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Self-organized criticality and absorbing states: Lessons from the Ising model

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We investigate a suggested path to self-organized criticality. Originally, this path was devised to "generate criticality" in systems displaying an absorbing-state phase transition, but nothing in its definition forbids the mechanism to be used in any other continuous phase transition. We used the Ising model as well as the Manna model to demonstrate how the finite-size scaling exponents depend on the tuning of driving and dissipation rates with system size. Our findings limit the explanatory power of the mechanism as it is to nonuniversal critical behavior, suggesting that the explanation of self-organized criticality in terms of absorbing-state phase transitions is incomplete.

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Self-organized criticality (SOC) refers to the spontaneous emergence of critical behavior in slowly driven dissipative systems [1,2]. Examples of physical systems believed to display SOC are rainfall [3] and earthquakes [4]. Most models are defined on lattices with local particle numbers z_i and thresholds z_i^c. They are driven discretely in time by increasing z_i at randomly chosen positions i until such an increase leads to $z_i > z_i^c$ somewhere in the system. Particles then topple to neighboring sites and can trigger avalanches of local redistribution propagating through the entire lattice. Dissipation typically takes place at the boundaries, where particles leave the system. When an avalanche has finished, the model is driven again [2]. The resulting avalanche size distributions obey simple scaling. Standard finite-size scaling (FSS) is expected if the ratio of correlation length and system size, ξ/L is constant or diverges as L increases.

In models displaying absorbing state (AS) phase transitions [5] a tuning parameter, such as the overall particle density, controls a transition between an inactive phase and a phase where activity in the system continues indefinitely.

From the introduction of SOC in 1987 [1], it was believed that SOC models maneuver themselves to the critical density between similar inactive and active phases. Tang and Bak suggested in 1988 that the density of "lattice sites on which $z > z_c$ [$\cdots$] may be viewed as the order parameter for this critical phenomenon" [6]. Such an identification of the activity with the order parameter implies a link to absorbing state phase transitions.

This link was formalized and made explicit about 10 years later [7–10]. Dickman *et al.* [8] introduced periodic boundaries to SOC systems, thereby turning them into AS models. Measuring the exponents characterizing the spreading of perturbations [11,12] or the roughness of the associated interface models [11,13], it has been observed that at the critical density the closed-model behavior resembles that of open SOC models [8,14].

The resulting interpretation of SOC is obvious [8–10]: Activity eventually leads to dissipation at the boundaries, which in turn reduces the particle density to below the critical value. Driving takes place whenever quiescence has been reached. SOC models therefore hover around the critical point, being pushed forth into the active state by driving and pushed back into the quiescent state by dissipation.

With this simple picture in mind one arrives at an equation of motion for the particle density ζ in the system [15,16]

$$\frac{d}{ds}\zeta(s) = h - \rho_a(s)\epsilon, \tag{1}$$

where s is the time, h is the driving rate and ϵ is called the (bulk) dissipation rate. The activity ρ_a is the order parameter, defined as the density of active sites, $z_i > z_i^c$, in the active phase. We will refer to this interpretation of SOC as "the AS approach."

Note that in the AS approach the system is not fixed at the critical point but only approaches it as L increases, implying no specific behavior of $\xi(L)/L$, wherefore simple scaling might not be expected to work. Furthermore, introducing bulk drive and bulk dissipation to SOC models does not result in a full correspondence to AS models, which operate at a constant density without external drive or dissipation. Given these difficulties in equating SOC and AS, it is sensible to ask what the limitations of the AS approach might be.

Clearly, the driving h must be very slow compared to the dissipation $\rho_a\epsilon$. Otherwise particles would be added while the system is active, leading to a fluctuating activity rather than distinct avalanches. The proponents of the AS approach point out that h, ϵ, and h/ϵ have to be tuned to zero in order to achieve the desired separation of timescales [7]. While the

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definitions of SOC models typically restrict dissipation to boundary sites and result in diverging avalanche sizes in the thermodynamic limit, leading to appropriately vanishing $\epsilon(L)$ and $h(L)$, so far no statement has been made as to how the limiting behavior is approached. But this turns out to be the all-important piece of information: The FSS behavior, the only scaling available in SOC, depends entirely on the scaling of the driving and dissipation rates with system size. Choosing $h(L)$ and $\epsilon(L)$ freely, arbitrary scaling behavior is produced.

In the following the relation between the scaling of h and ϵ and the resulting FSS is analyzed, using the two-dimensional Ising model as an example. However, the analysis is generally applicable and works equally well for standard SOC models and their AS counterparts, which is confirmed by simulations of the Manna model [13,17].

Translating Eq. (1) into magnetic language, ζ corresponds to the inverse temperature β and the activity ρ_a to the modulus of the magnetization density $|m|$. The parameters h and ϵ become cooling and heating rates, so that the temperature T is increased for large magnetizations and lowered otherwise,

$$\frac{d}{ds}\beta(s) = h - |m(s)|\epsilon. \tag{2}$$

The resulting model is an Ising model where the temperature is dynamically adapted according to the equation of motion (2). Therefore, the configurations are not sampled with Boltzmann-weight and the resulting "dynamical ensemble" is not canonical. However, by multiplying (2) by a small prefactor, corresponding to rescaling the time, the distribution of temperatures can be made arbitrarily narrow. For the sake of the following analysis, it is assumed that this "dynamical Ising model" is well characterized by a single effective (reduced) temperature, t_{eff}, allowing the formalism of FSS functions [18] to be brought to bear.

For the FSS analysis presented below we define the exponents ω and κ through the leading order approach of $h,\epsilon \to 0$ via

$$h = h_0 L^{-\omega} \text{ and } \epsilon = \epsilon_0 L^{-\kappa}, \tag{3}$$

where $\omega,\kappa > 0$. Assuming that the AS approach produces scale-free behavior requires $h(L)/\epsilon(L)$ to be a power law. Its physical motivation stems from the scaling of the density of (dissipative) boundary sites ($\propto L^{-1}$) and of the driving rate, which is bounded from above via the system-size dependent cutoff of avalanche durations. In the stationary state, $\langle (d/ds)\beta \rangle = 0$, (2) yields $\langle |m| \rangle = (h_0/\epsilon_0)L^{\kappa-\omega}$ with $\langle\ \rangle$ denoting the average over the dynamical ensemble introduced above. Clearly one must choose $\omega > \kappa$. To attain the prescribed $\langle |m| \rangle(L)$ the system settles at the effective (reduced) temperature $\langle T - T_c \rangle / T_c \equiv t_{\text{eff}}(L) \propto L^{-1/\mu}$ to leading order, see Fig. 1. Via $t_{\text{eff}}(L)$ all thermodynamic quantities depend only on L, which can be mistaken for standard FSS at temperature $T = T_c$. For the study of SOC models, it is vital to understand the difference because SOC systems are always critical, wherefore FSS is the only scaling available.

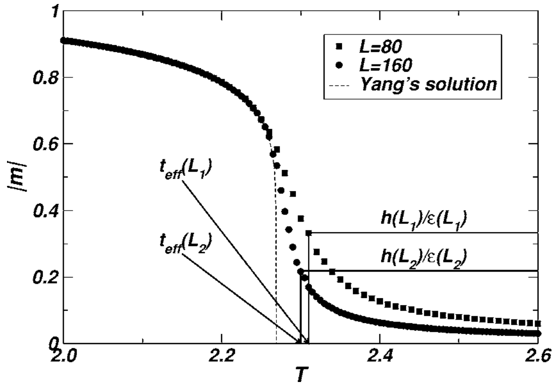

FIG. 1. Finite size behavior for the magnetization $|m|$ vs the temperature T in the regular, two-dimensional (2D) Ising model for $L_1 = 80$ (squares) and $L_2 = 160$ (circles). The dashed line shows Yang's solution [19]. In the dynamical Ising model, the cooling and heating rates, $h(L)$ and $\epsilon(L)$ prescribe the magnetization $|m(L)| = h(L)/\epsilon(L) \propto L^{\kappa-\omega}$, indicated by horizontal lines. The system is forced to move to an effective temperature $t_{\text{eff}}(L)$, indicated by arrows.

Around the critical point of a continuous phase transition, characterized by critical exponents α, β, γ, ν, etc., the singular part of the free energy leads to a simple scaling behavior of the magnetization density [18,27],

$$\langle |m| \rangle = -k_h L^{-\beta/\nu} Y'(k_t t L^{1/\nu}), \tag{4}$$

where t is the reduced temperature, negative in the low temperature phase (LTP) and positive in the high temperature phase (HTP), k_h and k_t are metric factors, and $Y'(x)$ is a universal scaling function, which becomes dependent on the boundary conditions and the geometry of the system in the limit of small arguments, case (5b) below. There are three qualitatively different (asymptotic) regimes

$$Y'(x) \to \begin{cases} \propto |x|^{\beta} & \text{for } x \to -\infty \quad \text{(a)} \\ \text{const} & \text{for } x \to 0 \quad \text{(b)} \\ \propto x^{-\gamma/2} & \text{for } x \to \infty \quad \text{(c)}, \end{cases} \tag{5}$$

where in the Ising model $\gamma = \nu d - 2\beta$.

The first line describes the asymptotic behavior of the magnetization in the LTP, the second line represents FSS, and the third line describes the HTP.

Setting $\langle |m| \rangle \propto L^{\kappa-\omega}$, the different regimes of (5) are accessed by three qualitatively different choices of $\kappa - \omega$, that is speeds at which $\langle |m| \rangle$ approaches zero:

(i) $\kappa - \omega > -\beta/\nu$ ("too slow"): In this case the magnetization approaches 0 slower than in a standard Ising model kept at temperature $T = T_c$ as the system size increases, so that $Y'[k_t t_{\text{eff}}(L) L^{1/\nu}] \propto L^{\kappa-\omega+\beta/\nu}$ is divergent in L. The only way to obtain a divergent $Y'(x)$ is via (5a), which requires a negatively divergent argument $x \to -\infty$. The effective temperature is therefore negative and scales like $|t_{\text{eff}}(L)|^{\beta} \propto L^{\kappa-\omega}$. Using $t_{\text{eff}}(L) \propto L^{-1/\mu}$ leads to

$$\mu = \beta/(\omega - \kappa) > \nu. \tag{6}$$

This implies that $t_{\text{eff}}(L)$ finally leaves the FSS region, whose width scales like $L^{-1/\nu}$, toward the LTP.

(ii) $\kappa-\omega=-\beta/\nu$ ("correct"): In this case $Y'(x)$ remains constant, so that its argument either remains constant or vanishes, according to (5b). Thus $t_{\text{eff}}(L)$ decays at least as fast as $L^{-1/\nu}$, *i.e.* $\mu \leq \nu$. To the order considered here the equality applies.

(iii) $\kappa-\omega<-\beta/\nu$ ("too fast"): $Y'(x)$ vanishes, following (5c). For $Y'[k_t t_{\text{eff}}(L)L^{1/\nu}] \propto L^{\kappa-\omega+\beta/\nu}$ and $t_{\text{eff}} \propto L^{-1/\mu}$ this implies

$$\mu = \frac{\gamma\nu}{2\nu(\kappa - \omega + \beta/\nu) + \gamma} > \nu, \tag{7}$$

provided that the denominator of Eq. (7) is positive. Hence the model leaves the FSS region toward the HTP. The special case of negative μ, implying divergent effective temperature, will be ignored.

Crucially, only for $\kappa-\omega=-\beta/\nu$ [case (ii)] does the model remain in the FSS region. To achieve this, the SOC system must tune h/ϵ exactly in the way the order parameter scales in a system displaying standard FSS while fixed at the critical temperature, $\langle |m| \rangle \propto L^{-\beta/\nu}$. In all other cases the scaling of the effective temperature eventually drives the model out of the FSS region: ξ/L vanishes in the thermodynamic limit. Nevertheless, $\langle T \rangle$ converges to T_c, so that the correlation length,

$$\xi \propto L^{\nu/\mu}, \tag{8}$$

diverges. With this scaling of ξ all observables will show standard finite-size scaling with ν replaced by μ [28].

To illustrate the above analysis we performed simulations of an Ising model with dynamics as described: Using Metropolis updating, the absolute magnetization density is calculated after each scan over the lattice. According to (2) a new temperature is then calculated to be used in the next sweep, $\dot{T}=-h+\langle |m| \rangle \epsilon$. Starting from $T=2.27$, systems of size $L=40,80\cdots 640$ were updated at least 10^6 times as transient and at least another 10^6 times for statistics.

Our numerical simulations fully confirm the above analysis: We observe the standard FSS exponents with ν replaced by μ for any reasonable choice of $\kappa-\omega$. The new scaling exponent μ (and T_c) can be determined from $\langle T \rangle - T_c \propto L^{-1/\mu}$. Using it in an FSS analysis allows us to identify all standard critical exponents.

For two reasons the method is very sensitive to the choice of ϵ_0 and h_0 in Eq. (3) [29]: Firstly, the amplitudes of the fluctuations in the effective temperature depend directly on h and ϵ; choosing h_0 and ϵ_0 too large, the system destabilizes. One can estimate these fluctuations by analyzing (2) and derive a lower bound for κ, ω. Secondly, if h and ϵ initially place the system close to T_c or in the wrong phase, the scaling function reaches its asymptotic behavior [(5a) or (5c)] only for very large system sizes.

Figure 2 shows the scaling of the effective temperature for the three qualitatively different choices of the driving exponents discussed above. The values of μ and γ derived from these data confirm the calculations. Depending on the choice of $\kappa-\omega$, the value of μ immediately determines either β, Eq. (6), or γ, Eq. (7). The FSS of specific heat and susceptibility produces the expected values of α/μ and γ/μ.

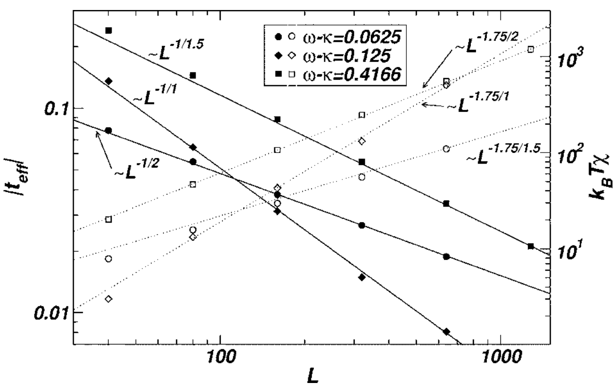

FIG. 2. The scaling of the effective temperature t_{eff} (filled symbols, full lines) and the susceptibility χ (open symbols, dashed lines) in the Ising model for different choices of the exponent κ and $\omega=1$. The symbols are numerical simulations, the lines show the slopes expected from theory.

Since our interest in the AS approach is due to its proposed role as an explanation for SOC, we repeated the analysis for a variant of the two-dimensional Abelian Manna model. This sandpilelike model has been used to exemplify the link between SOC and AS [11]. It is driven in the bulk and implements bulk dissipation, as suggested by (1). We were able to confirm the key equation (4), using the closed model values of $\beta=0.64(1)$ and $\nu=0.82(3)$ [11] and measuring $\gamma \approx 0.34$. We then obtained μ from (6) and (7). Unlike in the Ising model, the Manna model can get stuck when hitting an inactive state. This complicates the measurements especially of the susceptibilty in the third case discussed above. For all other cases the asymptotic behavior of $\rho_a \propto L^{-\beta/\mu}$ and $\chi \propto L^{\gamma/\mu}$ was reached for system sizes smaller than $L=1000$. In the AS approach also the avalanche size exponents show a clear, immediate dependence on the choice of the two exponents κ and ω.

The present study shows that the proposed explanation of SOC as "self-organized" AS criticality [8–10] needs to be supplemented with a mechanism that necessitates the "correct" value of $\omega-\kappa$, namely β/ν, in order to explain universality. The identification of this mechanism, which amounts to an explanation of SOC, should be the subject of future research.

The question whether or not SOC models have universal features is very important. Universality is a major justification for studying simple models and for disregarding the details of the physical processes they describe. Despite the importance of this issue, it is still unclear whether SOC systems can be grouped into universality classes; in fact, exponents can change due to small changes in the update rules (e.g., [20,21]), and SOC is notorious for its wide variety of critical exponents. Accepting the AS approach, this would be a consequence of implicitly setting the scaling of external drive and dissipation by the dynamical rules of the different models.

However, there is strong evidence in favor of universality in SOC. Many changes of the detailed dynamics do not affect the critical behavior [22–25]. Moreover, the ratio ξ/L appears to remain constant in direct measurements of some models [26] so that the "correct" FSS exponents are observed, which is in stark contrast to (8); similarly, simulations suggest that indeed $\omega-\kappa=\beta/\nu$.

Observations of the same exponents in SOC and AS models (such as [11,12,14]) suggest a link between the two. While the AS approach does establish a link between SOC and AS, it does not explain this identity. Quite the contrary, without further details the AS approach predicts almost surely [apart from case (ii)] a difference in exponents. Our investigation demonstrates clearly that the AS approach is incomplete and universality in SOC still needs an explanation. The key question is as follows: What forces the driving and dissipation rates to scale precisely so as to produce universality?

We have calculated the FSS behavior of a system approaching its critical point through a feedback mechanism between order parameter and tuning parameter. It works in the sense that it drives the models to the critical point and FSS is observed. Yet, ξ/L vanishes in the thermodynamic limit and the observed "critical behavior" is not universal. While scale-free distributions of responses such as those observed in the case of rainfall [3] or earthquakes [4], can be produced by such a process, it only yields critical behavior strongly dependent on the detailed dynamical rules of SOC models. There would be no universality and robustness against small changes in the dynamical rules. While the AS mechanism in its present form may produce further insight into potentially nonuniversal critical phenomena as observed in field experiments, it fails to explain the apparent universality of SOC models and is thus more limited than was previously thought.

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[27] For the remainder of this paper β will denote a critical exponent and is not to be confused with the inverse temperature. Also γ and ν denote the standard critical exponents, and d is the dimensionality of the lattice.
[28] To be precise, ξ enters the standard FSS equations as $L^{\nu/\mu}$, which leads for example to $\chi \propto L^{\gamma/\mu}$, $c_v \propto L^{\alpha/\mu}$, but also to $\langle |m| \rangle \propto L^{\kappa-\omega}$. However $\kappa-\omega \neq -\beta/\mu$ in the third case discussed above.
[29] As the dynamics do not provide a natural time scale, a rescaling of these quantities corresponds to a rescaling of time. In the limit of very slow dynamics the situation of a fixed-temperature simulation is recovered.