Synchronization and coarsening (without self-organized criticality) in a forest-fire model

K. E. Chan,* P. L. Krapivsky, and S. Redner
Center for BioDynamics, Center for Polymer Studies, and Department of Physics, Boston University, Boston, Massachusetts 02215
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We study the long-time dynamics of a forest-fire model with deterministic tree growth and instantaneous burning of entire forests by stochastic lightning strikes. Asymptotically the system organizes into a coarsening self-similar mosaic of synchronized patches within which trees regrow and burn simultaneously. We show that the average patch length $\langle L\rangle$ grows linearly with time as $t\to\infty$. The number density of patches of length L, $N(L,t)$, scales as $\langle L\rangle^{-2}\mathcal{N}(L/\langle L\rangle)$, and within a mean-field rate equation description we find that this scaling function decays as $\mathcal{N}(x)\sim e^{-1/x}$ for $x\to 0$, and as e^{-x} for $x\to\infty$. In one dimension, we develop an event-driven cluster algorithm to study the asymptotic behavior of large systems. Our numerical results are consistent with mean-field predictions for patch coarsening.

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I. INTRODUCTION

Forest-fire models [1–4] are simple archetypical examples of driven dissipative systems that exhibit intriguingly rich spatiotemporal structures [4–9]. These models provide a simple paradigm for cooperative time-dependent phenomena, such as epidemics, oscillatory chemical reactions, electrical neuron activity, cardiac dynamics, and turbulence [2,10–13]. As the name indicates, the forest-fire model ostensibly describes the time evolution of burning trees in a forest. In typical models of this genre, trees are located at regular lattice sites and each can exist in one of three states: burnt, alive, or burning. The dynamics involves the following elements: (i) A burnt tree turns into a living tree at some specified rate; (ii) a living tree can be ignited, either by a lightning strike or by fire spreading from a neighboring burning tree; (iii) after a specified time interval a burning tree is consumed and the fire at this location is extinguished.

Depending on which of these processes are operative, as well as their relative rates, different dynamical behaviors can arise, ranging from self-organized critical behavior with fires of all sizes occurring [2–4], to spiral fire-front wave propagation [2,14]. While forest-fire models have been extensively investigated, there is still uncertainty about their long-time properties even after extensive numerical simulations in many realizations of the model [9].

This work is focused on a specific version of the forest-fire model which exhibits coarsening [15] rather than self-organized criticality or complex fire front propagation. Because of this phenomenological simplicity, we can apply the rate equations in a natural way to determine the evolution of the system. The model itself was first introduced by Drossel [16]. Its crucial feature is that tree growth is deterministic; a tree that has just burned remains dead for exactly one time unit and then a new tree reappears. This particular regrowth rule is the mechanism that gives rise to a coarsening mosaic of growing synchronized forests. This is a generic feature and does not require the tuning of model parameters to critical values.

Let us define a "patch" as a coherent region of the system that is either occupied by live trees or by burnt trees. This patch evolves by tree regrowth and by the burning of trees due to lightning strikes. The mechanism for coarsening is that adjacent patches must eventually synchronize, after which they evolve in phase [16]. While neighboring patches begin their existence as distinct, eventually the burnt trees in one patch will regenerate while the adjacent patch is still forested. When this occurs, all the trees in these two patches become incorporated into an augmented patch which then evolves as a single unit. (Fig. 1).

In Ref. [16], the process was investigated numerically and the total number of patches $N(t)$ was found to decay with time. However, more quantitative observations were not reported. We will show that $N(t)\propto t^{-1}$ and we will investigate the patch length distribution, both analytically and numerically. For our analytic study, we will employ the classical rate equation of aggregation kinetics [17]. This approach is ideally suited to treat the coarsening behavior of the system under investigation.

Another important feature of our approach is that we treat the dynamics at a mesoscopic level in which the basic units

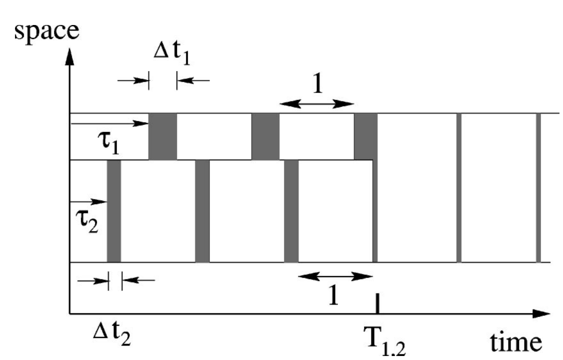

FIG. 1. Illustration of the merging of two adjacent patches of lengths L_1 and $L_2>L_1$. White space indicates a burned patch while the shaded region indicates a forested patch. The times until the first regrowth events are τ_1 and τ_2. Each forested patch survives for a time $\Delta t_j\propto L_j^{-1}$ until lightning strikes and instantaneous burning occurs. The patch then regrows after exactly one time unit elapses. Because Δt_1 and Δt_2 are different, the two patches will eventually synchronize at the joining time $T_{1,2}$.

*Present address: Siebel Systems Inc., San Mateo, CA.

are synchronized patches rather than individual trees. In the long-time limit, we will argue that the lifetime of such patches can be viewed as deterministic. By tracking only the merging events of adjacent patches we are able to investigate systems of effectively much larger size and to much longer times than those accessible by tree-based simulations. In addition to obtaining a power-law growth of the average patch length, our method yields clean results for the probability distribution of patch lengths. This distribution is found to obey scaling, with no memory of the initial state retained and with the asymptotes of the distribution in good agreement with rate equation predictions. While we focus on the particular case of one dimension, our approach should also apply in higher dimensions.

In the next section, we define the model and outline the effective mesoscopic picture for the evolution of patches. In Sec. III, a rate equation description for this evolution is presented and basic results about the patch length distribution are obtained. We explain our simulational approach and describe the results that follow in Sec. IV. Our basic conclusions are given in the last section.

II. FOREST EVOLUTION IN ONE DIMENSION

The evolution of the system is governed by the competition between two fundamental time scales. Suppose that each tree in the system may be struck by lightning at a rate Λ. Then a forest of length L has a characteristic lifetime $t_{\rm occ} \sim (\Lambda L)^{-1}$ before one of its trees is struck by lightning. We assume that the time to burn the forest completely is much less than any other time scale in the problem, so that we view the burning of a forest as instantaneous. There is also the deterministic time interval t_{emp} between the instant that a forest burns down and the reappearance of trees in this burnt patch. We assume that this refractory time is the same for all trees, so that regrowth of trees in a single burnt patch occurs simultaneously. Without loss of generality, we take this refractory time to be $t_{emp}=1$.

At early times, where $t_{emp} \leqslant t_{\rm occ}$, the stochastic nature of lightning events is important. However, as we shall soon show, patches naturally grow with time. Thus t_{emp} eventually becomes much larger than $t_{\rm occ}$ and the fire dynamics becomes nearly deterministic in the long-time limit. To describe this late-stage dynamics, we ignore individual trees and treat the system mesoscopically as a contiguous array of patches, each with length L_j. Each patch can either be forested or burnt. If two distinct but adjoining forested patches arise by the regrowth of one patch next to a forested patch, they immediately join to form a larger patch (Fig. 1).

At long times, patches have a small lifetime and they are almost always in the burnt state. Without loss of generality we initialize the system so that it is effectively in this long-time state. That is, at $t=0$, we consider all patches to be burnt, and we define τ_j (with $0<\tau_j<1$) to be the time at which the jth burnt patch first becomes a forest. Consider now two adjacent patches, and let $\phi_j=\tau_{j+1}-\tau_j$ be the time difference between the appearance of forest j and forest $j+1$. After these two forests undergo one cycle of regrowth and subsequent burning, ϕ_j changes by $\Delta t_{j+1}-\Delta t_j$, where $\Delta t_j=(\Lambda L_j)^{-1}$ is the lifetime of the jth forest. This shift in the difference of burning times continues until ϕ_j reaches either $\phi_j=0$ or $\phi_j=1$. When this occurs, the forests necessarily join and are subsequently synchronized (Fig. 1).

Since lightning strikes a forest of length L_j at rate ΛL_j, the lifetime Δt_j of each forest is a stochastic variable whose average value is $\langle \Delta t_j\rangle=(\Lambda L_j)^{-1}$. In the long-time limit, these lifetimes become very small. Therefore we may replace the sum of a large number of these lifetimes by the average lifetime times the number of cycles. Thus for two adjacent patches j and $j+1$ that satisfy $L_j<L_{j+1}$ and $\tau_j<\tau_{j+1}$, the joining time $T_{j,j+1}$ is

$$T_{j,j+1}=\begin{cases} \dfrac{\tau_{j+1}-\tau_j}{\langle \Delta t_{j+1}\rangle-\langle \Delta t_j\rangle}, & \tau_j<\tau_{j+1}, \\ \dfrac{1-\tau_j+\tau_{j+1}}{\langle \Delta t_j\rangle-\langle \Delta t_{j+1}\rangle}, & \tau_j>\tau_{j+1}. \end{cases} \tag{1}$$

An analogous result holds when $L_j>L_{j+1}$. From these joining times between adjacent patches, we conclude that the typical joining time scales with the average patch length as $T\sim\langle \Delta t\rangle^{-1}\sim\Lambda\langle L\rangle$. Thus in a time interval $dt\sim T$, a typical patch grows by an amount $dl\sim L$. Consequently $(d/dt)\langle L\rangle \sim T^{-1}\langle L\rangle$, and we obtain

$$\langle L\rangle\sim\Lambda^{-1}t. \tag{2}$$

In previous forest-fire models that are driven by stochastic tree growth and by stochastic lightning strikes [3,4], the latter rate needs to be very small to ensure nontrivial dynamics. There is no need for such parameter tuning in the present model, as the magnitude of Λ determines only the overall scaling of the typical forest length. Therefore, we set $\Lambda=1$ henceforth.

III. RATE EQUATION DESCRIPTION

A natural approach to determine the evolution of the patch length distribution is the rate equations. Let $N(L,t)$ be the number density of patches of length L at time t, and let $N(t)=\int_0^\infty N(L,t)dL$ be the total number of patches of any length. In the rate equation description of these quantities, we make the mean-field assumption that there are no correlations between adjacent patches. In a similar spirit, we also ignore the initial phase difference between two patches in Eqs. (1), so that the joining rate of two patches is simply proportional to $\langle \Delta t_j\rangle-\langle \Delta t_{j+1}\rangle \propto |L_j^{-1}-L_{j+1}^{-1}|$. With these approximations, the patch length distribution evolves according to the rate equation

$$\frac{\partial N(L,t)}{\partial t}=\frac{1}{2N(t)}\int_0^L dl K(l,L-l)N(l,t)N(L-l,t) - \frac{N(L,t)}{N(t)}\int_0^\infty dl K(L,l)N(l,t). \tag{3}$$

Here $K(x,y)=|x^{-1}-y^{-1}|$ is the rate at which a patch of length x joins with a patch of length y.

This rate equation is nearly identical in form to the corresponding equation for the kinetics of aggregation [17], except for the overall factor of $1/N(t)$. This difference arises because we consider a finite system and we track the number of forests of a given length rather than the corresponding probability. However, this factor can be absorbed into a rescaled time variable defined by

$$\mathcal{T}=\int_0^t \frac{dt'}{N(t')}, \tag{4}$$

to reduce Eq. (3) to the standard form of the rate equation for aggregation. This can then be analyzed by well-established methods [17].

While the rate equation with reaction rate $K(x,y)=|x^{-1}-y^{-1}|$ has not been solved, basic features about the long-time solution can be inferred from a scaling approach. In general, the long-time behavior of the rate equations with homogeneous reaction rates that satisfy (i) $K(ax,ay)=a^{\lambda}K(x,y)$ and (ii) $K(x,y)\sim x^{\mu}y^{\nu}$ for $x\ll y$ $(\lambda=\mu+\nu)$, have been generically classified according to whether $\mu>0$, $\mu<0$, or $\mu=0$ [17,18]. In all cases, the asymptotic length distribution approaches a scaling form:

$$N(L,t)\simeq\langle L(t)\rangle^{-2}\mathcal{N}(L/\langle L(t)\rangle), \tag{5}$$

in which the average patch length grows algebraically with time, $\langle L\rangle\sim\mathcal{T}^{1/(1-\lambda)}$, when $\lambda<1$. However, the scaling function $\mathcal{N}$ exhibits different behaviors in the three cases.

The reaction rate for our forest-fire model, $K(x,y)=|x^{-1}-y^{-1}|$, is homogeneous with homogeneity exponent $\lambda=-1$, while $\mu=-1$ and thus $\nu=0$. Therefore, the scaling theory prediction for the average patch length becomes $\langle L\rangle\sim\mathcal{T}^{1/2}$. Consequently, $N(\mathcal{T})\sim\langle L\rangle^{-1}\sim\mathcal{T}^{-1/2}$, and from Eq. (4), we recover $\langle L\rangle\sim t$, in agreement with the qualitative argument preceding Eq. (2). According to the general classification scheme of van Dongen and Ernst [17,18], since $\mu=-1$, the scaling function $\mathcal{N}(x)$ should vanish exponentially in the limits of small and large x:

$$\mathcal{N}(x)\sim\begin{cases} e^{-1/x}, & x\to 0,\\ e^{-x}, & x\to\infty. \end{cases} \tag{6}$$

For short patches, this result therefore predicts $N(L,t)\sim e^{-t/L}$. This behavior can also be established directly from Eq. (3). When $L\ll t$, the gain term in the rate equation can generally be ignored. Additionally, in this limit the reaction rate $K(L,l)$ simplifies to L^{-1}. Hence the density of short patches satisfies $\partial N/\partial t=-L^{-1}N$, which indeed implies the above exponential decay.

IV. SIMULATION RESULTS

In our simulations, we start the system with a random array of patches of lengths $\{L_j(t=0)\}$. As discussed in Sec. II, it is asymptotically exact to replace the stochastic forest lifetime Δt_j by its average value $\langle\Delta t_j\rangle=(L_j)^{-1}$. The time for regrowth of a tree is always equal to 1. Thus the dynamical steps become deterministic and randomness enters only through the initial conditions.

For convenience, we assume that each patch is initially in the burned state. From these regrowth and burning processes, a simulation of forest fires should be based on the following steps.

(1) Initialize the line with patches of random lengths L_j and with random times τ_j, for $j=1,2,\ldots,N$ at which the jth patch first turns into a forest. Assign a lifetime $(L_j)^{-1}$ to a forest of length L_j.

(2) Using Eq. (1), compute the joining times $T_{j,j+1}$ for all neighboring pairs of patches.

(3) Sort the list of joining times $\{T_{j,j+1}\}$ in ascending order. A standard sort algorithm [19] requires a time of the order of $N\ln N$ for a set of N elements.

(4) Join the pair of patches $(j,j+1)$ with the minimal joining time and increment the time accordingly. Recompute the joining times $T_{j-1,j}$ and $T_{j,j+2}$ of the patches adjacent to the newly joined forest.

(5) Decrement the total number of patches by 1 and return to step 3.

Such an algorithm is perforce inefficient because of the resorting of joining times after each event. However, this step is typically unnecessary for two reasons. First, we "cache" only a small subset of the joining times with $T_{j,j+1}$ less than a judiciously chosen cutoff time T_c and sort only this subset in step 3. We need not consider joining times $T_{j,j+1}>T_c$ because these events in the far future will never be considered before the current list of joining times needs to be resorted. This restriction significantly reduces the time needed to sort the joining time list for large N.

Second, it is unnecessary to resort this reduced joining time list after each joining event because only the two joining times $T_{j-1,j}$ and $T_{j,j+2}$ are modified. If these updated joining times are greater than the elements in the already sorted list, there is no need for resorting. One can simply continue to use the joining times from the presorted list to define joining events until one of the newly created joining times becomes less than the next joining time in the presorted list. Only when such a misordering occurs is it necessary to return to step 3 and resort the joining time list. These steps are completely analogous to those employed in Ref. [20] to simulate the kinetics of one-dimensional ballistic annihilation reactions efficiently.

We initialize the system with $N=5\times10^6$ patches whose lengths are randomly drawn from the distribution $N(L,t=0)=L_0^{-1}\exp(-L/L_0)$, with $L_0=0.1$. This initial length should be viewed as much larger than the lattice spacing between individual trees. We are interested in the intermediate asymptotic regime, where the average patch length is growing systematically with time and before finite-size effects begin to play a role. Figure 2 indicates that this intermediate asymptotic regime begins when $t>t^*\approx10$. An important feature of the system at long times is that there is only a very short-range spatial correlation in the lengths of neighboring patches (Fig. 3). In particular, the normalized correlation function $(\langle L_iL_{i+k}\rangle-\langle L\rangle^2)/\sigma^2$, with $\sigma^2=\langle L^2\rangle-\langle L\rangle^2$, quickly approaches zero for $k\geqslant2$. This provides empirical justification for the validity of the mean-field approxi-

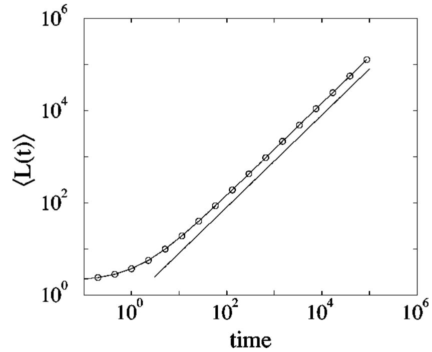

FIG. 2. Average patch length $\langle L(t)\rangle$ versus time t for initial length distribution $N(L,t=0)=L_0^{-1}\exp(-L/L_0)$, with $L_0=0.1$. A straight line of slope 1 is shown for comparison.

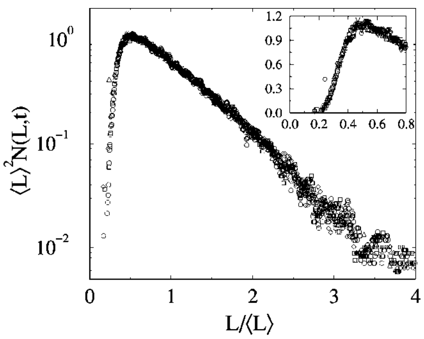

FIG. 4. Number of patches $N(L,t)$ of length L at $t=1478$, 3325, 7482, and 16 384 ($t=2.25^n$ with $n=9-12$) plotted in scaled form. The main plot shows the data on a semilogarithmic scale to illustrate the exponential decay of the large-length tail. The inset shows the same data in the small-length limit on a linear scale to highlight the essentially singular form.

mation of the rate equations. It is worth remarking that such a lack of spatial correlations appears ubiquitously in many one-dimensional coarsening processes [15,21].

We now examine the behavior of $N(L,t)$ for representative values of $t>t^*$ (Fig. 4). The different sets exhibit data collapse according to the scaling ansatz of Eq. (5). The large-length tail of this distribution appears to be consistent with a simple exponential decay. However, there is a very small but apparently systematic downward curvature to the data for which we do not have an explanation. The small-length tail decays extremely rapidly near the origin and there are essentially no patches with scaled length less than 0.2. This is consistent with the essential singularity predicted by Eq. (6). Again, it is worth remarking that in almost all coarsening processes that are controlled by an underlying diffusion process, the small-length tail of the patch-length distribution is a linear function. The one well-known example of an essential singularity in the small-size tail of a cluster distribution is the aggregation of Brownian particles [17].

V. CONCLUSIONS

We have developed a mesoscopic description for a forest-fire model with stochastic lightning strikes and deterministic tree growth. Instead of treating the system at the level of single trees, the basic element in our description is a patch of synchronized trees. Each patch undergoes periodic cycles of burning and regrowth, and in the long-time limit, the lifetime of patches can be viewed as deterministic. Whenever two adjoining patches are simultaneously in the forested state, the patches join and remain synchronized forever. The number of distinct patches decreases while their typical length grows continuously with time as in classical coarsening processes [15].

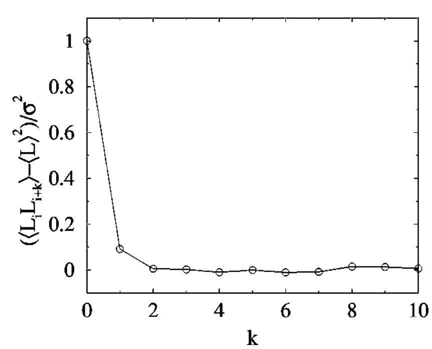

FIG. 3. Dependence of the normalized correlation function $(\langle L_iL_{i+k}\rangle-\langle L\rangle^2)/\sigma^2$, where $\sigma^2=\langle L^2\rangle-\langle L\rangle^2$, on patch separation k at $t=2.25^9\approx1478$.

This mesoscopic description is well suited to a rate equation approach for the evolution of the patch-length distribution, as well as efficient simulations. Numerically, we find that the average patch length grows linearly with time, while the number of patches correspondingly decreases as $1/t$. The patch-length distribution is sharply peaked, with an exponential large-length tail and an essential singularity in the small-length tail. These features are consistent with the rate equation predictions. Even though the system has nearest-neighbor interactions only, there are essentially no spatial correlations in the lengths of neighboring patches. Because of this lack of spatial correlation, we can anticipate that the rate equation predictions, which are based on no correlations between patch lengths, should provide an accurate account of the one-dimensional simulations.

Most aspects of our approach can be extended to higher spatial dimensions d. A complicating factor in developing numerical simulations is that the number of neighbors for a given patch is variable. Nevertheless, the same updating rule given by Eqs. (1) will still apply, with patch length being replaced by patch volume. As a result, we expect that the average patch volume should grow linearly with time. Under the assumption that patches remain compact, this would imply that the typical length scale of a growing patch would grow in time as $L\sim t^{1/d}$. It is, of course, far from obvious that patches remain compact. Understanding of the forest-fire model in higher dimensions appears to be an interesting challenge.

Finally, we want to stress that the seemingly innocent change of the tree growth rule from stochastic to deterministic drastically affects the dynamics. Almost all earlier work focused on stochastic tree growth. According to the literature on this model, self-organized critical behavior should occur in the limit of infinitesimal rate of lightning. However, after considerable effort, the understanding of the scaling laws is still incomplete, and even their very existence has recently been questioned [9]. In contrast, when the tree growth is deterministic, there is no need to tune the rate of lightning strikes to zero, and the model exhibits simple coarsening rather than complex time-dependent phenomena. It is, of course, possible that the higher-dimensional version of the model will offer some surprises.

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