt{\log n})}$-time and -register algorithm [Kuperberg 2003]
- Subset Sum gives the *optimal* multiregister measurement [Bacon, Childs, van Dam 2005]

More Abstractly...

- If $H = \{1, m\}$, there is a *missing harmonic*:

$$\sum_{h \in H} \pi(h) = 0$$

- Weak sampling gives random two-dimensional irreps σ_j ; think of these as integers $\pm j$.
- Tensor products: $\sigma_j \otimes \sigma_k \cong \sigma_{j+k} \oplus \sigma_{j-k}$
- Find subset that gives $\sigma_0 \cong \mathbb{1} \oplus \pi$.

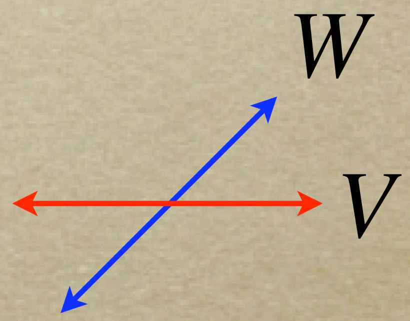
Subsets in General

- Suppose H has a missing harmonic τ.
- For each subset I, consider the subspace W_τ^I resulting from applying the group action to I. (In D_n, this flips the integers j in this subset.)
- If the hidden subgroup is a conjugate of H, then the state is perpendicular to W_τ^I for all I.
- How much of $\mathbb{C}[G^k]$ does this leave? What fraction is spanned by the W_τ^I?

Independent Subspaces

- Say that two subspaces *V*, *W* of a space *U* are *independent* if, just as for random vectors in *U*,

$$\mathrm{Exp}_{v \in V} \|\Pi_W v\|^2 = \frac{\dim W}{\dim U}$$



or equivalently

$$\frac{\mathrm{tr}\,\Pi_V \Pi_W}{\dim U} = \frac{\mathrm{tr}\,\Pi_V}{\dim U} \frac{\mathrm{tr}\,\Pi_W}{\dim U}$$

- Being in *V* or *W* are “independent events.”

Each Subset Contributes

- For $I \neq J$, W_τ^I and W_τ^J are independent.
- Therefore, $W_\tau = \mathrm{span}_I W_\tau^I$ is large:

$$\frac{\dim W_\tau}{\dim \mathbb{C}[G^k]} \geq 1 - \frac{1}{1 + 2^k/|G|}$$

- If $k \geq \log_2 |G|$, probability of "some subset being in τ" is $\geq 1/2$ if the hidden subgroup is trivial, but is zero if it is a conjugate of H.
[M., Russell 2005]

Find An Informative Subset!

- Divide $\mathbb{C}[G^k]$ into subspaces; for each one, find a subset I for a large fraction of the completely mixed state is in W_τ^I: e.g. $\sigma_0 \cong \mathbb{1} \oplus \pi$ in D_n.
- "Pretty Good Measurement" (i.e., Subset Sum for D_n) is optimal for Gel'fand pairs... [MR 2005]
- ...but it is not optimal for S_n [Childs]. What is? And, is it related to Subset Something?

The Hunt Continues

vs.

Beauty and Truth

The Adversary

Acknowledgments