Bounds on the Quantum Satisfiability Threshold

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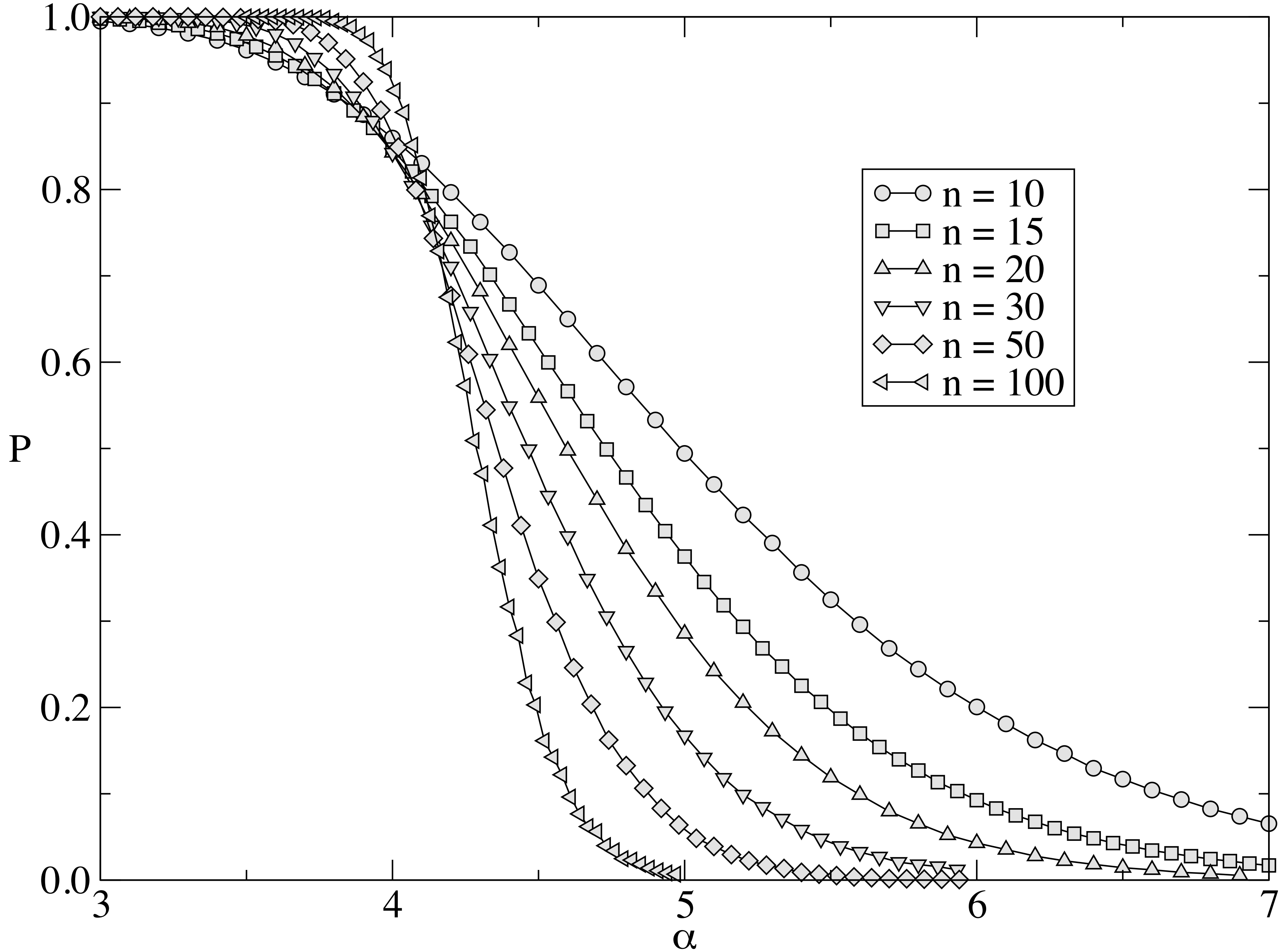
joint work with Sergey Bravyi (IBM) and Alexander Russell (Connecticut)

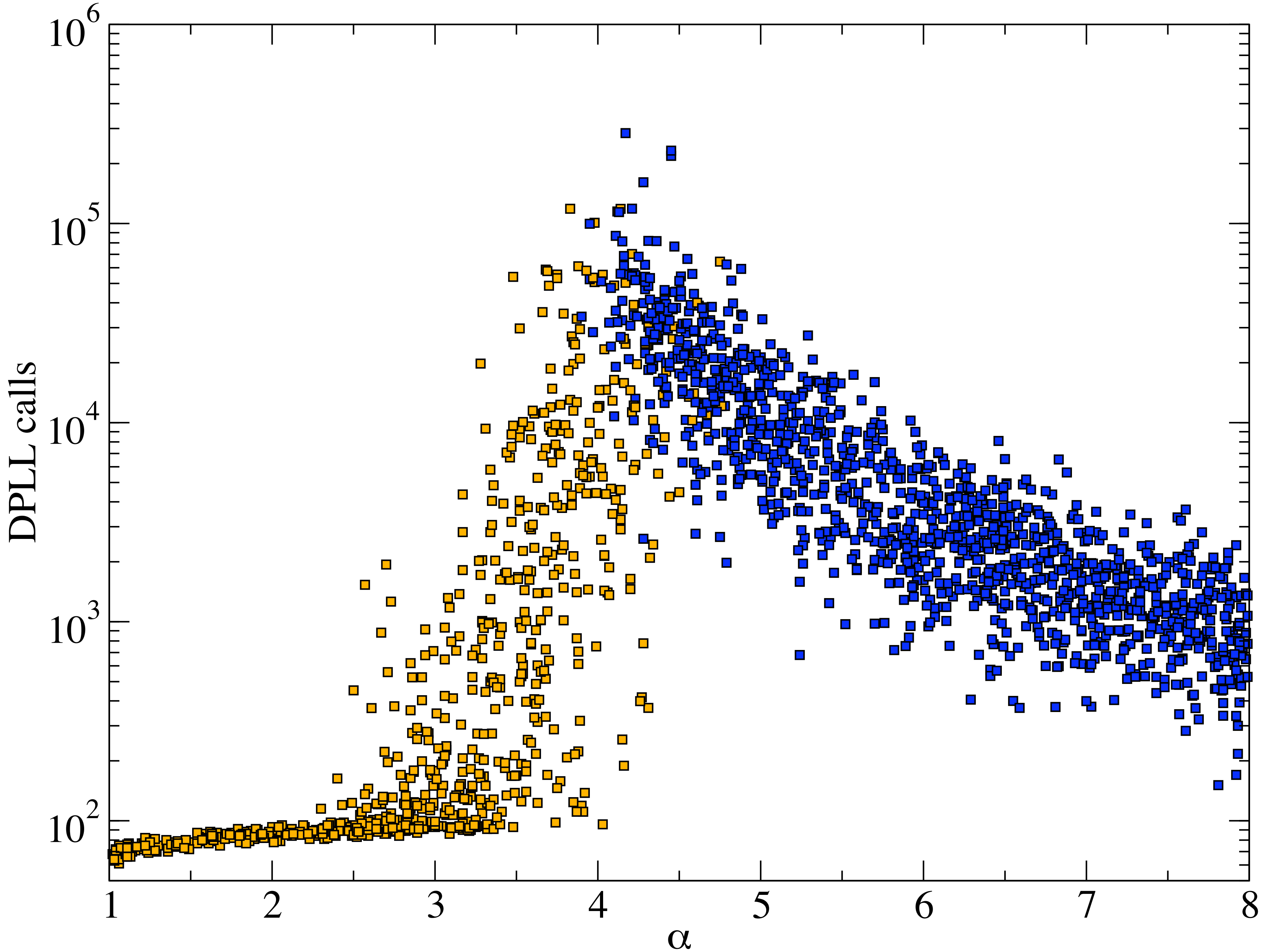
A phase transition

- Random 3-SAT formulas with *n* variables and α*n* clauses



A phase transition

- Search times appear to peak at the transition



Quantum k-SAT [Bravyi]

- Classical SAT: each clause forbids one out of 8 truth values. Think of this as forbidding a basis vector:

$$(x_1 \vee \overline{x_2} \vee x_3) \;\Leftrightarrow\; \langle 010|x\rangle = 0$$

- Quantum SAT: forbid an arbitrary vector in $\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2$,

$$\langle v|x\rangle = 0$$

- For each clause *c*, we have $\Pi_c|\psi\rangle = |\psi\rangle$ where

$$\Pi_c = (1 - |v\rangle\langle v|) \otimes \mathbf{1}_{n-3}$$

A local Hamiltonian

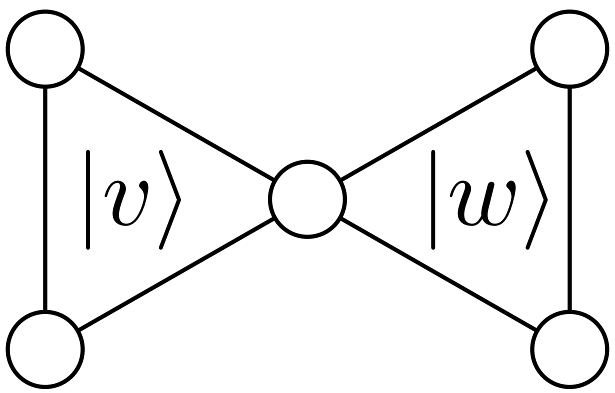
- Alternately, ask whether there is a zero-energy state $|\psi\rangle$ of a local, disordered Hamiltonian:

$$H = \sum_c |v\rangle\langle v| \otimes \mathbf{1}$$

- What is its ground state energy? QMA_1-complete [Bravyi]

- When are its ground states entangled?

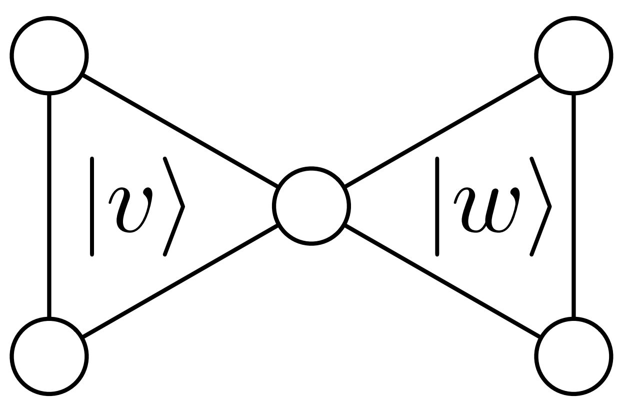
Forbidden and satisfying subspaces



$$V_{\text{forbidden}} = \text{span}\left\{\begin{array}{c} |\ v\ \rangle \otimes |00\rangle \\ |\ v\ \rangle \otimes |01\rangle \\ |\ v\ \rangle \otimes |10\rangle \\ |\ v\ \rangle \otimes |11\rangle \\ |00\rangle \otimes |\ w\ \rangle \\ |01\rangle \otimes |\ w\ \rangle \\ |10\rangle \otimes |\ w\ \rangle \\ |11\rangle \otimes |\ w\ \rangle \end{array}\right\}$$

- The satisfying subspace is $V_{\text{sat}} = V_{\text{forbidden}}^{\perp}$
- With probability 1, $\text{rank}\, V_{\text{forbidden}} = 8$, so $\text{rank}\, V_{\text{sat}} = 32 - 8 = 24$

Generic clause vectors



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- These ranks take generic values with probability 1
- Coincidences can only decrease $\operatorname{rank} V_{\text{forbidden}}$, and increase $\operatorname{rank} V_{\text{sat}}$
- For a given hypergraph, if *any* choice of clause vectors make it unsatisfiable, it is generically unsatisfiable [Laumann et al.]

Random quantum k-SAT formulas

- Two sources of randomness:
  - A random hypergraph with *n* vertices and *m* hyperedges (clauses), where

$$m = \alpha n$$

  - Random clause vectors, chosen uniformly from unit-length vectors in $\mathbb{C}_2^{\otimes k}$
- Threshold conjecture:

$$\lim_{n \to \infty} \Pr[H(n, m = \alpha n) \text{ is generically satisfiable}] = \begin{cases} 1 & \alpha < \alpha_c \\ 0 & \alpha > \alpha_c \end{cases}$$

A classical upper bound

- Compute the expected number of satisfying assignments. For k-SAT,

$$\mathbb{E}[X] = 2^n \left(1 - 2^{-k}\right)^m = \left(2(1 - 2^{-k})^\alpha\right)^n$$

- This is an upper bound on the probability of satisfiability:

$$\Pr[X > 0] \leq \mathbb{E}[X]$$

- This becomes exponentially small when α is large enough:

$$\alpha_c \leq \log_{1/(1-2^{-k})} 2 \approx 2^k \ln 2$$

- This is asymptotically tight [Achlioptas&Moore, Achlioptas&Peres]

A simple quantum upper bound

- Number of solutions is analogous to $\operatorname{rank} V_{\text{sat}}$

- Expectation of a clause projector:

$$\mathbb{E}_v \Pi_c = (1 - \mathbb{E}_v |v\rangle\langle v|) \otimes \mathbf{1} = (1 - 2^{-k})\mathbf{1}$$

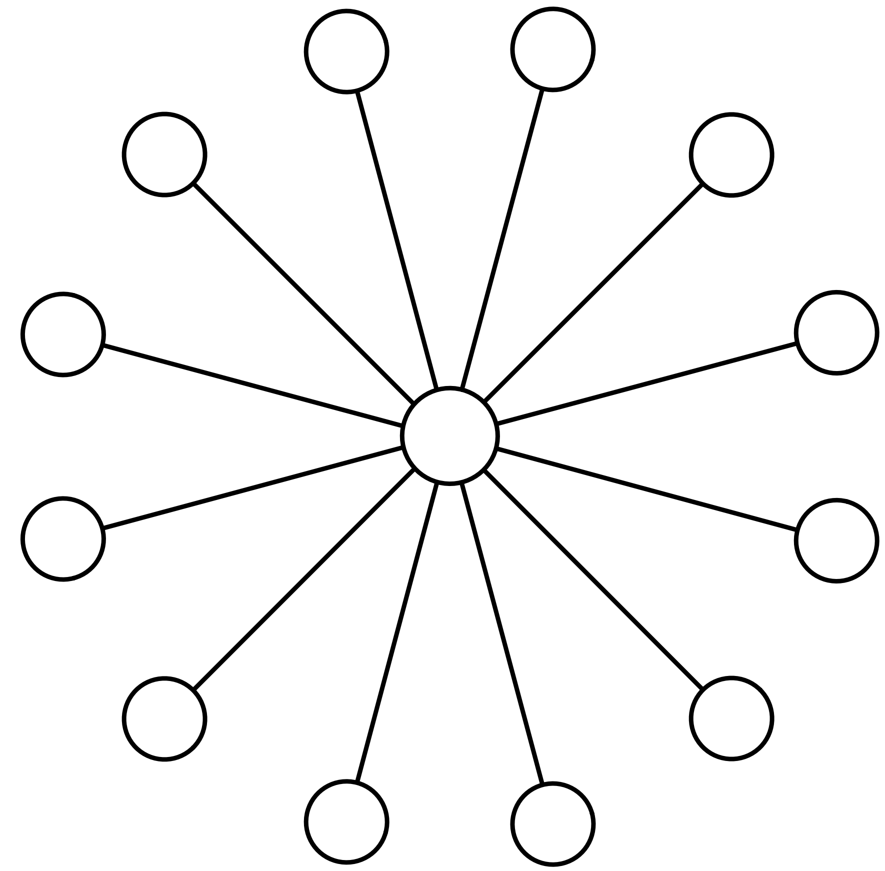
- Since the clauses are independent, if $\Pi_\phi = \prod_c \Pi_c$ then

$$\operatorname{rank} V_{\text{sat}} \leq \mathbb{E}_{\{v\}} \operatorname{tr} \Pi_\phi^\dagger \Pi_\phi = 2^n (1 - 2^{-k})^m$$

- So, the quantum bound is at most the classical one: $\alpha_c^q \leq \alpha_c$

Quantum SAT is more restrictive

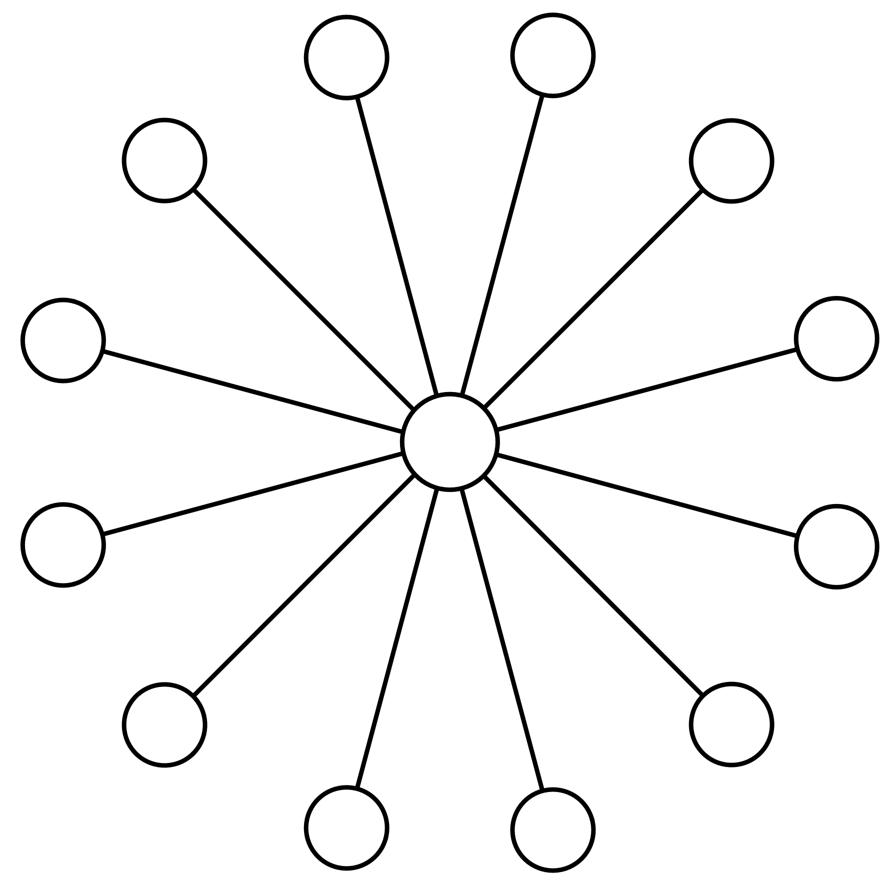
- 2-SAT problem on a star of degree d



- Classical: at least $2^{\lfloor d/2 \rfloor} + 2^{\lceil d/2 \rceil}$ solutions
- Quantum: only $n + 1 = d + 2$

Quantum SAT is more restrictive

- Remember that any choice of forbidden vectors gives an upper bound

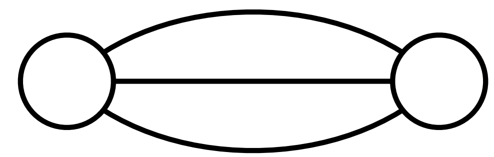


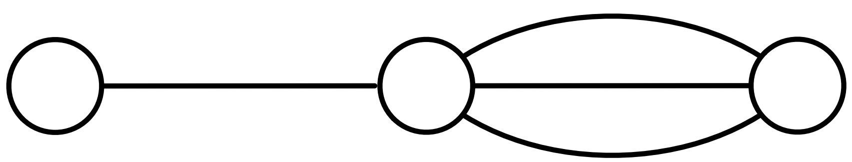
- Forbid singlets: $|v\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

- $\langle v|\psi\rangle = 0$ if and only if $|\psi\rangle$ is symmetric under transpositions

- If the graph is connected, $|\psi\rangle$ must be symmetric under all permutations

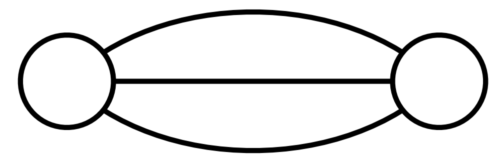
Entangled states

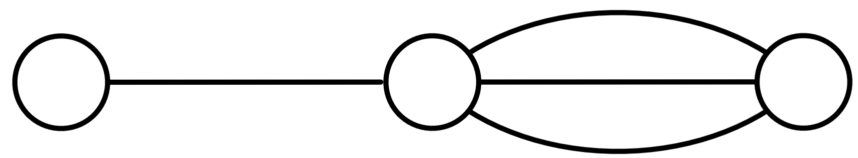




Entangled states

- This 2-SAT formula is satisfiable:





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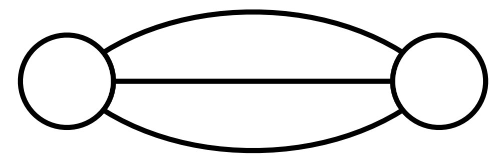

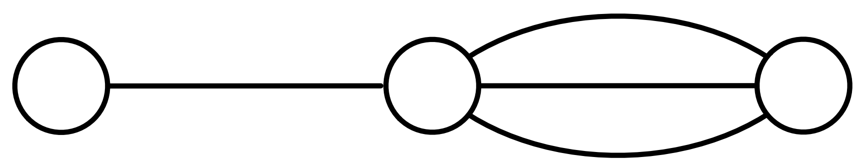

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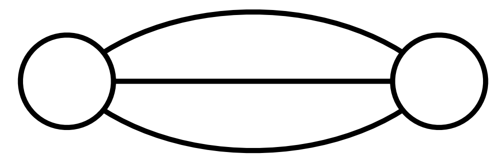
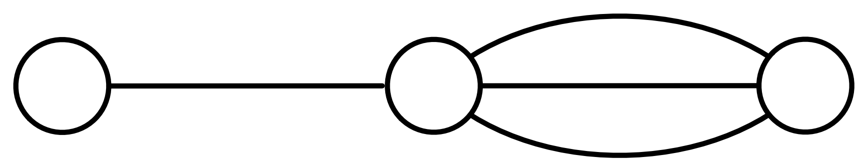


- Is this one?



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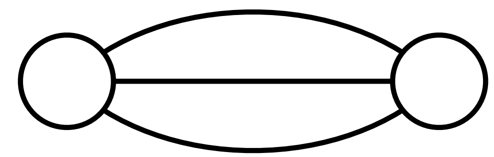

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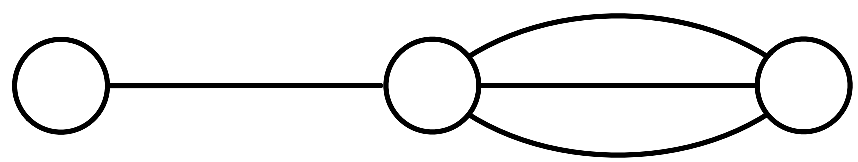

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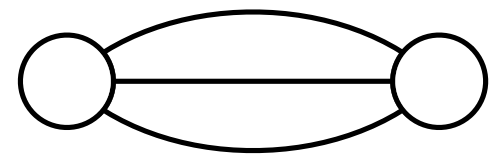

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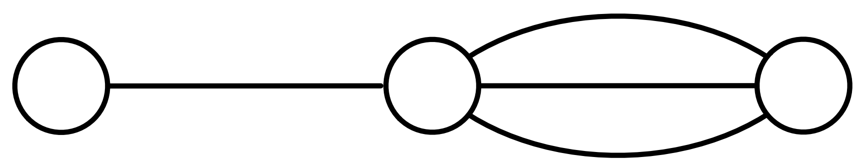

- Classical: of course! Use the new variable to satisfy the new clause.

Entangled states

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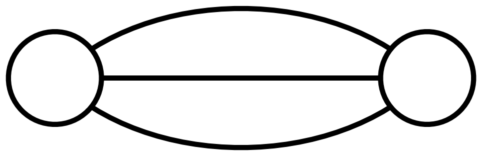

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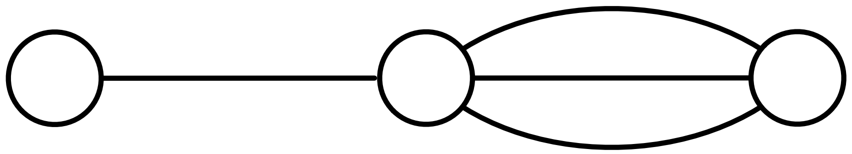


- Classical: of course! Use the new variable to satisfy the new clause.
- Quantum: no! In entangled states, single variables don’t have values. Similarly, single variables can’t satisfy entangled clauses.

Better upper bounds

- For any gadget *H* on *t* vertices,

$$\mathbb{E}[\Pi_H] = \frac{\operatorname{rank} V_{\mathrm{sat}}(H)}{2^t}\,\mathbf{1}$$

- Any time we add a gadget, we reduce the generic rank. With probability 1,

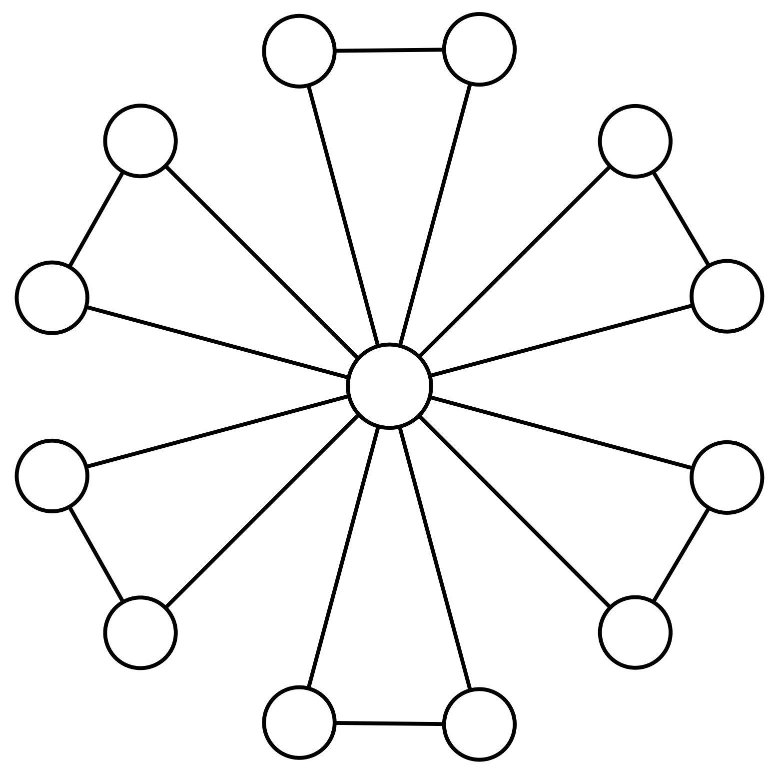
$$\operatorname{rank} V_{\mathrm{sat}}(G \cup H) \le \frac{\operatorname{rank} V_{\mathrm{sat}}(H)\operatorname{rank} V_{\mathrm{sat}}(G)}{2^t}$$

- Partition a random hypergraph into gadgets:

$$\operatorname{rank} V_{\mathrm{sat}} \le 2^n \prod_i \frac{\operatorname{rank} V_{\mathrm{sat}}(H_i)}{2^t}$$

The Sunflower

- Partition the hypergraph into n_d sunflowers of degree d:



- This gives

$$\operatorname{rank} V_{\text{sat}} \leq 2^n \prod_{d=1}^{\infty} \left(\left(\frac{3}{4}\right)^d \left(\frac{d}{6}+1\right)\right)^{n_d}$$

Sunflower partitions

- Naive: at each step, choose a random vertex, declare it and its clauses to be a sunflower, and remove them

- Continuous time: give each vertex an index $t \in [0,1]$, and remove in decreasing order

- The degree of a sunflower of index *t* is the number of clauses whose variables all have index < *t*. Poisson distribution with mean $k\alpha t^{k-1}$

- Setting $\frac{\ln \operatorname{rank} V_{\text{sat}}}{n} = 0$ gives $\alpha_c^q \leq 3.894$

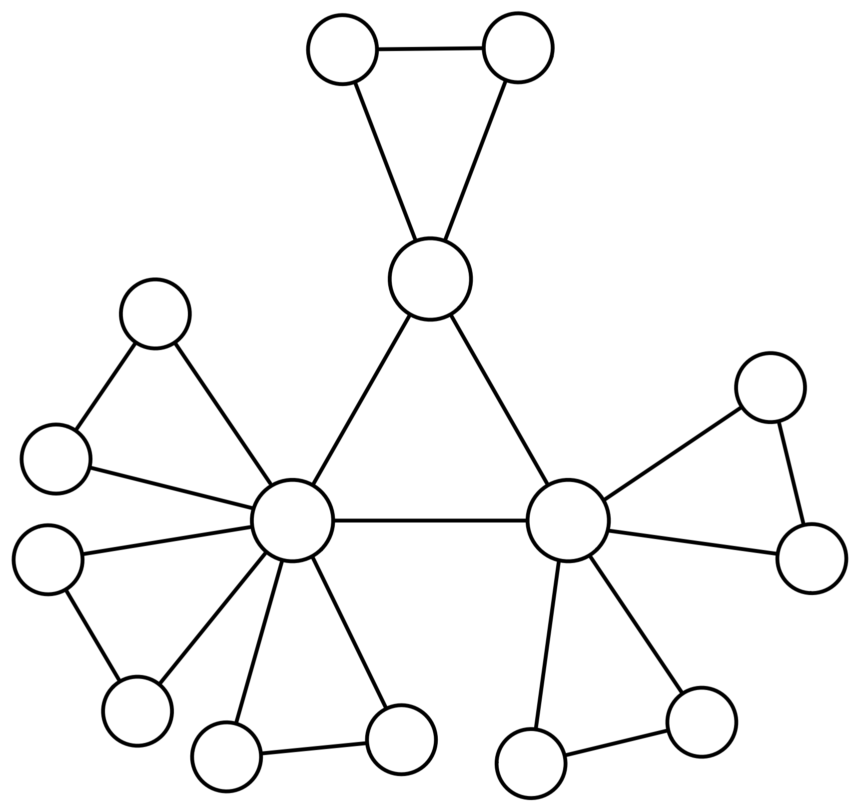
- Greedier partition: taking high-degree vertices first gives $\alpha_c^q \leq 3.689$ (analyze with system of differential equations)

The Nosegay

nosegay |ˈnōzˌgā|

noun
a small bunch of flowers, typically one that is sweet-scented.

- Bigger gadgets: more conflict, smaller rank



- At each step, choose a random clause, and take it and its neighbors
- Gives $\alpha_c^q \leq 3.594$, far below the classical $\alpha_c \approx 4.267$

When k is large

- Asymptotically, we have

$$\alpha_c \leq 2^k b$$

- where $b \approx 0.573 < \ln 2$ is the root of $\ln 2 - 2b + \ln(b+1) = 0$

- Classically,

$$\alpha_c = (1 - o(1)) \leq 2^k \ln 2$$

so the quantum threshold is a constant smaller.

Open questions

- Classical: counting satisfying assignments of a 3-SAT formula is *#P-complete.*
  Quantum analog: computing $\mathrm{rank}\, V_{\mathrm{sat}}$. What it its complexity?
  Might not be in #P: entanglement again.

- Similarly, is generic satisfiability of a hypergraph in NP? Is it NP-hard?

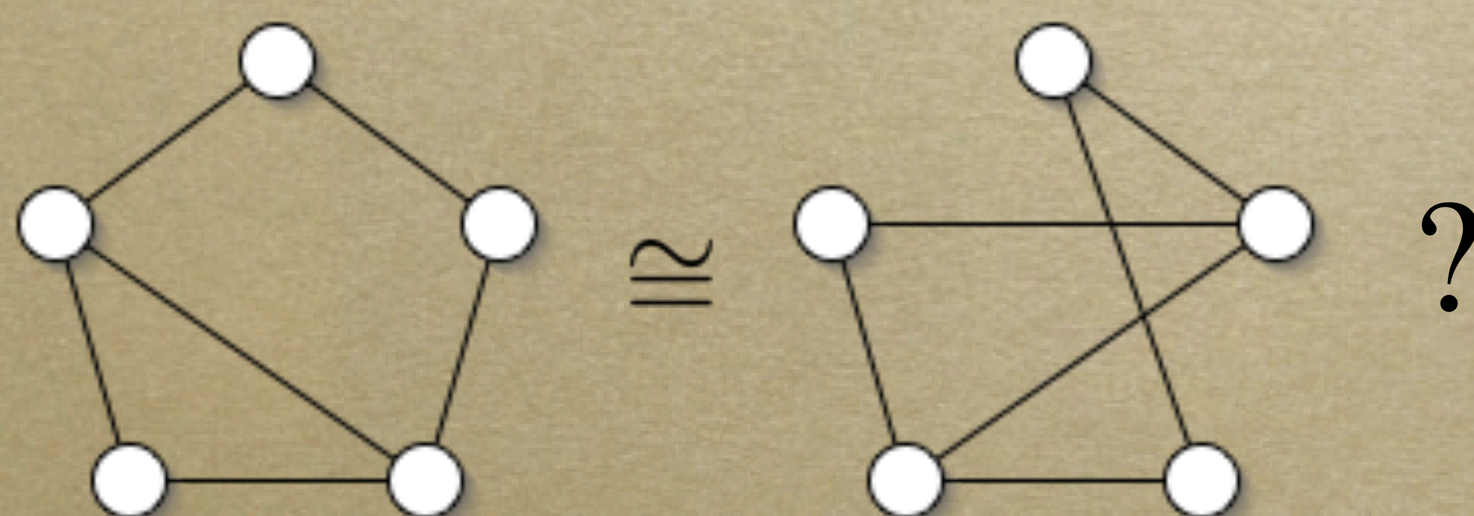
- Is there a satisfiable-but-entangled phase, in which random formulas are satisfiable, but all satisfying states are highly entangled?

- Assuming there is a transition, does α_c^q grow as 2^k ? Does it even grow without bound as *k* increases? Best lower bounds so far are less than 1!

- What is the *adversarial* classical threshold, where the hypergraph is random, but the adversary chooses which literals to negate?

Graph Isomorphism

- Factoring appears to be outside P, but not NP-complete. (Indeed, we believe that BQP does not contain all of NP.)
- Another candidate problem like this:



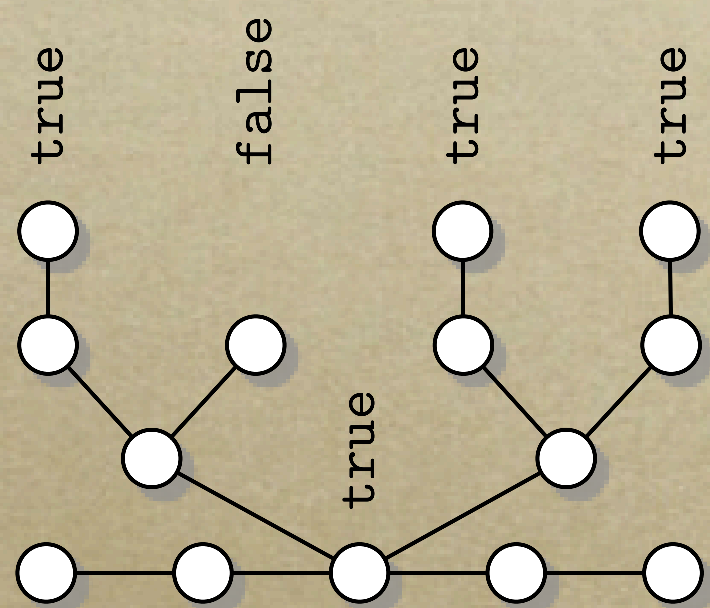
The Hidden Subgroup Problem

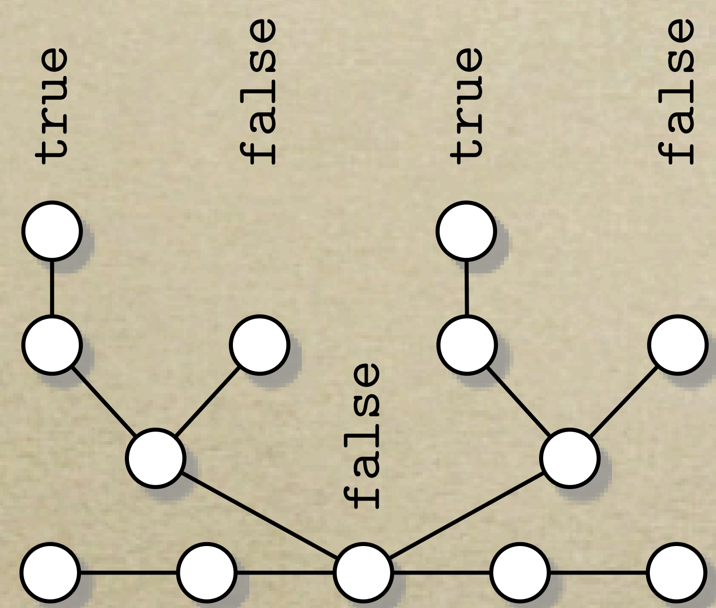
- We have a function $f : G \to X$
- We want to know its symmetries $H \subseteq G$
- Essentially all quantum algorithms that are exponentially faster than classical are of this form:
    - $\mathbb{Z}_n^*$= factoring
    - S_n= Graph Isomorphism
    - D_n= some cryptographic lattice problems

The Story So Far...

- It turns out that this naïve generalization of Shor’s algorithm doesn’t work: the permutation group S_n is “too non-Abelian.”
- Tantalizingly, we know a *measurement* exists, but we don’t know if we can do it efficiently.
- How much can quantum computing really do? How “special” is factoring?

Scattering Algorithms





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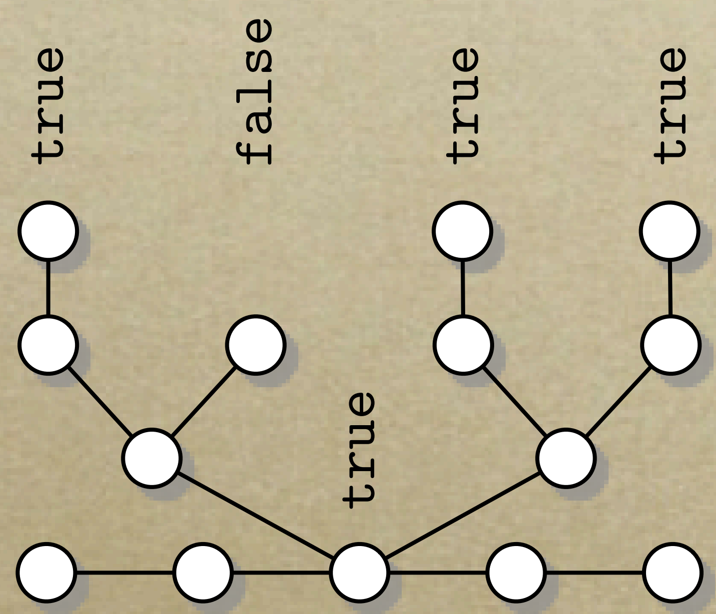

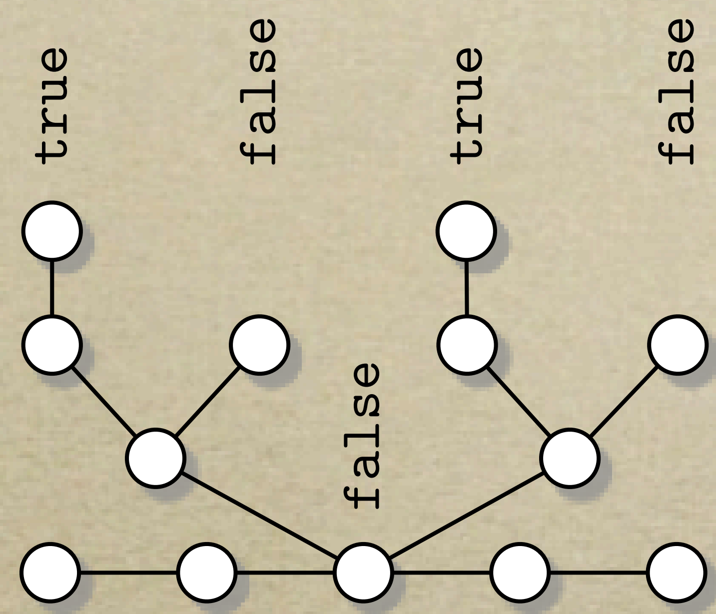

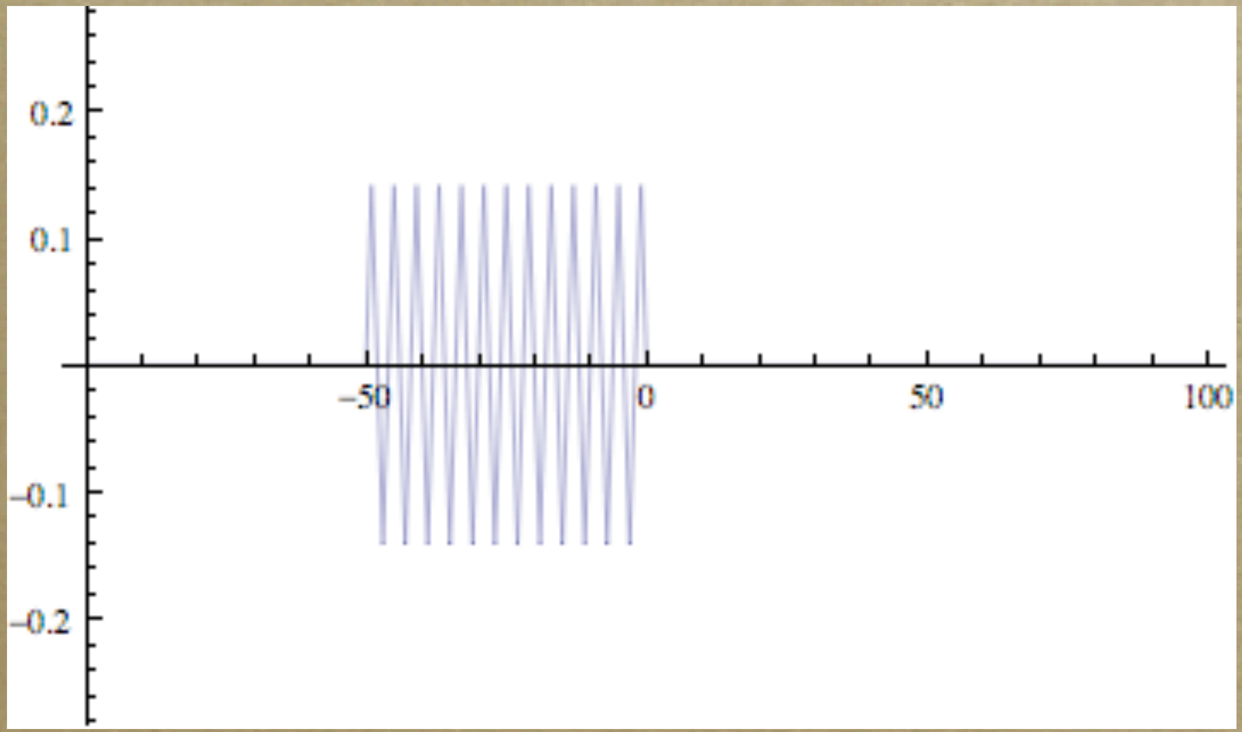

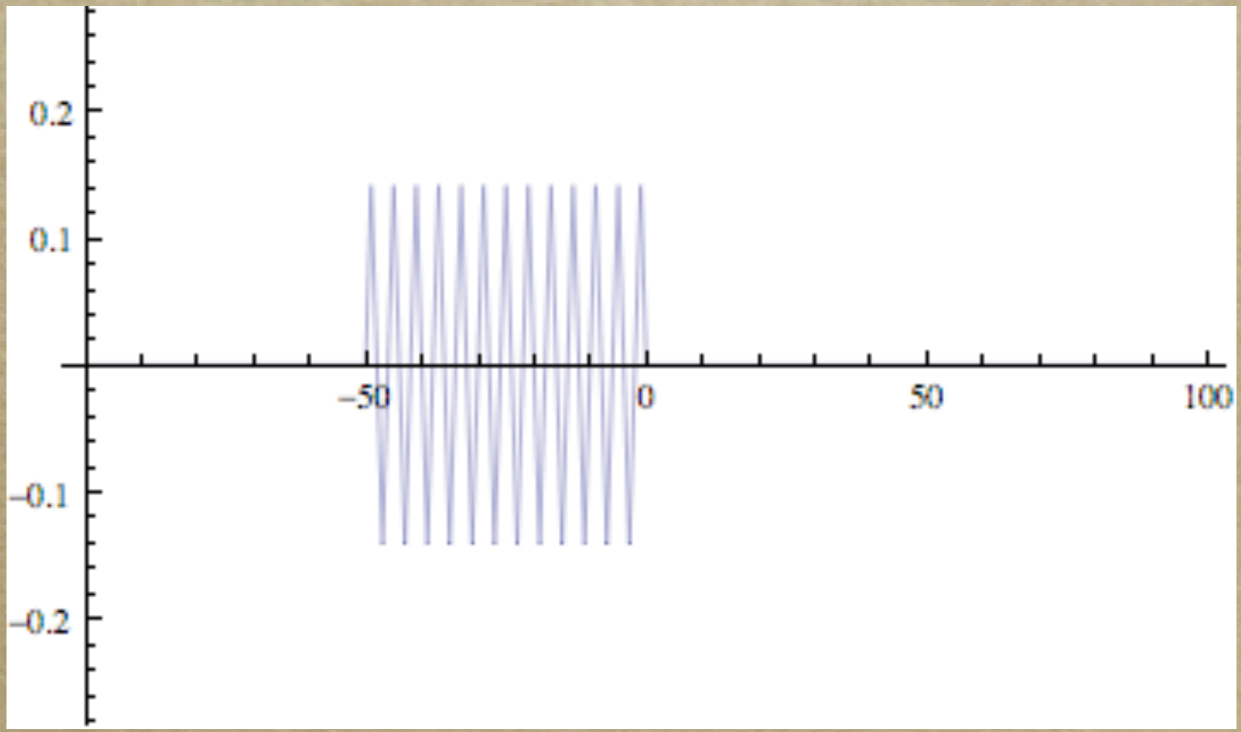

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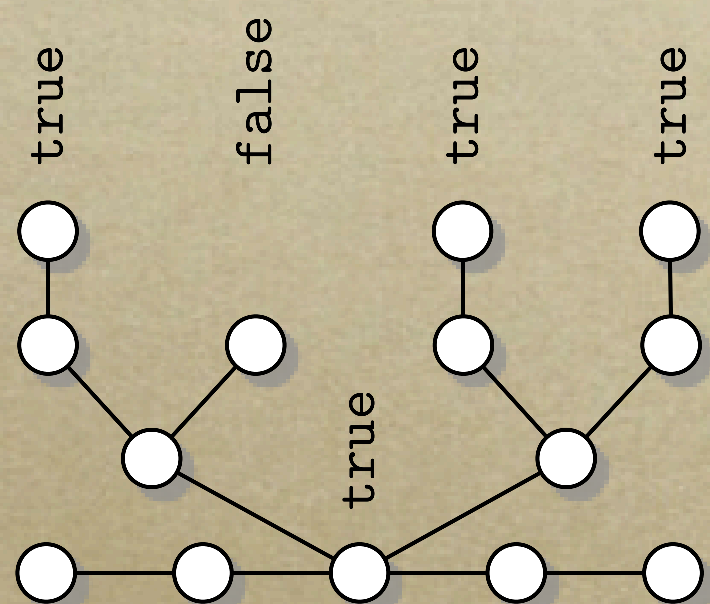
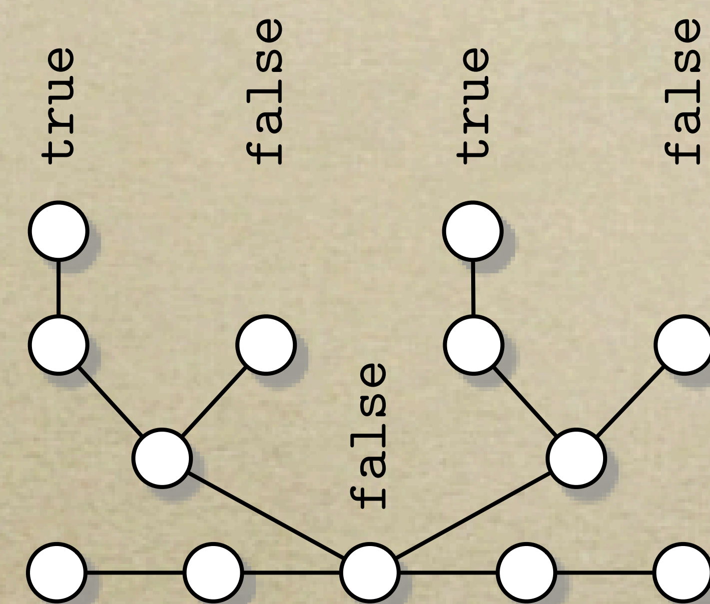
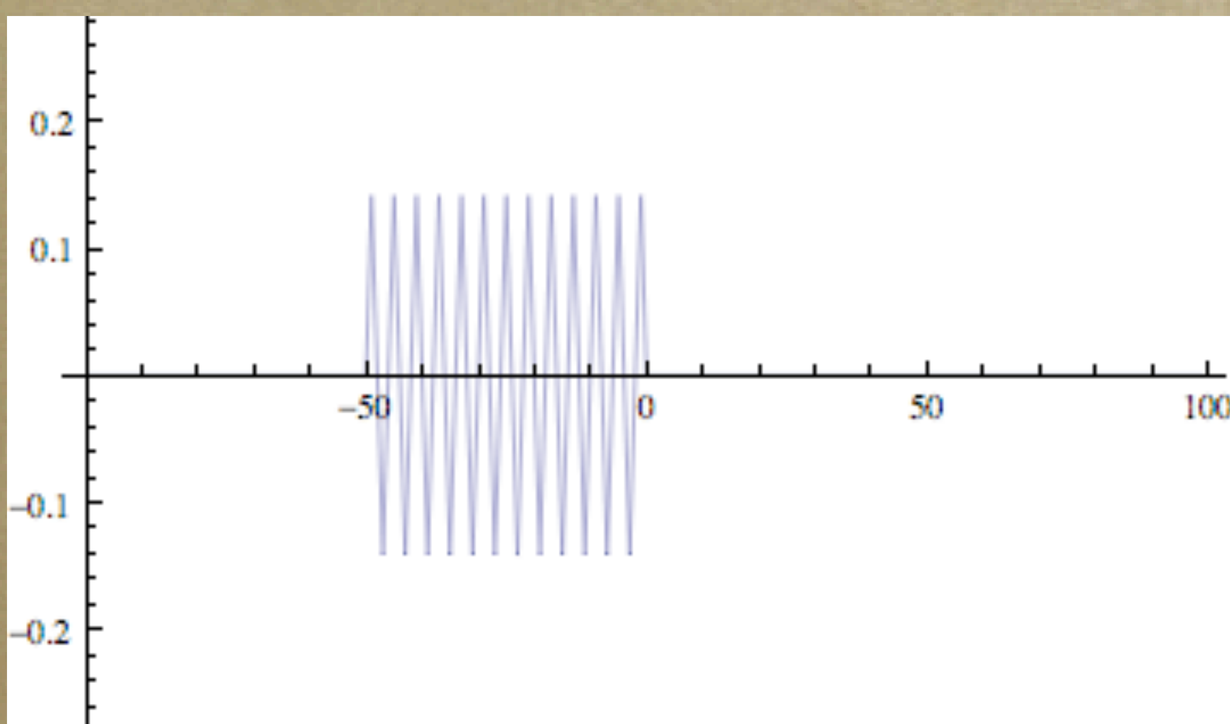
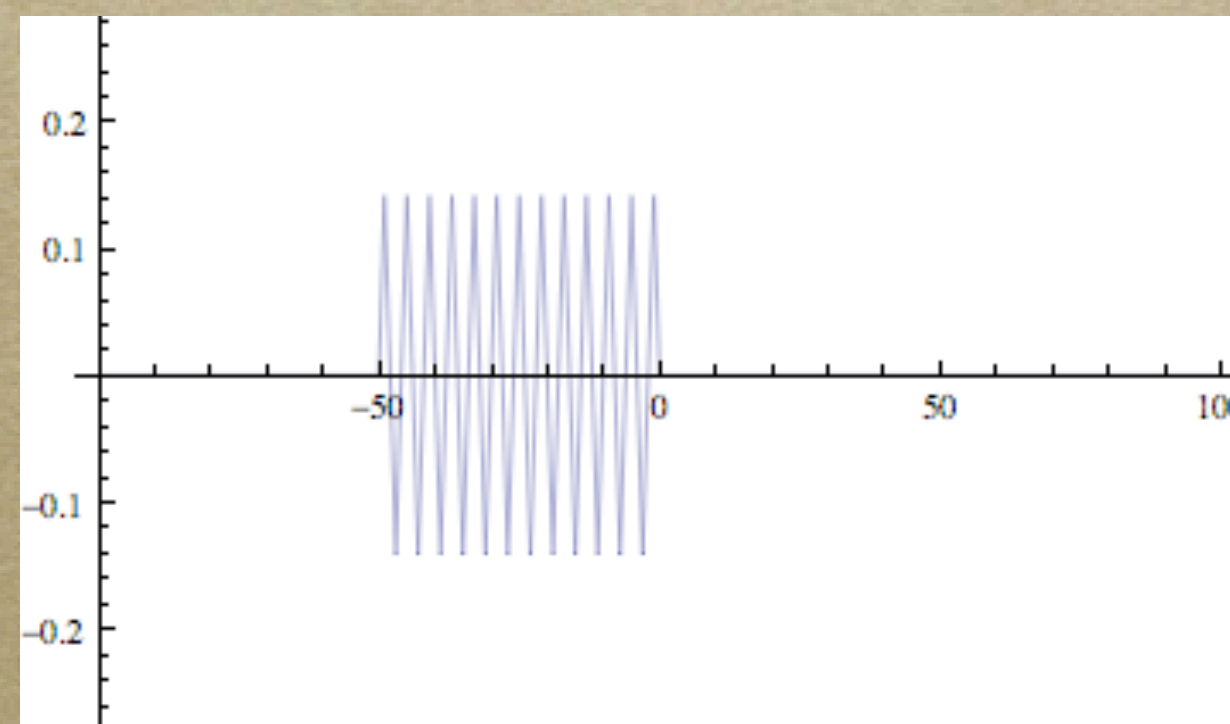
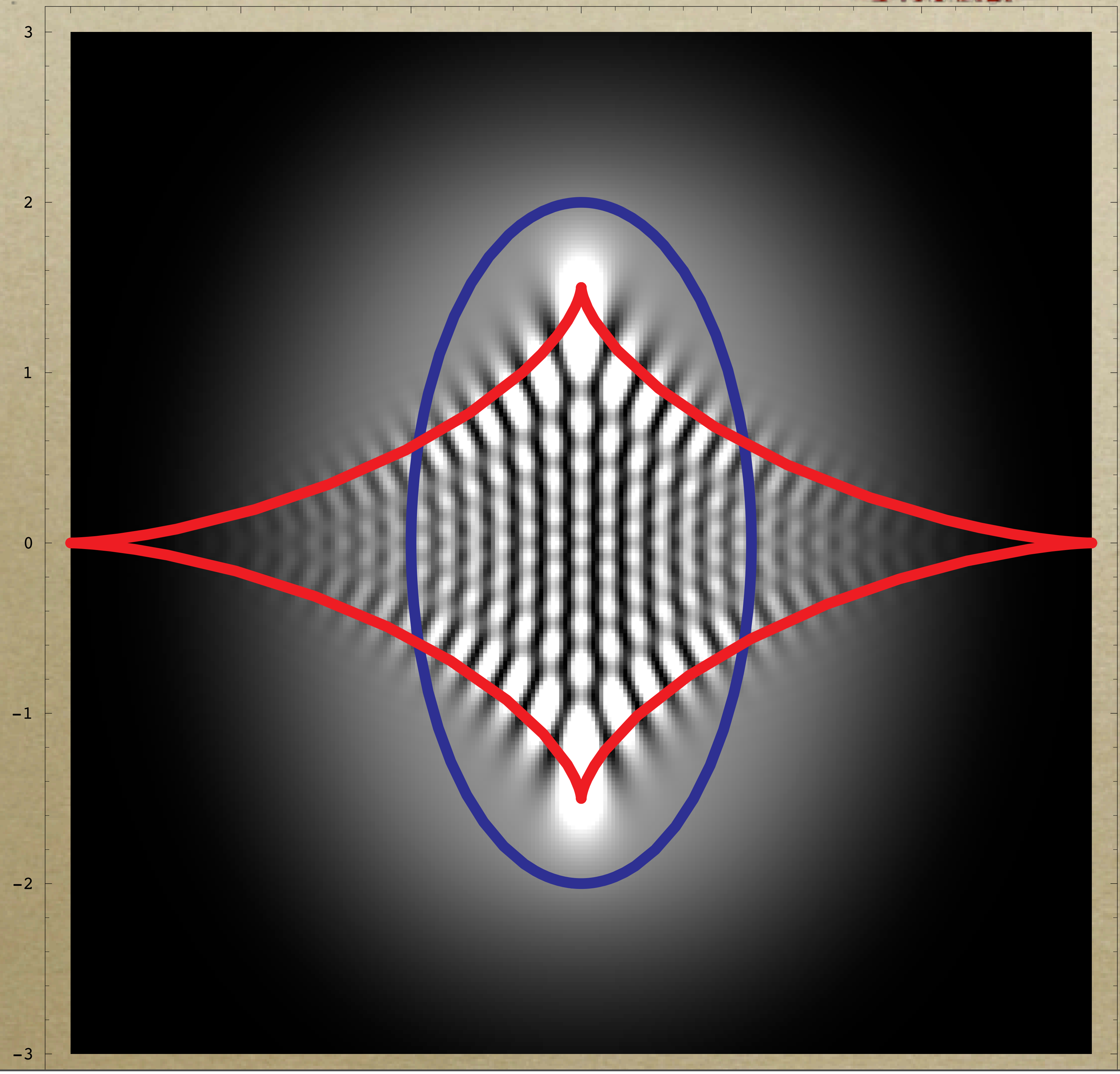






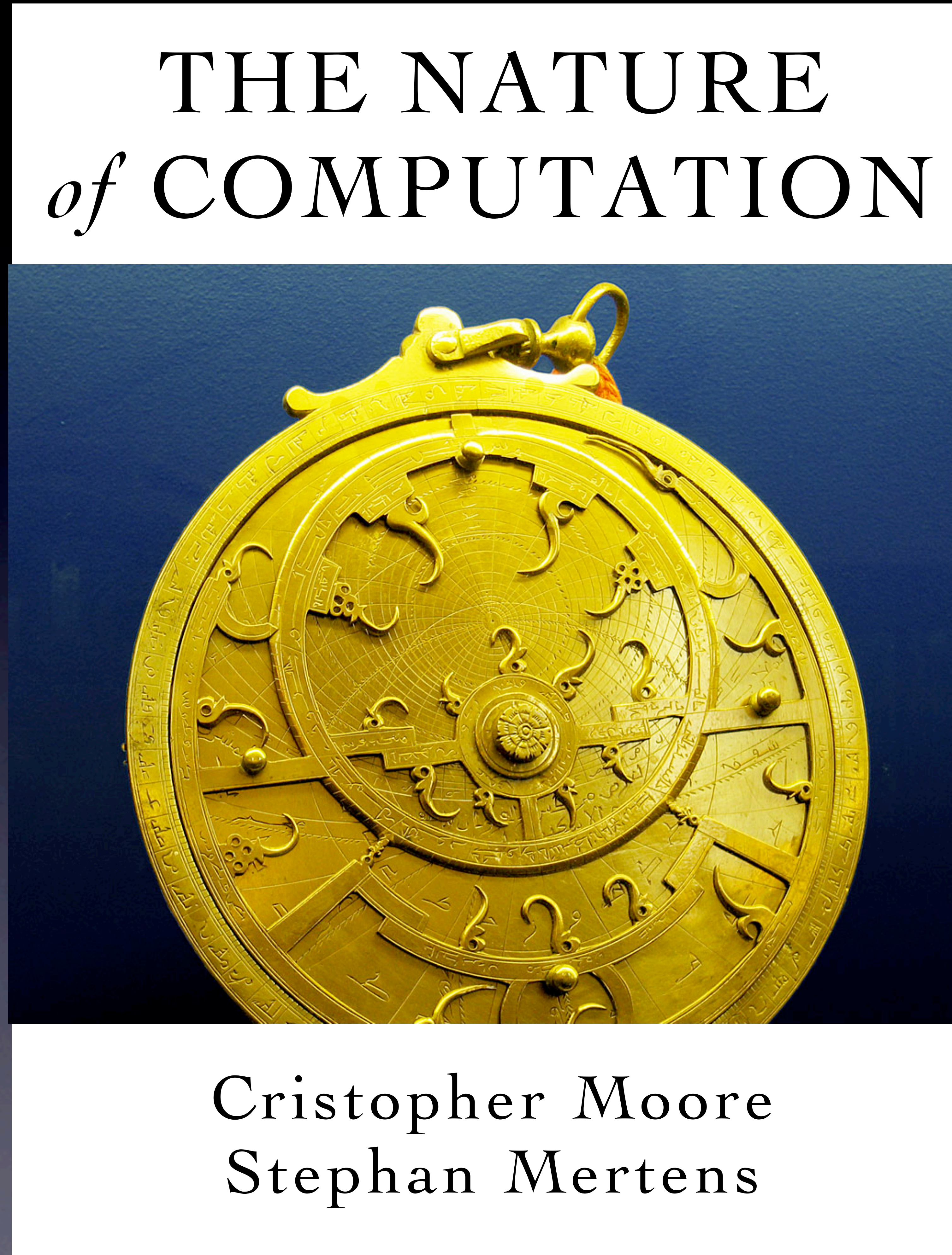


Schrödinger's Equation, Diffraction, and Evolutes



Shameless Plug

Oxford University Press, 2010



Acknowledgements

- Also, NSF and DTO