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LETTER TO THE EDITOR

A connection between linear and nonlinear resistor networks

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Abstract. We explore the connection between the higher moments of the current (or voltage) distribution in a random *linear* resistor network, and the resistance of a *nonlinear* random resistor network. We find that the two problems are very similar, and that an infinite set of exponents are required to fully characterise each problem. These exponent sets are shown to be identical on a particular hierarchical lattice, a simple model which accurately describes the geometrical properties of the backbone of the infinite cluster at the percolation threshold and also the voltage distribution on this structure.

The critical behaviour of random resistor networks has been extensively studied in the past few years (see e.g., Zabolitsky 1984, Herrmann *et al* 1984, Hong *et al* 1984, Lobb and Frank 1984, and references therein). However, it has been only very recently that attention has turned to the distribution of voltage drops across each conductor in a resistor network (de Arcangelis *et al* 1985). The importance of this distribution is that it provides detailed microscopic information about the structure of the network, in addition to providing fundamental information such as the network conductivity. Our goal, in this letter, is to point out a connection between the higher moments of the voltage distribution on *linear* resistor networks and the resistance of *nonlinear* networks.

To define the voltage distribution on a linear network, consider a hypercubic cell of linear dimension L which contains a random resistor network at the percolation threshold. If a unit voltage drop is applied across opposite faces of the network, a total current I_{tot} will flow. Since the voltage drop V between opposite faces is unity, I_{tot} simply equals G, where G is the conductance of the system. Furthermore, notice that I_{tot} coincides with the current flowing through the links, or cutting bonds of the backbone. These are defined as the bonds, which, if cut, cause the opposite faces of the network to become disconnected.

Each bond of the network can be characterised by the fraction of the total current flowing through it, $\alpha = I/I_{tot}$. For example, the cutting bonds are characterised by $\alpha = 1$, while bonds which belong to very large blobs are mostly characterised by very small values of α. An interesting feature of this bond characterisation is that an infinite set of *independent* lengths, $\mathscr{L}_k$, and corresponding exponents, $\tilde{\zeta}_k$, can be defined at the percolation threshold via

$$\mathscr{L}_k = \sum \alpha^k N(\alpha) \sim L^{\tilde{\zeta}_k} \qquad (1)$$

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where $N(\alpha)$ is the number of bonds corresponding to the value α, and the tilde refers to a critical exponent measured in terms of the correlation length, i.e., $\tilde{\zeta}_k = \zeta_k/\nu$. Thus $\mathscr{L}_k$ is related to the moments of the current distribution, or equivalently, by the linear relation between the current and the voltage; (1) is also related to the moments of the voltage distribution. From the basic distribution defined by (1), several connections with familiar physical quantities can be made. For example, $\mathscr{L}_2$ equals the resistance, while $\mathscr{L}_\infty$ is the number of cutting bonds. Therefore $\tilde{\zeta}_2 = \tilde{\zeta}_R$, where $\tilde{\zeta}_R$ is the resistance exponent, and the latter is related to the conductivity exponent $\tilde{t}$ through $\tilde{t} = d - 2 + \tilde{\zeta}_R$, while $\tilde{\zeta}_\infty = 1/\nu$. Furthermore, Rammal *et al* (1985a, b) have shown that ζ_4 is related to the amplitude of the noise in a random resistor network.

In a previous paper, we have demonstrated the independence of the exponents $\tilde{\zeta}_k$ and the lengths $\mathscr{L}_k$ both by analytical calculations on a hierarchical model (figure 1), and by numerical simulations in two dimensions (de Arcangelis *et al* 1985). This is a rather striking result since it is customary in critical phenomena that the higher moments of a microscopic distribution, such as the cluster size distribution in percolation, are described in terms of only one 'gap' exponent. Only in the mean-field limit do all the exponents $\tilde{\zeta}_k$ coincide, because in this limit the backbone of the percolating cluster is made of links only, and therefore $\alpha = 1$ for all bonds.

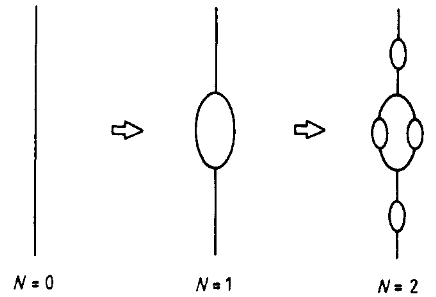

Figure 1. Hierarchical lattice for the case $\lambda = 1$, corresponding to two dimensions. Starting with a single bond (level $N = 0$), it is replaced by the unit cell shown at level $N = 1$. This procedure is then repeated indefinitely.

For the hierarchical lattice, the explicit expression for ζ_k is (de Arcangelis *et al* 1985)

$$\zeta_k = \nu\tilde{\zeta}_k = 1 + \ln(1 + \lambda/2^k)/\ln 2/\lambda \qquad (2)$$

where λ is the ratio of the number of bonds in the blobs to the number of bonds in the links within one unit cell of the hierarchical lattice. If the hierarchical lattice is used as a description of the percolating backbone, then clearly λ must go to zero for spatial dimensions $d \to 1$ and for $d \to 6$, and λ should achieve a maximum in $d = 2$ where blobs are relatively most important. These conditions can be satisfied by the simple choice $\lambda = \frac{1}{4}(6-d)$ for $d \geq 2$, and $\lambda = d - 1$ for $d < 2$. With this connection between the hierarchical lattice structure and the spatial dimension d, exponent predictions can be given for all d. In figure 2, we have plotted ζ_k against k for spatial dimensionality 1-6, and the agreement between the best available numerical data and the predictions of the model are quite good (see, e.g., table 1 of de Arcangelis *et al* 1985 and references therein).

A problem which is reminiscent of the voltage distribution is the nonlinear resistor network, in which the relation between the voltage drop across a particular bond ij

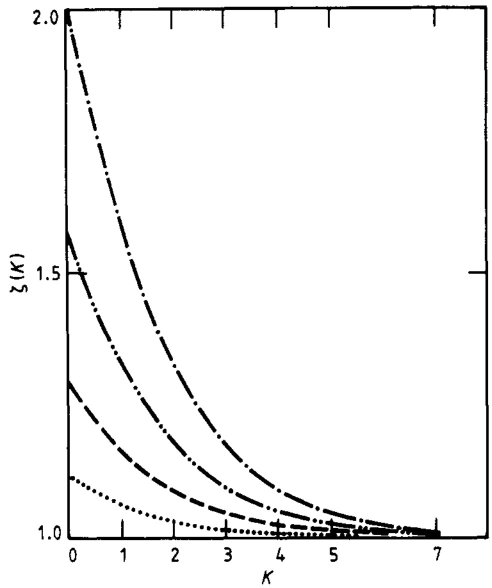

Figure 2. Plot of the exponent ζ_k as a function of k for spatial dimensions $d = 1$-6, based on the predictions of the hierarchical model. —— $d = 1, 6$; $\cdots\cdot$ $d = 5$; $---$ $d = 4$; $-\cdot\cdot\cdot-$ $d = 3$; $-\cdot-\cdot-$ $d = 2$.

and the current is given by

$$V_{ij} = rI_{ij}^{\beta}.$$

The critical behaviour of a random network of such conducting elements was first investigated by Kenkel and Straley (1982) and more recently by Blumenfeld and Aharony (1985). For such a nonlinear network of linear dimension L, one can define a generalised resistance, $R(\beta)$, through

$$V = R(\beta)I^{\beta}$$

where V is the voltage difference between the opposite faces of the network, and I is the total current flowing. If the concentration of conducting bonds is at the percolation threshold, a family of exponents for the nonlinear network can be defined by

$$R(\beta) \sim L^{\tilde{x}(\beta)}$$

where $\tilde{x}(1)$ coincides with the resistance exponent ζ_{R}, $\tilde{x}(\infty) = 1/\nu$, and $\tilde{x}(0)$ is the exponent characterising the length of the shortest path or chemical distance in the backbone (Blumenfeld and Aharony 1985).

The question that we now address is whether the higher moments of the voltage distribution in the linear network are somehow related to the β dependent resistance of the nonlinear network. We note that the exponents, $\tilde{x}(\beta)$ and $\tilde{\zeta}_k$, coincide for $\beta = 1$ and $k = 2$, and for $\beta = k = \infty$. Furthermore, both sets of exponents tend to $1/\nu$ as the spatial dimension approaches 6 from below (the mean-field limit).

To test for a general connection between the exponents, we solve the nonlinear network problem on the hierarchical lattice. To calculate the critical exponent $\tilde{x}(\beta)$ on the hierarchical lattice, we first evaluate $R(\beta)$ on the unit cell of the model using the following combination rules for series and parallel nonlinear resistors,

$$r_{\text{series}} = r_1 + r_2$$
$$r_{\text{parallel}}^{-1/\beta} = r_1^{-1/\beta} + r_2^{-1/\beta}.$$

This gives

$$R(\beta) = 2/\lambda + 2^{-\beta}$$

if it is assumed that each conductor has a unit value of nonlinear resistance. It is possible to generalise this calculation for an Nth-order hierarchical lattice in order to obtain $\tilde{x}(\beta)$, but this procedure is unnecessarily complicated. For obtaining the critical exponent $\tilde{x}(\beta)$, it is much more convenient to use a simple renormalisation procedure which is exact for the hierarchical lattice. We use as a rescaling parameter, L_1, the number of singly connected bonds. Then we formally can write the following connection between $R(\beta)$ and L_1 for the unit cell of the hierarchical lattice

$$R(\beta) \sim L_1^{x(\beta)}.$$

Since in the unit cell, there are $2/\lambda$ links, we can easily solve for the critical exponent

$$x(\beta) = \ln(2/\lambda + 2^{-\beta})/\ln 2/\lambda. \tag{3}$$

Using the exact relation between the number of links and the linear size of the lattice (Coniglio 1981, 1982), $L_1 \sim L^{1/\nu}$, we can also obtain $\tilde{x}(\beta) = x(\beta)/\nu$.

As a result of this line of reasoning, we find

$$\tilde{x}(\beta) = \tilde{\zeta}_{\beta+1}. \tag{4}$$

Although the correspondence of (4) is exact for the hierarchical lattice, it is dependent on the high degree of symmetry for this model. That is, the current flowing along each side of any bubble within a blob is the same. Clearly such an exact symmetry will not hold in general, but we do expect it to hold on the average. Thus we believe that the predictions (3) and (4) arising from the hierarchical model should be quite close to the exponents of the appropriate nonlinear resistor network. The largest discrepancy is expected for $d = 2$ and $\beta = 0$, where the influence of blobs with large non-symmetric current paths, should be the largest.

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