

LETTER TO THE EDITOR

Exact solution of the one-dimensional percolation problem with further neighbour bonds†

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Abstract. We obtain an exact solution for the one-dimensional site percolation model with bonds connecting the L th nearest neighbours. The critical exponents are found to depend on L : $2 - \alpha_p = L$, $\gamma_p = L$, $\nu_p = L$, $\eta_p = 1$, $\beta_p = 0$ and $\delta_p = \infty$. By mapping the percolation problem onto an Ising-like model with multi-spin interactions, we argue that the dependence of the percolation exponents on L can be understood from consideration of Ising model universality classes.

The percolation problem (Shante and Kirkpatrick 1971) has attracted considerable interest in recent years. The concepts of scaling and universality, which are so useful in understanding the behaviour at the critical point, have also proved to be important in the investigation of the percolation transition. In this Letter we solve a one-dimensional ($d = 1$) site percolation problem with the addition of further neighbour bonds.

The generating function (Fisher and Essam 1961) which is the mean number of clusters per site, is

$$\hat{G}(p) \equiv \sum'_{s \geq 1} \langle n_s \rangle, \quad (1)$$

where $\langle n_s \rangle$ is the average number of s -site clusters per lattice site, and p is the probability that a site is occupied. The prime on the sum indicates that the infinite cluster is omitted. First we introduce a variable that plays a role analogous to the magnetic field in the thermal problem (Kasteleyn and Fortuin 1969). To each site in the lattice we associate a 'ghost site'‖ which is occupied with probability h (cf figure 1). All the ghost sites are connected to one another, so that if a site on the lattice is occupied, it is part of the infinite cluster if its corresponding ghost site is occupied (Reynolds *et al* 1977). Hence,

$$\langle n_s \rangle = [D_s(q)(1-h)^s] p^s, \quad (2)$$

where $D_s(q)$ is the perimeter polynomial for $h = 0$, and $q \equiv 1 - p$. The factor $(1-h)^s$ in the generalised ($h \neq 0$) perimeter polynomial ensures that the s -site cluster is isolated.

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‖ We could also define a single ghost spin connected by ghost bonds to each site and obtain an identical formalism. See also Reynolds *et al* (1977).

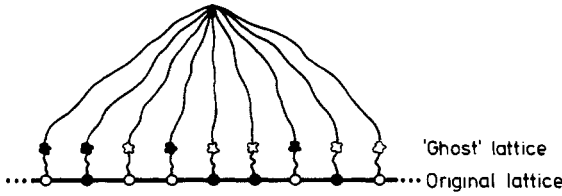


Figure 1. A segment of the $d = 1, L = 1$ lattice with the corresponding ghost lattice. Note that site 2 is a one-site cluster if $h = 0$, but may belong to the infinite cluster if $h \neq 0$.

We now calculate $\hat{G}_L(p, h)$ for a linear chain with bonds connecting up to the L th nearest neighbours. First we treat the $L = 2$ case in detail, and then we indicate how the argument is generalised to arbitrary L . (For the case $L = 1$, see Reynolds *et al* (1977) and references therein.) The average number of one-site clusters is $\langle n_1 \rangle = [q^4(1-h)]p$, since four lattice sites and one ghost site must be empty in order to isolate a single occupied site (cf figure 2(a)). There are two distinct two-site clusters, as in figures 2(b) and 2(c), and hence $\langle n_2 \rangle = [q^4(1-h)^2]p^2 + [q^5(1-h)^2]p^2$. For general s , there are $s - 1$ possible places between pairs of occupied sites where a site can be empty without destroying the cluster connectivity. We call such a configuration (an empty site flanked by two occupied ones) a one-site gap. If an s -site cluster has k gaps, there are $\binom{s-1}{k}$ possible configurations. Hence,

$$\langle n_s \rangle = \left[q^4(1-h)^s \sum_{k=0}^{s-1} \binom{s-1}{k} q^k \right] p^s = [q^4(1+q)^{s-1}(1-h)^s] p^s, \quad (3)$$

where the factor q^4 arises from the exterior perimeter of the cluster, and q^k comes from the interior gaps. Substituting (3) into (1), we obtain for $L = 2$

$$\hat{G}_2(p, h) = \frac{q^4(1-h)p}{1 - (1-q^2)(1-h)}. \quad (4)$$

To extend this argument to the general- L case, we note that now gaps of up to $L - 1$ sites can occur without destroying the cluster connectivity (see figure 2(d)). To

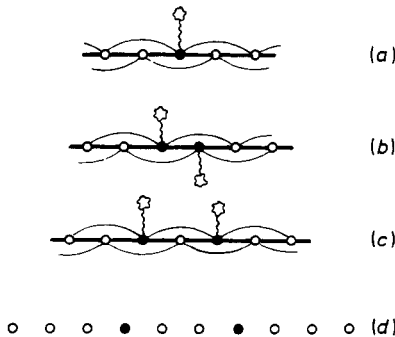


Figure 2. (a) A one-site cluster on the $d = 1, L = 2$ lattice with four lattice perimeter sites and one ghost perimeter site. (b) and (c) two possible two-site clusters that may occur for $L = 2$, each containing four exterior perimeter sites and two ghost perimeter sites; in (c) there is an additional interior perimeter site. (d) Gaps of up to $L - 1$ sites can occur without destroying the cluster connectivity; the two occupied sites form a two-site cluster for $L > 2$, and two one-site clusters for $L \leq 2$.

calculate $\langle n_s \rangle$ we must sum over all possible ways of distributing the various size gaps among the $s - 1$ possible places between the s occupied sites. The combinatorial factors that occur are the coefficients in a multinomial expansion of order $(s - 1)$. Therefore, in general

$$\hat{G}_L(p, h) = q^{2L} \sum_s p^s (1-h)^s (1+q+\dots+q^{L-1})^{s-1} = \frac{q^{2L}(1-h)p}{1-(1-q^L)(1-h)}. \quad (5)$$

From equation (5) it follows that, for all L , the percolation threshold is at $(q = 0, h = 0)$. To obtain the critical exponents we note that $\hat{G}_L(p, h)$ is a generalised homogeneous function (GHF) of the scaling fields q and h in the neighbourhood of $q = h = 0$,

$$\hat{G}_L(p, h) = \frac{-q^{2L}h + q^{2L} + q^{2L+1}h - q^{2L+1}}{q^L + h - hq^L} \sim \frac{q^{2L}}{q^L + h}. \quad (6)$$

The scaling powers of $\hat{G}_L(p, h)$ are therefore†

$$a_q = \frac{1}{L}, \quad a_h = 1, \quad (7)$$

leading (Hankey and Stanley 1972) to the exponents

$$\beta_p = 0, \quad \delta_p = \infty, \quad \gamma_p = L, \quad 2 - \alpha_p = L, \quad \nu_p = L, \quad \eta_p = 1, \quad (8)$$

where the last two equalities follow from the fact that the fractal dimensionality (Stanley *et al* 1976) $d_f = \gamma_p / \nu_p = 1$, and from the weak scaling relation $\gamma_p / \nu_p = 2 - \eta_p$. The GHF characteristic of equation (6) also allows us to derive several scaling laws (e.g. $\alpha_p + 2\beta_p + \gamma_p = 2$) analogous to those in thermal critical phenomena.

We have also computed the percolation probability and the mean size directly. From equation (5), we obtain by differentiation with respect to h (Reynolds *et al* 1977),

$$P(p) = 1 - \frac{q^{2L}(1-h)}{[1 - (1-q^L)(1-h)]^2} \quad (9)$$

and

$$S(p) = \frac{1 + (1-q^L)(1-h)}{1 - (1-q^L)(1-h)}, \quad (10)$$

leading again to the exponents of equation (8).

Having exponents which depend on the 'bond range'‡ is at variance with what we have come to expect from critical points. In order to gain some understanding of this result we present a different way of looking at percolation. As we will see, in this picture the percolation generating function is a thermodynamic derivative of a *thermal* free energy for an Ising-like model. In order to describe this picture of percolation we must first obtain an expression for the *total* number of clusters.

† Of course, in equations (7) and (8) we choose the scaling fields to be q and h for all L .

‡ Equation (5) is also a GHF in terms of q^L and h . With this choice of variables we obtain the exponents $\gamma_p = \nu_p = 2 - \alpha_p = 1$, $\beta_p = 0$, $\delta_p = \infty$, $\eta_p = 1$. From equation (5) there is no reason to choose one set of variables over the other. One can appeal to the renormalisation group (RG) and express exponents only in terms of RG scaling fields. This however does not resolve the problem as the RG does not distinguish between q and q^L as scaling fields. If we choose our scaling fields as q and h for all L , we have a breakdown of universality.

We stress the distinction between \hat{N}_L the number of clusters, which is a microscopic quantity, and $\langle \hat{N}_L \rangle$, the *average* number of clusters. Furthermore, we must distinguish between $\hat{G} = \lim_{N \rightarrow \infty} N^{-1} \langle \hat{N}_L \rangle$ *excluding* the zero-site clusters, and $G \equiv \lim_{N \rightarrow \infty} N^{-1} \langle N_L \rangle$ *including* the zero-site cluster contribution[†]. This latter quantity is required in order to map the site percolation problem onto an Ising model, and is just the analogue of the free energy introduced by Kasteleyn and Fortuin (1969) for the *bond* percolation problem. The zero-site cluster contribution is understood by considering the $L = 1$ bond problem, for which the free energy is obtained by counting the number of clusters of a given site size. In this counting procedure, isolated single-site clusters arise from a pair of adjacent empty bonds (cf figure 3). Under the covering transformation—which takes the bond problem to a site problem on a covering lattice—the one-site cluster in the bond problem maps into a bond, isolated by two adjacent empty sites, in the site problem. Thus, if we wish to obtain the free energy for site percolation, we *must* count the contribution of two empty adjacent sites, and we call these the ‘zero-site’ clusters[‡].

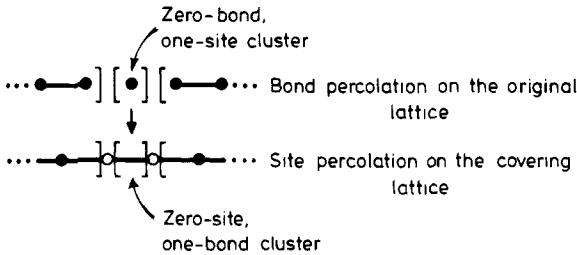


Figure 3. Two empty adjacent bonds on the original lattice map into an isolated bond on the covering lattice. This isolated bond contributes to the Kasteleyn–Fortuin free energy, and hence, for the site percolation problem two empty adjacent sites must be counted as a cluster. We call such clusters zero-site clusters.

We now introduce a spin language for the percolation problem (Kasteleyn and Fortuin 1969). Associate with each site of our lattice an Ising spin: an occupied site corresponds to an up-spin, and an empty site corresponds to a down-spin. These spins do not interact with each other, but do respond to an applied external magnetic field $k_B TH$. We can introduce projection operators (Coniglio 1976) for the two spin states at each site

$$\pi_j^\pm \equiv \frac{1 \pm s_j}{2}, \quad (11)$$

and also define averages of products of projection operators; for example

$$\langle \pi_j^+ \pi_{j+1}^- \rangle \equiv \frac{\sum_{\text{config}} \pi_j^+ \pi_{j+1}^- \exp(H \sum_k s_k)}{\sum_{\text{config}} \exp(H \sum_k s_k)} = pq. \quad (12)$$

We next obtain an expression for the number of clusters N_L in terms of the π_j^\pm . For $L = 1$, an occupied site at position j on the lattice is isolated from all sites to its right if there exists an empty site at $j + 1$. If we imagine walking along the lattice from left to

[†] Here, and in what follows, we exclude the infinite cluster.

[‡] The number of zero-site clusters is q^2 even for $h \neq 0$, since an empty site cannot be connected to a ghost site.

right, we must increase the cluster number by one each time we encounter an occupied site followed by an empty one. This method, however, does not count ‘zero-site’ clusters, and hence

$$\hat{N}_1 = \sum_j \pi_j^+ \pi_{j+1}^- . \quad (13)$$

In order to calculate N_1 , the *total* number of clusters (including those of zero site size), we walk along the lattice as before; however, we now increase the cluster count *every* time we encounter an empty site, and thereby count each configuration of two adjacent empty sites as a cluster. Therefore

$$N_1 = \sum_j \pi_j^- , \quad (14)$$

and hence the contribution of the zero-site clusters is

$$N_1 - \hat{N}_1 = \sum_j \pi_j^- \pi_{j+1}^- . \quad (15)$$

where we have used the identity $1 - \pi_j^+ = \pi_j^-$.

To generalise (15) to arbitrary L , we prove by induction the identity

$$N_L = \sum_j \pi_j^- \pi_{j+1}^- + \hat{N}_L , \quad (16)$$

where $\hat{N}_L \equiv \sum_j \pi_j^+ \pi_{j+1}^- \dots \pi_{j+L}^-$. Equation (15) is the case $L = 1$. Now assume (16) is true for $L = l$. To obtain (16) for $L = l + 1$ we note that N_{l+1} equals N_l minus the number of l -site gaps, $\sum_j \pi_j^+ \pi_{j+1}^- \dots \pi_{j+l}^- \pi_{j+l+1}^+$, since such a gap separates clusters for $L = l$ but not for $L = l + 1$ (cf figure 2(d)):

$$N_{l+1} = N_l - \sum_j \pi_j^+ \pi_{j+1}^- \dots \pi_{j+l}^- \pi_{j+l+1}^+ = \sum_j \pi_j^- \pi_{j+1}^- + \sum_j \pi_j^+ \pi_{j+1}^- \dots \pi_{j+l}^- \pi_{j+l+1}^- . \quad (17)$$

Including the zero-site cluster contribution, we can now write the ‘free energy’ for the percolation problem at $h = 0$,

$$\langle N_L \rangle = \frac{\sum_{\text{config}} (\sum_j \pi_j^- \pi_{j+1}^- + \sum_j \pi_j^+ \pi_{j+1}^- \dots \pi_{j+L}^-) \exp(H \sum_k s_k)}{\sum_{\text{config}} \exp(H \sum_k s_k)} . \quad (18)$$

Using equation (12), (18) reduces to

$$G_L(p, h = 0) = q^2 + pq^L = q^2 + \hat{G}_L(p, h = 0) , \quad (19a)$$

where $\hat{G}_L(p, h = 0)$ can be obtained from equation (5). In general,

$$G_L(p, h) = q^2 + \hat{G}_L(p, h) . \quad (19b)$$

Equation (18) can be rewritten as

$$G_L(p, h = 0) = \lim_{N \rightarrow \infty} N^{-1} \frac{d}{d\alpha} \ln \left[\sum_{\text{config}} \exp \left(H \sum_k s_k + \alpha \left(\sum_j \pi_j^- \pi_{j+1}^- + \sum_j \pi_j^+ \pi_{j+1}^- \dots \pi_{j+L}^- \right) \right) \right]_{\alpha=0} , \quad (20)$$

where the argument of the logarithm plays the role of a ‘partition function’, and the argument of the exponential the role of a ‘Hamiltonian’. That is, the percolation free energy is obtained from the free energy of a system of Ising spins via a thermodynamic

derivative. This implies that the partition function in (20) is singular[†] at a 'thermal' critical point ($H = H_c$, $\alpha = 0$), and that we can understand the singularities of the percolation transition by studying this 'thermal' point.

By examining the interactions contained in the 'Hamiltonian' of (20), the breakdown of universality can be understood. For $L = 1$, the Hamiltonian contains only single-spin interactions. For $L = 2$, expansion of the products of the π_i^\pm results in a three-spin interaction, as well as nearest- and next-nearest-neighbour interactions. In general, for $L = l$, an expansion of the projection operators results in the addition of a new $(l + 1)$ -spin interaction.

Although the complete relation between symmetry properties of Ising-like Hamiltonians and universality classes is not understood, there are results which indicate that the addition of multi-spin interactions to Ising Hamiltonians *does* change the universality class. In particular, the triangular lattice with three-spin interactions solved by Baxter and Wu (1973) has different exponents than the Ising model with two-spin interactions. The eight-vertex model solved by Baxter (1972), which can be mapped into an Ising system with four-spin interactions, is in still another universality class. Even though universality breaks down there is a weaker sense in which it still holds (Suzuki 1974): the renormalised exponents

$$\hat{\gamma} \equiv \gamma/\nu, \quad \hat{\beta} \equiv \beta/\nu, \quad \hat{\eta} \equiv \eta, \quad \hat{\delta} \equiv \delta, \quad \hat{\phi} \equiv (2 - \alpha)/\nu \quad (21)$$

are universal. Similarly, we find that the renormalised percolation exponents, defined in analogy with (21), are L -independent.

In summary, we have solved the site percolation problem on a one-dimensional lattice with bonds connecting L th nearest neighbours, and noted that the critical exponents are not L -independent. This behaviour can be illuminated by relating the percolation problem to a corresponding 'thermal' problem with multi-spin interactions.

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[†] Strictly speaking the partition function (20) is not a singular function of p . However, the point $H = H_c$, $\alpha = 0$, is associated with an infinite length which determines the physics at $p = p_c$. The partition function in (20), generalised to $h \neq 0$ (unpublished) is singular in the limit $h \rightarrow 0$.