J. Phys. A: Math. Gen. 14 (1981) L349–L354. Printed in Great Britain

LETTER TO THE EDITOR

Percolation and conduction in a random resistor-diode network

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Received 13 July 1981

Abstract. We study percolation and conduction in a randomly diluted network in which the occupied bonds may be either resistor-like, transmitting connectivity in both directions along a bond, or *diode*-like, transmitting in only one direction. This system exhibits novel phase transitions signalling the onset of infinite clusters with either isotropic or unidirectional connectivity. Position-space renormalisation group methods are applied to map out the phase diagram and calculate exponents associated with these phase transitions.

The percolation problem has been extensively investigated, partly because it is an extremely simple system exhibiting the intriguing complexities of continuous phase transitions, and also because of its many physical realisations (Stauffer 1979, Essam 1980). Recent work has focused on developing generalisations of percolation which are of fundamental theoretical interest, as well as models which more realistically describe the properties of particular physical systems. Many such generalisations are contained implicitly in the pioneering work of Broadbent and Hammersley (1957). They proposed that neighbouring lattice sites may be joined by two randomly occupied *directed* bonds, one 'transmitting' connectivity or information in one direction, and the other transmitting in the reverse direction. In this sense, the directed bonds act as *diodes*, in contrast to pure bond percolation in which the bonds act as *resistors*, transmitting in both directions.

One special case of this Broadbent–Hammersley model is directed bond percolation, in which, on the square lattice, randomly occupied directed bonds may transmit only upward or to the right. Above the percolation threshold, an infinite cluster forms which may be traversed from the lower-left to the upper-right, but not vice versa. This model has different critical behaviour from pure bond percolation (Blease 1977a, b, c, Kertész and Vicsek 1980, Dhar and Barma 1981), exhibiting an anisotropic structure for both the infinite cluster and the decay of correlations (Obukhov 1980, Kinzel and Yeomans 1981). Further interest in this model stems from the fact that directed percolation can be mapped into a Reggeon field theory (Cardy and Sugar 1980), and the latter model can then be related to Markov processes with absorption, branching and recombination (Grassberger and Sundermeyer 1978, Grassberger and de la Torre 1979) which are of relevance for describing many chemical and biological processes (Schlögl 1972). Reynolds (1981) treated a more general problem in which the diodes could 'break down', and conduct in both directions. By varying such a breakdown

† Supported in part by grants from the ARO and AFOSR.

probability, Reynolds studied the crossover between directed and isotropic percolation, and argued that the two models are in different universality classes.

In this Letter, we consider a more general percolation process on the square lattice, mediated by both resistors and *randomly oriented* diodes. We define positive diodes to be transmitting either upward or to the right and vice versa for negative diodes. Resistors transmit in both directions, and vacancies are non-transmitting (figure 1).

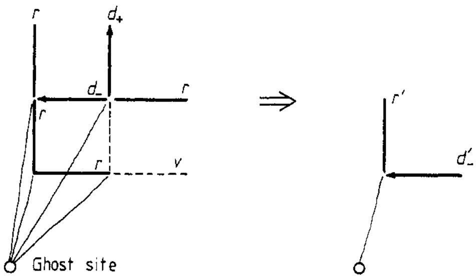

Figure 1. A typical configuration in a 2×2 cell of the square lattice. Diodes, resistors and vacancies are represented by arrows, heavy lines and broken lines respectively. The probability of traversing the cell vertically requires only the four vertical bonds and the d_- horizontal bond. Under rescaling, the configuration maps into the state shown on the right. These five bonds are also used to calculate the probability of finding a path that joins the top edge of the cell with the ghost site via ghost bonds (wavy lines).

These elements occur with random probabilities d_+, d_-, r and v respectively. For this system, the diode 'polarisation' points along an axis inclined at 45° to the horizontal, taking on any value between -1 and $+1$. Our work indicates that this model exhibits novel phase transitions due to the formation of infinite clusters which support either isotropic information flow or unidirectional flow, either with or against the diode polarisation.

We have applied the position-space renormalisation group (PSRG) to study the properties of this random resistor-diode network. Our procedure is based on the simplest approximation of rescaling a 2×2 bond cell as shown in figure 1 (Reynolds *et al* 1977). For the 4^5 configurations of the cell, we calculate the probabilities of traversing the cell in both directions, and in only one direction. These give respectively, the probability of a renormalised resistor, and the probability of a renormalised diode oriented in the direction of traversing. From these recursion relations, we obtain the phase diagram in the probability space spanned by d_+, d_-, r and v (figure 2). Four phases exist: a 'vacancy' phase containing only finite clusters, a 'resistor' phase in which information flows isotropically within the infinite cluster, and two unidirectional 'diode' phases in which information flows only along the direction of diode polarisation. The phase diagram exhibits two symmetries which arise from the invariance of the recursion relations under inversion ($d_+ \leftrightarrow d_-$), and under a duality in which $v \leftrightarrow r$ and $d_\pm \leftrightarrow d_\pm$. The latter symmetry appears to be exact, similar to the dual symmetry of pure bond percolation.

Of the six non-trivial fixed points (figure 2), there are two directed fixed points which occur at d_+^* (or d_-^*) $= 0.5550$, $v^* = 1 - d_\pm^*$, and $r^* = 0$. They describe the onset of an infinite cluster which transmits only along the direction of diode polarisation. Our value of d^* is a reasonable approximation to the estimates $d_c = 0.643$–0.645 for directed percolation found by more accurate numerical methods (Blease 1977a, b, c, Kertész

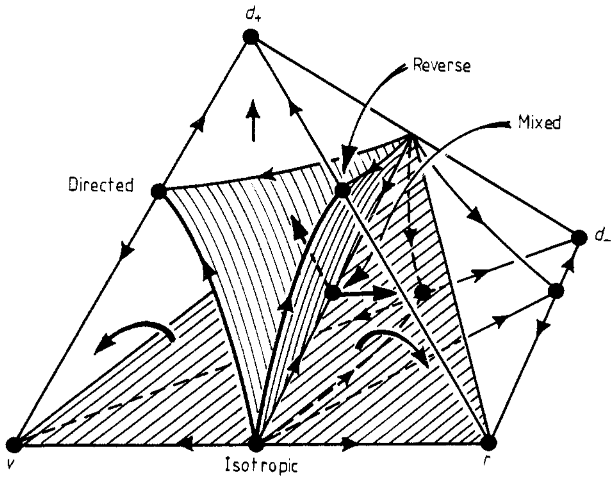

Figure 2. Schematic phase diagram of the random resistor-diode network in the probability space d_+, d_-, r and v. Fixed points are shown as heavy dots, and arrows indicate the direction of flow under renormalisation. The phase diagram is symmetric across the plane defined by $d_+ = d_-$ (cross-hatched), and across the plane $v = r$. The two shaded surfaces form a wedge which originates from the central line of the symmetry plane. The wedges on the d_+ and d_- side of the symmetry plane meet on this line of higher-order critical points, and divide the tetrahedron into regions of positive diode, negative diode, resistor and vacancy phases. At the mixed fixed point, independent exponents arise when we approach in the r and d directions (within and perpendicular to the symmetry plane respectively).

and Vicsek 1980, Dhar and Barma 1981, Kinzel and Yeomans 1981). From the thermal eigenvalue, $\lambda_{d_\pm} \equiv \partial d_\pm / \partial d_\pm |_{d^*} = 1.567$, we obtain a correlation length exponent $\nu_\| = \ln 2 / \ln \lambda_{d_\pm} = 1.543$ describing the divergence of the length along the polarisation. This is consistent with the estimate $\nu_\| = 1.65$–1.70 found by more accurate methods. There also exist two reverse fixed points which signal the onset of infinite paths, mediated by resistors, which transmit opposite to the direction of polarisation. By duality, these fixed points occur at $d_\pm = 0.5550$, $r^* = 1 - d_\pm$, and $v^* = 0$, with the same exponents as those at the directed fixed points. Both pairs of fixed points are domains of attraction for the two surfaces of second-order transitions which separate the diode and vacancy phases and the diode and resistor phases respectively (figure 2).

The isotropic fixed point occurs at $r^* = \frac{1}{2}$ (which is exact by duality), $v^* = 1 - r^*$, and $d_\pm^* = 0$. This is a higher-order critical point where the vacancy, resistor and two diode phases become simultaneously critical. Within the symmetry plane defined by $d_+ = d_-$, the linearised recursion relation is diagonal, leading to an isotropic correlation length exponent $\nu_i = 1.428$ as the fixed point is approached in any direction. Finally, a 'mixed' fixed point occurs at $d_+ = d_- = 0.2457$, $r^* = v^* = 0.2543$. Here an isotropic infinite cluster forms whose connectivity requires both diodes and resistors, and which is described by two diverging correlation lengths. One is an isotropic length scale as the fixed point is approached in the 'r' direction. In our 2×2 cell approximation, the exponent governing the divergence of this length equals ν_i of the isotropic fixed point. The second length scale is associated with connected paths oriented opposite to the diode polarisation as the fixed point is approached in the 'd' direction. As the polarisation approaches zero, this reverse length diverges with an exponent of 1.710.

The domain of attraction of the mixed fixed point is a line of higher-order critical points in the symmetry plane, where the four phases are simultaneously critical (figure 2).

In addition to finding thermal exponents, we introduce a magnetic-field-like variable to calculate magnetic exponents as well. This may be accomplished by having a ghost site which may connect to all lattice sites via randomly occupied ghost bonds (Kasteleyn and Fortuin 1969, Reynolds *et al* 1977, 1980). When the ghost bond probability, h, is greater than zero, an infinite cluster exists for all non-zero lattice bond probabilities in a manner analogous to the magnetisation induced in a ferromagnet by a field for all temperatures. To renormalise h, we calculate the probability that there exists a connected path joining one edge of the cell to the ghost site. Configurations with such a path renormalise to the combination of an occupied lattice bond and ghost bond, leading to a recursion relation of the form $r'h' = R_h(r, h)$ for pure percolation. We obtain the best numerical results when we employ a five-bond cell. This procedure is *ad hoc* since six bonds of the cell contribute to the probability of reaching the ghost. Our approximation yields a magnetic eigenvalue $\lambda_h = \partial h'/\partial h|_{r^*,h^*} = 3.75$, a magnetic scaling power of $y_h = \ln 2/\ln \lambda_h = 1.907$, and exponents $\beta = (d - y_h)/y_p = 0.1242$, $\gamma = (2y_h - d)/y_p = 2.418$, in good agreement with more accurate estimates (see e.g. Stauffer 1979, Essam 1980).

For directed percolation, the orientation of connected paths on the lattice suggests that we introduce directed ghost bonds which go *from* the ghost *to* all the lattice sites (Obukhov 1980). This construction induces an orientation for connected paths in finite field in which the ghost is 'earlier' than all lattice sites. Consistent with this 'ordering', we renormalise a diode ghost bond by calculating the probability of getting from the ghost to a site in the cell, and finally to a cell edge. From this recursion relation, and with the numerical estimates $d^*_\pm = 0.634$ and $\nu_\| = 1.70$, we find $\beta = 0.2156$ and $\gamma = 2.968$. Our values of β and γ should be compared with the series estimates $\beta = 0.28 \pm 0.02$ and $\gamma = 2.27 \pm 0.04$ (Blease 1977a, b, c). For the reverse threshold, we calculate the probability that we can go in the reverse direction *from* the cell edge *to* the ghost. Since this goes against the diode polarisation, both resistor-like lattice and ghost bonds are required to construct such a path. From the recursion relation for the resistor ghost bond probability, and from d^* and $\nu_\|$ quoted above, we find $\beta = 0.0633$ and $\gamma = 3.273$.

At the mixed threshold, there exist two independent magnetic exponents related to approaching the fixed point in the r and d direction. In the former case, we measure an isotropic mean cluster size by calculating the probability of getting from the ghost to the cell edge *or* vice versa. This renormalises the resistor ghost bond probability, and we obtain a magnetic eigenvalue $\lambda_{h_r} = 3.731$. Under the assumption that the thermal exponent in the r direction is the same as that in pure bond percolation, $y_p = \frac{3}{4}$, we obtain $\beta = 0.1342$, $\gamma = 2.398$, very close to the magnetic exponents found in pure percolation. In the d direction, the mean size of clusters transmitting only opposite to the diode polarisation diverges as d_+ (or d_-) $\to d^*$. In addition, the probability that a particular diode belongs to a unidirectional infinite cluster transmitting parallel to the diode vanishes as the polarisation approaches zero. These processes may be described by renormalising the diode ghost bond probability. We find a magnetic scaling power $\lambda_{h_d} = 3.756$, and associated exponents, $\beta = 0.1550$ and $\gamma = 3.109$.

We have also examined the conductivity properties of a random network of resistors and 'ohmic' diodes. The latter elements have zero conductivity in the reverse direction, and *ohmic* response in the forward direction (equivalent to a real resistor and diode in series). An oriented network of ohmic diodes, resistors and vacancies can be treated self-consistently within our approximate PSRG. However, with both positive and negative diodes allowed, a new circuit element appears after rescaling: a bidirectional bond with different conductivities in either direction. We treat these new elements as resistors by taking the average of the conductivities in the two directions, thereby truncating the parameter space to the four variables we have already considered.

To obtain the critical behaviour, we employ the simplest approximation and calculate the arithmetic mean of the conductivities of all traversing cell configurations at the fixed point, under the assumption that each resistor and each forward-biased diode has a conductivity σ. From this rescaling, we find a conductivity exponent, t, through $t=\nu[d-2+\ln b/\ln(\sigma/\sigma')]$ (Rosman and Shapiro 1977, Bernasconi 1978). For bond percolation, this rescaling yields $\sigma'/\sigma=0.566\ldots$, and an isotropic conductivity exponent of $t=1.093$, in reasonable agreement with recent estimates of t (Kirkpatrick 1979). At the directed fixed point, we find that the forward-bias conductivity approaches zero as $d\rightarrow d^*$ from above, with an exponent of 1.292. At the reverse threshold, the conductivity opposite to the diode polarisation approaches zero as $r\rightarrow r^*$ from below with an exponent of 1.498. At the mixed fixed point, there are two critical conductivity processes. One is the vanishing of the isotropic conductivity as the fixed point is approached in the 'r' direction, while the other is the vanishing of the conductivity along the direction of diode polarisation as the fixed point is approached in the 'd' direction. Employing the truncation discussed above, we find that the former process is governed by an exponent of 1.111, while the latter is governed by an exponent of 1.463.

In conclusion, we have studied percolation and conduction in a random resistor-diode network. This system exhibits new types of phase transitions due to the formation of infinite clusters which are either isotropic or unidirectional, with the directionality being either parallel or antiparallel to the diode polarisation. Position-space renormalisation group methods have been applied to map out the phase diagram of this system, and to calculate the exponents associated with various novel percolation and conduction processes. Many of our predictions may be tested quantitatively by more accurate numerical methods.

I thank S Alexander, A Brown, P G de Gennes, W Klein, W Kinzel, H Nakanishi, J F Nicoll, P Pincus, P J Reynolds and J Y Yeomans for informative discussions, and F Y Wu for a helpful correspondence. I am also grateful to A Brown for checking several of my calculations by computer. After I initiated this work, I learned that Reynolds (1981) has generalised his work on oriented resistor-diode networks, and treated a randomly oriented resistor-diode network similar to the one considered in this paper.

Note added in proof. Recently Cardy (private communication) pointed out a very simple alternative approach for obtaining the recursion relation $(d'_{\pm}+r')$. Through this method several numerical errors were found and corrected.

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