DIRECTIONALITY EFFECTS IN PERCOLATION

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ABSTRACT

The percolation properties of random networks containing resistors (two-way streets) and/or diodes (one-way streets) are considered. The directionality constraints of the diodes are found to lead to novel geometrical behaviour. As a simple example, various random cluster models with a preferred axis, such as directed random walks or directed lattice animals, are shown to be anisotropic in character. The critical behavior of directed percolation is then treated and the connection with branching Markhov processes is explained. A closely related "reverse" percolation problem, a transition from a one-way percolation to isotropic percolation, is introduced. Finally, the geometrical properties of a network containing arbitrarily oriented diodes is treated. Symmetry and duality arguments are applied to yield exact results for certain aspects of its critical behaviour.

1. Introduction

Although the classical percolation problem has been extensively studied over the past decade (see, e.g., Stauffer 1979, Essam 1980, Domb 1983, Hammersley 1983 for reviews and comprehensive references), there are still many features of the model which have yet to be addressed. In particular, several interesting and relatively unexplored variations of percolation have been proposed recently both from the fundamental and applications point of view. In this paper, we discuss a

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particular generalization in which diodes, which connect or permit transport in only one direction along their length, are incorporated into the random network (Fig. 1). In contrast, the conventional bonds of isotropic percolation permit transport in either direction, thus acting as resistors. The directionality constraint gives rise to a broader range of critical phenomena than in classical percolation (see, e.g., Kinzel 1983, Redner 1983, and references therein). In particular, if the diodes have a preferred orientation, a random network exhibits anisotropic critical behaviour. Furthermore, when the orientation of the diodes is varied, a novel orientational percolation transition occurs.

This paper describes recent work aimed at understanding these new phenomena. First, some simple phenomenological results for directed walks and lattice animals are outlined to illustrate the effect of a global bias on geometrical models. We then introduce several resistor-diode percolation models of increasing degrees of generality and complexity. The nature of the directed percolation transition and a "reverse" transition--from one-way to two-way percolation--are then elucidated. The position-space renormalization group is used to map out the phase diagram of these and more general resistor-diode percolation models and calculate critical properties. Finally, symmetry and duality are exploited to derive exact results for several critical properties of resistor-diode networks.

2. Simple Phenomenology of Biased Systems

As an example of the effects of a preferred orientation on a geometrical model, consider a discrete random walker which is biased to move along one axis. For any bias, no matter how small, the average shape of the random walk trajectory is fundamentally changed from the isotropic case (Fig. 2(a)). With no bias, the root-mean-square displacement scales as N^{ν}, where N is the number of steps in the walk and the correlation length exponent ν has the value 1/2 for any spatial dimension d. On the other hand, the biased walk has an average longitudinal displacement, $\langle \xi_{\parallel} \rangle$, varying as $N^{\nu_{\parallel}}$ with $\nu_{\parallel} = 1$, and an average transverse width $\langle \xi_{\perp} \rangle$ which varies as $N^{\nu_{\perp}}$ with $\nu_{\perp} = 1/2$, again valid for all d.

For a self-avoiding walk, defined as a random walk in which each site may not be visited more than once, a similar effect occurs. When no bias is present, an excellent approximation for ν is the Flory formula, $\nu = 3/(d+2)$, valid for $1 \leq d \leq 4$ (see e.g., de Gennes 1979 and references therein). An infinite bias may be represented by a directed self-avoiding walk model which can be solved by a transfer-matrix method (in two dimensions only). For any dimension, however, one may derive $\nu_{\parallel} = 1$ and $\nu_{\perp} = 1/2$. Scaling arguments indicate that these exponent values will also hold for any finite, but non-zero value of the bias. The coincidence of these exponents with those of directed random walks stems from the fact that for directed self-avoiding walks the excluded-volume effect is no longer fully operative.

More interesting results are obtained for randomly branched structures or lattice animals (Fig. 2(c)). Early work showed that for an isotropic model with no excluded volume interaction, $\nu = 1/4$ for all d (Zimm and Stockmayer 1949). Once a bias is introduced, heuristic arguments suggest $\nu_{\parallel} = 1/2$ and $\nu_{\perp} = 1/4$, again valid for all d (Redner and Yang 1982, Day and Lubensky 1982). With excluded-volume interactions, a Flory-type theory for an isotropic model gives $\nu = 5/2(d+2)$, valid for $d \leq 8$ (Issacson and Lubensky 1981). A similar approach for directed animals yields $\nu_{\parallel} = (d+11)/4(d+2)$ and $\nu_{\perp} = 9/4(d+2)$ for $d \leq 7$ (Redner and Coniglio 1982, Lubensky and Vannimenus 1982). Notice the resemblance between the directed animal in Fig. 2(c) and the shapes of trees in a forest, or the topography of river networks; these two connections have been noted in the literature (Day and Lubensky 1982, Green and Moore 1982). Finally, the Flory method is easily extended to directed percolation, yielding $\nu_{\parallel} = (d+9)/4(d+2)$ and $\nu_{\perp} = 7/4(d+2)$, valid for

$d \leq 5$.

These results for $\nu_{\parallel}$ and $\nu_{\perp}$ are in excellent agreement with all numerical work, in spite of the extreme simplicity of the Flory approach. The largest discrepancy is in two dimensions, where the generating function for directed animals can be calculated exactly (Dhar et al 1982, Nadal et al 1982, Dhar 1982a, Hakim and Nadal 1983). From these analyses, it is possible to derive $\nu_{\perp} = 1/2$, but nothing for $\nu_{\parallel}$.

In summary, a preferred orientation strongly affects the large-scale behaviour of a wide variety of geometrical models. There is preferential growth along the special axis, and a corresponding transverse shrinking of the structure. This anisotropic behaviour can be realized, for example, by diffusion with biased hopping rates (see e.g., Stephen 1981, Barma and Dhar 1983) or by a purely geometrical problem such as directed animals.

3. Resistor-Diode Percolation Models

The bias discussed in the previous section may also be realized by directed percolation. On the square lattice, for example, each lattice edge is randomly occupied by a bond constrained to "point" upward (if it is vertical) or to the right (if it is horizontal). Such an orientational constaint leads to an anisotropic percolation transition. This behaviour and the critical properties of more general networks with arbitrarily oriented diodes are the focus of the remainder of this article.

We therefore introduce a general resistor-diode percolation model which contains directed percolation as a special case. For concreteness, consider again the square lattice whose edges may be occupied by the following bond elements: positive diodes which point either upward or to the right, negative diodes which point in the opposite directions, or resistors (Fig. 3). The occupation probabilities for each of these elements are defined to be p_+, p_- and p

respectively, while each bond may be vacant with probability $q \equiv 1 - p - p_+ - p_-$. We shall discuss the features of particular cases of the general model, in systematically increasing order of generality and complexity. These are:

(i) $p, q \neq 0$ and $p_{\pm} = 0$: classical isotropic percolation.

(ii) $p_+, q \neq 0$ and $p_- = p = 0$: directed percolation

(iii) $p_+, p \neq 0$ and $p_- = q = 0$: "reverse" percolation.

(iv) $p_+, p, q \neq 0$ and $p_- = 0$: an "oriented" resistor diode network containing resistors and diodes of one orientation.

(v) $p_{\pm}, p, q \neq 0$: a "random" resistor-diode network containing resistors and diodes of arbitrary orientation.

(i) Isotropic Percolation: As the concentration of resistors is increased to a critical value, the network undergoes a transition from a non-percolating to an isotropically percolating phase. If a point fluid source is introduced into a percolating network, the fluid will isotropically spread and "wet" a finite fraction of the lattice.

(ii) Directed Percolation: This model exhibits a transition from one-way percolation to no percolation as the concentration of positive diodes is decreased below a critical threshold value (Fig. 4). For $p_+ = 1$, a fluid source at the origin wets the entire first quadrant of the lattice. As p_+ is decreased and q is correspondingly increased, the opening angle ϕ of this wetted region begins to shrink. Near the transition ϕ vanishes as $\xi_\perp / \xi_\parallel$, and fluid flow propogates predominantly along the diagonal. Finally, below the transition, only finite-sized clusters occur. They become increasingly anisotropic in shape as the threshold is approached from below, as discussed in section 2.

There is an extremely interesting connection between directed percolation and branching Markov processes which are useful in describing diffusion-reaction processes (see, e.g., Schlögl 1972, Nicolis and Prigogine 1977, Grassberger and de le Torre 1979, Griffeath 1979). For the anisotropically-shaped cluster at the bottom of the figure, imagine that the diagonal is a "time" axis while one spatial dimension is defined by an axis transverse to the time. Then the equilibrium cluster configuration is equivalent to the trajectory of a diffusing-reacting particle system in one space and one time dimension, where particles may diffuse, split ($\uparrow_{\rightarrow}$), recombine ($^{\rightarrow}\!\uparrow$), or "die". The reaction rates for each of these processes may be arbitrary, and directed percolation is obtained when the rates are chosen to correspond to random bond occupation probabilities (Kinzel 1983). The lifetime of the diffusing particles is equivalent to $\xi_{\parallel}$ in directed percolation, while the spatial extent of the population is equivalent to $\xi_{\perp}$.

(iii) <u>Reverse Percolation</u>: This is a transition from one-way to two-way percolation as the concentration of resistors increases to a critical value. Starting from the top right of Fig. 4, the opening angle defining the region of wetted sites increases as p_+ decreases and the concentration of resistors increases. Due to the presence of the latter circuit element, fluid introduced at the origin may enter the second and the fourth quadrants. As the transition is approached, ϕ increases to π and when this angle is surpassed, the fluid wets all lattice sites.

In two dimensions, directed and reverse percolation are related by duality (Dhar et al 1981, Redner 1982a). Therefore the manner in which $\phi \longrightarrow \pi$ in the reverse transition is identical to the way in which $\phi \longrightarrow 0$ for directed percolation. This equivalence may also be used to derive very accurate rigourous bounds for the critical concentrations of the two models (Dhar 1982b) and qualitatively describe the phase diagram of the network.

(iv) <u>Oriented Resistor-Diode Network</u>: This network displays features from both the directed and reverse transitions, as well as new behaviour where the two transitions coalesce. The phase diagram of the network may be described conveniently by a triangle in the composition space spanned by p, q, and p_+ (Fig. 5). This triangle represents the intersection of the half-spaces $p,q,p_+ \leq 1$, with the plane $p + q + p_+ = 1$. Each corner of the triangle corresponds to a lattice completely filled with one type of bond element, while an interior point corresponds to a lattice with all three elements present. For such a point, the relative concentration of the i^{th} element is given by the perpendicular distance from the point to the edge opposite the i^{th} corner of the triangle. Directed percolation corresponds to the left edge of the triangle where p_+ and q are non-zero but p is zero, while the right edge of the triangle describes the reverse problem where p_+ and p are non-zero, while q is zero.

A particularly simple, but powerful method to study this model is the position-space renormalization group (see e.g., Reynolds et al 1977, 1980, and Stanley et al 1982 for a review). This technique was described for isotropic percolation in an earlier article of this volume. For models with directed bonds, the renormalization group can be readily generalized by calculating the probability of traversing a finite cell from bottom to top and vice-versa, and the probability of traversing the cell in one sense only. These probabilities define the recursion relations for p' and p'_+ resectively, the renormalized probabilities that a cell maps to a resistor in the former case, or to a diode in the latter case.

From these recursion relations, the phase diagram of figure 5 is obtained. Two second-order phase transition lines divide the area into three phases characterized by isotropic percolation, one-way percolation, or no percolation. It is useful to think of these situations in terms of forward or reverse conductances, G_+ or G_- respectively, being zero or non-zero. The second-order line on the left is the directed transition where G_+ vanishes with a power-law singularity, while G_- remains zero. The other line is the reverse tranition where G_+ is finite and varies smoothly as the line is crossed, while G_- vanishes with a power-law singularity.

These two lines meet at the isotropic percolation threshold where G_+ and G_- both vanish. In the theory of critical phenomena, this is a tricritical point as three phases in the system are becoming simultaneously critical (see e.g., Pfeuty and Toulouse 1977). The values of the critical exponents at this point are distinct from the exponents measured anywhere along the two transition lines.

In addition to calculating geometrical exponents, a network conductivity problem in which the I-V response of each bond is a general asymmetric function of V may be treated (Redner 1982b). Two natural choices for this response are $G_+ \neq 0$ and $G_- = 0$, or $G_+ = \infty$ and $G_- \neq 0$, but finite. A wide variety of interesting conductivity properties may be derived.

While the renormalization group method has proved to be of great utility for obtaining global information, the method suffers from the basic flaw that the rescaling procedure is intrinsically isotropic, while the network is geometrically anisotropic along the second-order lines. The construction of a rescaling that correctly treats the anisotropy and gives accurate estimates for both critical exponents, $\nu_\parallel$ and $\nu_\perp$, is still an open problem.

(v) Random Resistor-Diode Network: When negative diodes are included, the three-component triangle from above generalizes to a four-component tetrahedron. The renormalization group predicts a very symmetric phase diagram (Fig. 6), with a broad range of geometrical transition phenomena. To gain an intuition for the phase diagram, it is helpful to focus on the multicritical line that joins the point marked I to the point marked RM. The former is the isotropic percolation threshold where $p = q = 1/2$ and $p_\pm = 0$, and the latter is "random Manhattan", where $p_\pm = 1/2$ and $p = q = 0$. (This would be the state of midtown Manhattan if one-way street signs are assigned random directions for every block.) This line represents the intersection of the reflection symmetry plane with a dual symmetry plane. The first plane is defined by equal concentrations of positive and negative diodes ($p_+ = p_-$), while the second is defined by equal concentrations of resistors and vacancies ($p = q$).

Emanating upward from the multicritical line are two second-order surfaces which enclose a wedge-shaped volume. These two surfaces are the continuations of the second-order lines of figure 5 into the interior of the tetrahedron. The enclosed volume defines the positive diode phase region of the diagram. An identical structure below the reflection symmetry plane encloses a second volume which defines the negative diode region. These four second-order surfaces commonly meet along the multicritical line, and symmetrically divide the tetrahdron into the positive diode, negative diode, resistor and non-percolating phases.

To help visualize this, consider a two-dimensional projection of the tetrahedron onto a plane perpendicular to the multicritical line (Fig. 7). The four second-order surfaces collapse into an X-shaped structure that separates the regions of no percolation (0), two-way percolation (+ -), or one-way percolation (either + or -). Moving along a horizontal path in this figure corresponds to exchanging resistor with vacancies, while moving along a vertical path corresponds to exchanging positive and negative diodes. Crossing two of the lines yields a directed transition (0 to +, or 0 to -), while crossing the other two causes a reverse transition (+ to + - or - to + -). The point at the center is a multicritical, fourth-order transition. When this point is approached, there is new critical behaviour which is distinct from that of the directed and reverse transitions. The renormalization group predicts that the exponents at this point, and hence along the entire multicritical line, are just those of isotropic percolation. Thus the critical behaviour of random Manhattan is identical to that of isotropic percolation. This striking result is a partial consequence of the duality arguments that follow.

4. Duality for Resistor-Diode Percolation

The dual transformation is an extremely simple but useful tool for deducing exact critical probabilities of a variety of two-dimensional percolation models (see e.g., Essam 1972 and references therein). In this section, we indicate how these arguments can be generalized to random resistor-diode networks.

As a preliminary, we outline duality for isotropic percolation on the square lattice. This hinges on a one-to-one mapping between a cluster configuration on a lattice $\mathcal{L}$ and a closely related configuration on the dual lattice $\mathcal{L}^D$. The mapping is defined by every occupied bond in $\mathcal{L}$ (heavy line) being replaced by an empty bond in $\mathcal{L}^D$ (dashed line), and vice-versa (Fig. 8). For each bond in $\mathcal{L}$, the corresponding dual bond is placed perpendicular and midway across the original one. The top half of figure 8 shows that a percolating configuration on $\mathcal{L}$ maps into a non-percolating configuation on $\mathcal{L}^D$, and vice-versa. Because of this fact, the critical concentration on the original lattice, p_c, must equal $1 - p_c^D$. Since the square lattice is self-dual by construction, it immediately follows that $p_c = 1 - p_c$, or $p_c = 1/2$.

This mapping can be extended to include networks with diodes (Dhar et al 1981, Redner 1982a). The rule for transforming resistors and vacancies is kept unchanged, but a diode on $\mathcal{L}$ is replaced by dual diode on $\mathcal{L}^D$ which is rotated by $\pi/2$ clockwise with respect to the original. If the "time" axis on $\mathcal{L}$ is defined to be the upper-right diagonal, then the time axis on $\mathcal{L}^D$ points to the lower right. The dual mapping is therefore defined by $p \longleftrightarrow q$ and $p_{\pm} \longleftrightarrow p_{\mp}$.

Under this extended duality, a non-percolating configuration maps to a two-way percolating configuration just as in pure percolation. However, consider the effect of this dual transformation on a one-way percolating configuration (bottom row of Fig. 8). On the left a simple one-way configuration is shown, and its dual counterpart is shown on the right. Since vacancies map to resistors, the empty space on the right-hand side should be completely occupied by resistors. They have not be drawn to illustrate that the one-way path on $\mathcal{L}$ has mapped to a one-way "barrier" on $\mathcal{L}^D$. There may be transport across the barrier to the lower right, but not to the upper left. Thus a one-way percolating configuration maps to another one-way configuration of the same "temporal" sense.

This result implies that the phase diagram must be symmetric across the self-dual plane p = q (the vertical dashed line in figure 7). Therefore the intersection of the self-dual with the reflection symmetry plane (the horizontal dashed line in figure 7), must be a locus of percolation transitions. Along this line the critical concentration of resistors is given by $p_c = 1/2 - p_+$. This generalizes the result $p_c = 1/2$ for isotropic percolation, to the random resistor-diode network.

Furthermore, the use of duality together with exact results for the functional dependence of the pair-connnectedness function shows that the critical exponents anywhere along this line are just those of isotropic percolation (Redner 1982c). This establishes the equivalence between the critical properties of isotropic percolation and random Manhattan.

5. Concluding Remarks

The study of random media has been generally confined to situations where symmetry with respect to reversal of direction holds. Such a system may be described by a random network of resistors. However, when directionality constraints, such as diodes, are introduced, novel geometrical and transport properties result. In the special case where the diodes have an overall orientation, there may be a transition from no percolation to one-way percolation, and also a transition from one-way to two-way percolation. The former is anisotropic in character, and it may be accurately described by a simple Flory-type theory. There is considerably less quantitative information concering the properties of the latter transition.

A random network containing resistors and arbitrarily oriented diodes displays a wealth of geometrical transition phenomena which may be treated qualitatively by a renormalization group approach. In addition, duality arguments can be formulated to derive exact results for the critical concentration of the network. While some global information has been obtained, there is much less quantitative information available. This may be a promising area for future investigations by rigourous mathematical methods, and by the numerical tools of statistical mechanics such as Monte Carlo and series expansions.

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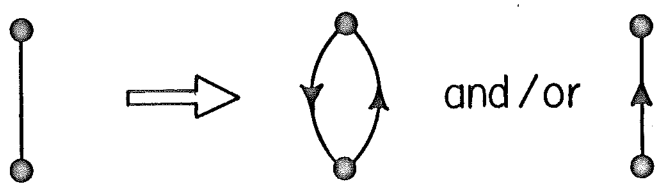
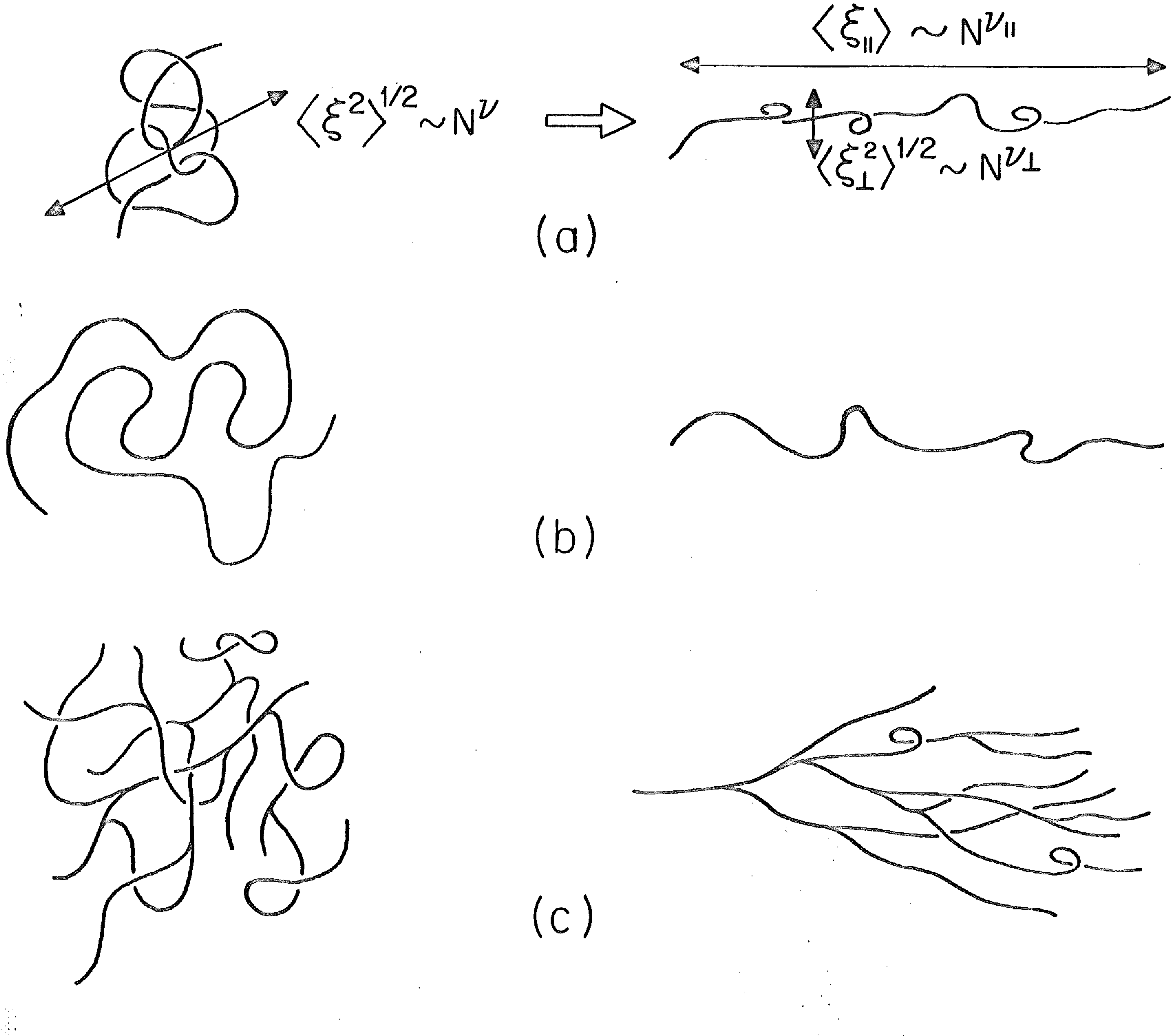


Fig 1.



$\langle \xi^2 \rangle^{1/2} \sim N^{\nu}$
$\langle \xi_{\parallel} \rangle \sim N^{\nu_{\parallel}}$
$\langle \xi_{\perp}^2 \rangle^{1/2} \sim N^{\nu_{\perp}}$
(a)
(b)
(c)

Fig 2

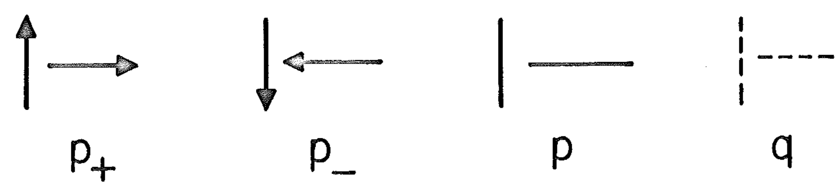


Fig 3

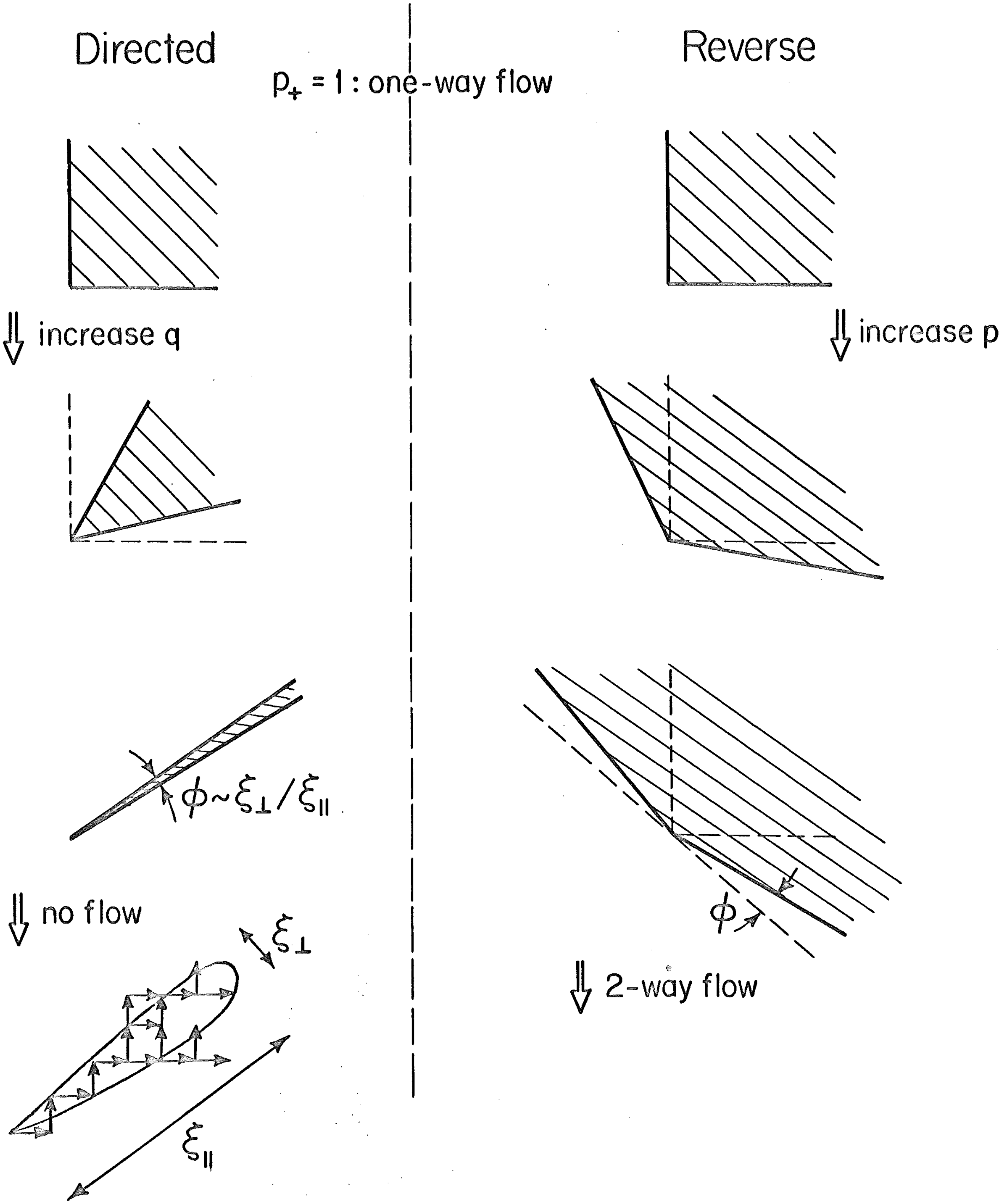

Directed
Reverse
p+ = 1 : one-way flow
increase q
increase p
φ~ξ⊥/ξ∥
no flow
ξ⊥
ξ∥
φ
2-way flow

Fig 4

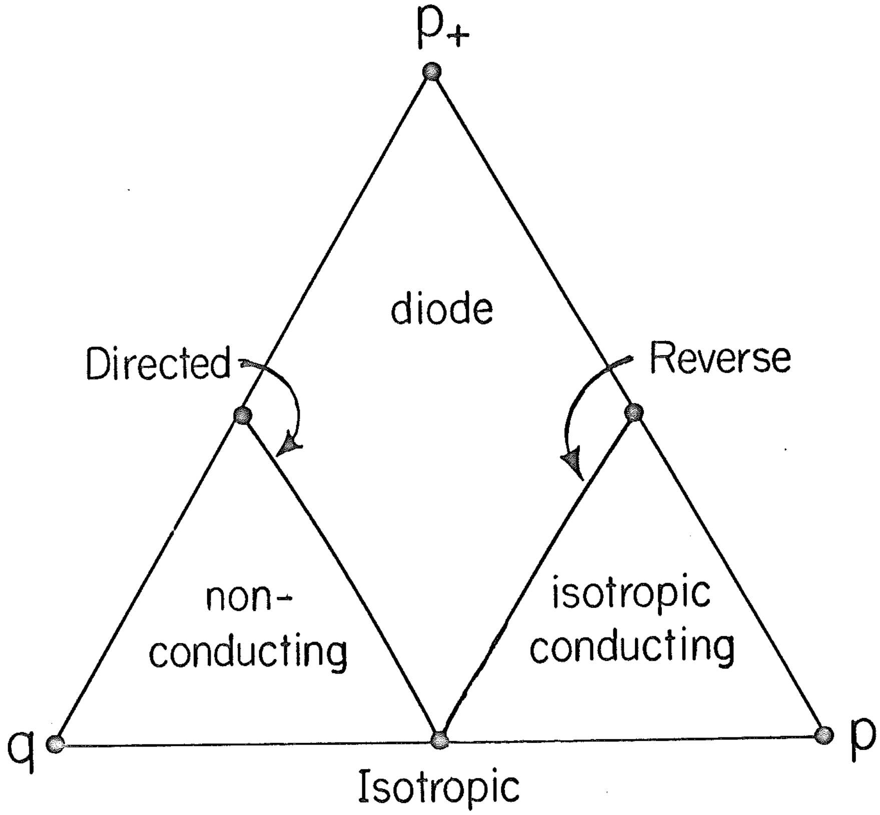

p+
diode
Directed
Reverse
non-
conducting
isotropic
conducting
q
p
Isotropic

Fig 5

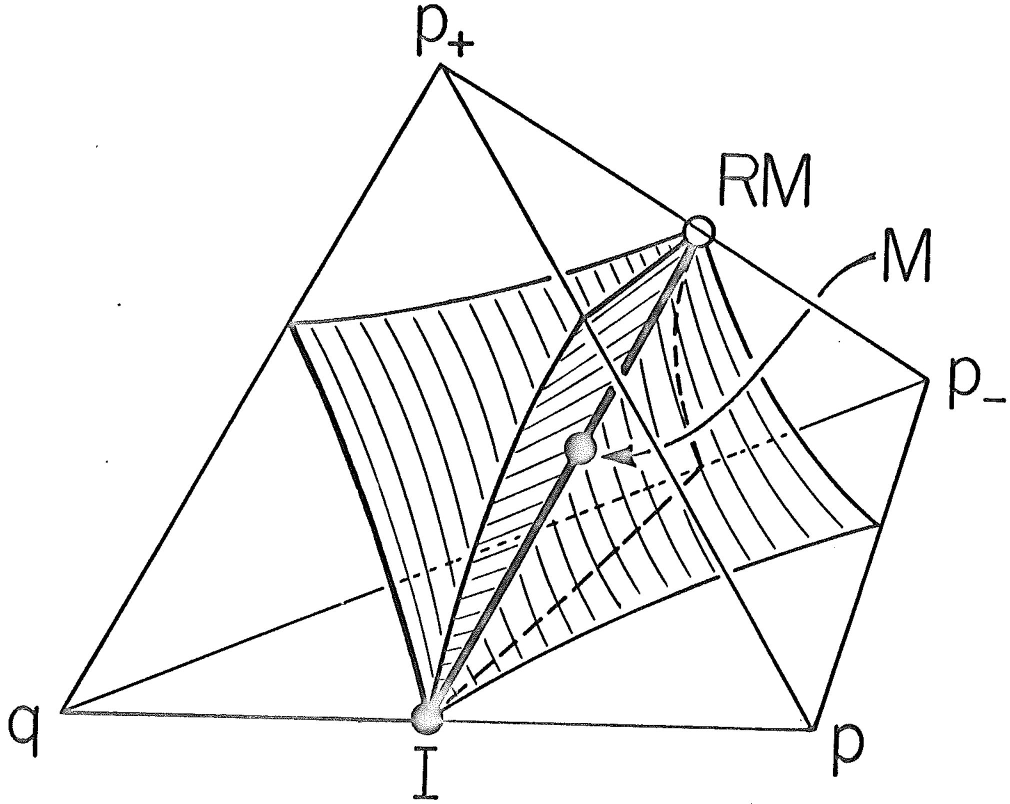


Fig 6

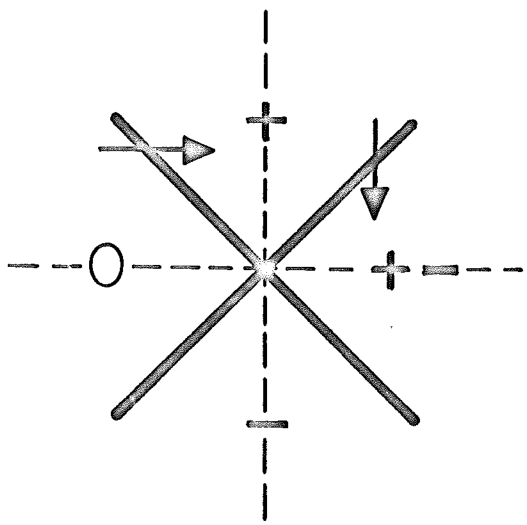
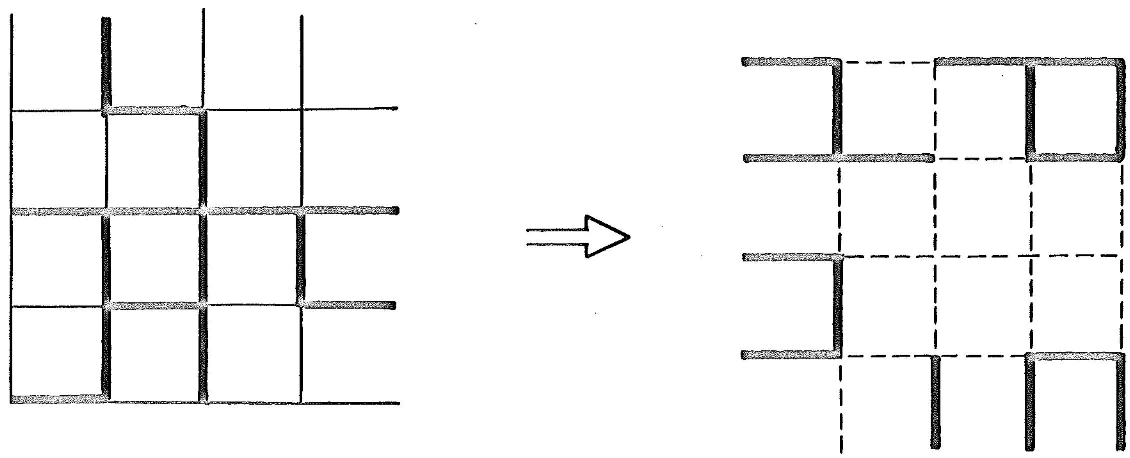


Fig 7



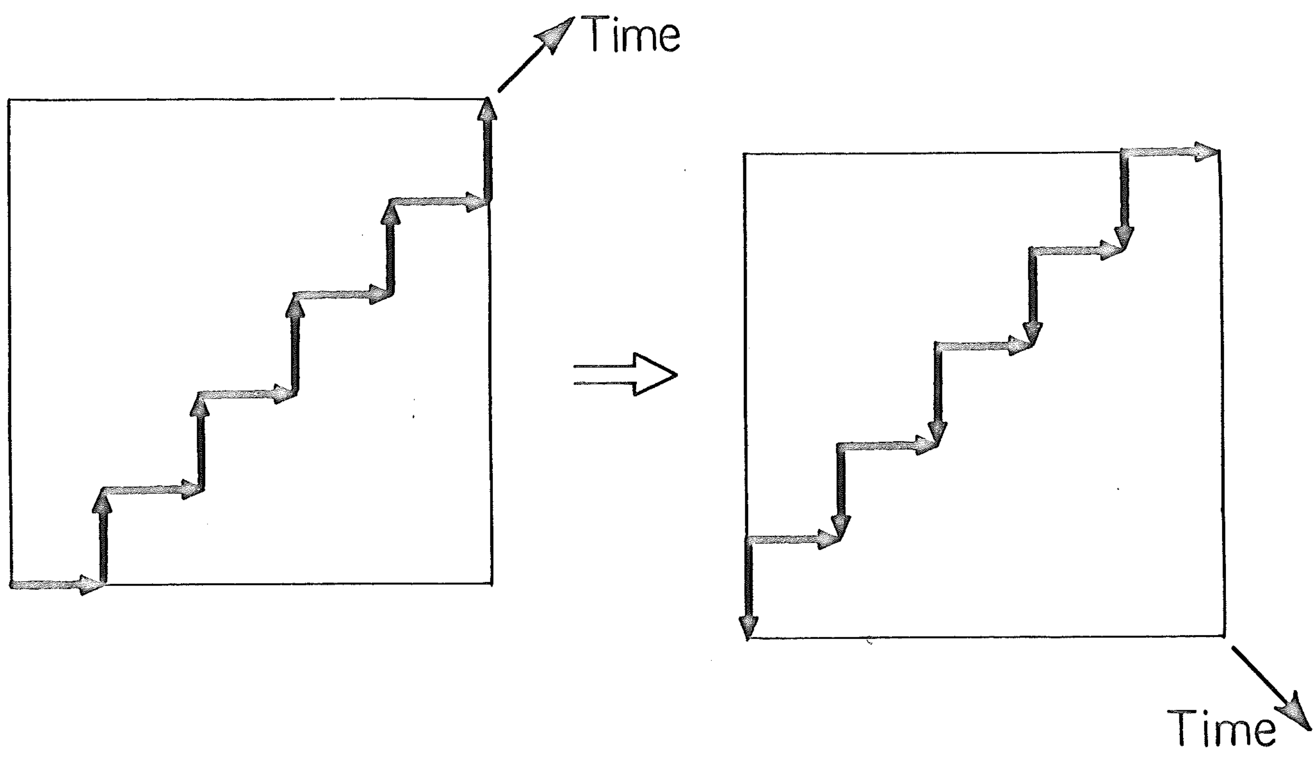


Fig 8