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# The magic of networks grown by redirection

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**Abstract:** We highlight intriguing features of complex networks that are grown by redirection. In this mechanism, a target node is chosen uniformly at random from the pre-existing network nodes and the new node attaches either to this initial target or to a neighbor of this target. This exceedingly simple algorithm generates preferential attachment networks in an algorithmic time that is linear in the number of network nodes $N$. Even though preferential attachment ostensibly requires global knowledge of the network, redirection requires only local knowledge. We also show that changing just a single attachment rate in linear preferential attachment leads to a non-universal degree distribution. Finally, we present unexpected consequences of redirection in networks with undirected links, where highly modular and non-sparse networks arise.

**Keywords:** Random recursive tree; Complex networks; Redirection; Emergent modularity; Anomalous scaling

## 1. Introduction

Redirection is a natural mechanism to create growing networks. In a social setting, you may meet somebody and ultimately befriend of one the friends of your initial acquaintance. This redirection also underlies a growth mechanism in Facebook, where you are encouraged to create new links to some of the friends of your initial Facebook friend [1, 2]. The simplest implementation of redirection for networks where each link has a prescribed directionality is the following (Fig. 1):

1. A new node `n` picks a pre-existing node `x` from the network uniformly at random.
2. With probability $0<1-r<1$, `n` attaches to `x`.
3. Otherwise, with probability $r$, `n` attaches to the (unique) ancestor node `y` of `x`.

These steps are repeated until a network of a desired size is generated.

By construction, a network with a tree topology always remains a tree. While it is straightforward to generalize to networks with loops by the new node choosing multiple provisional targets, we focus on trees both for their simplicity and because they illustrate many of the intriguing features of networks that are grown by preferential attachment.

Without the redirection step, the above growth rules define a model that is known as the random recursive tree (RRT). We discuss this fundamental null model [3] in Sect. 2. Redirection represents a minimalist extension of the RRT; this idea was suggested in [4] and developed in [5]. (Alternative extensions of growth mechanisms that are still local in character [6–9] have also yield networks with broad degree distributions). As we will discuss, standard redirection is equivalent to shifted linear preferential attachment [5], in which the rate of attaching to a pre-existing network node of degree $k$ is proportional to $k+\lambda$, with $\lambda=\frac{1}{r}-2$. This connection highlights an fascinating aspect of redirection—it transforms a purely local growth mechanism—namely the RRT plus redirection to the ancestor—into the global mechanism of linear preferential attachment. The motivation for preferential attachment stems from the "rich get richer" parable [10, 11]; that is, popular high-degree nodes are more likely to attract additional links merely by virtue of being popular. While enormous effort has been devoted to understanding the properties of these types of networks (see, e.g., Refs. [12–19]), we will present, in Sect. 3, a number of surprising and under-appreciated features of preferential attachment.

As we will discuss in this section, networks that are built by the redirection mechanism of Fig. 1 have a degree distribution that possesses a non-universal algebraic tail,

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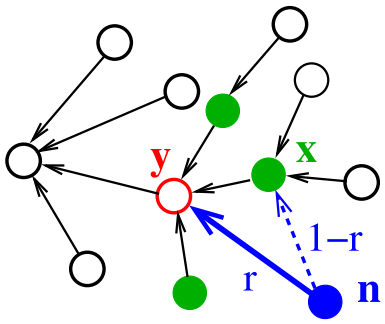


**Fig. 1** Illustration of redirection. The rate at which a new node **n** attaches to the ancestor **y** of node **x** by redirection is proportional to the number of upstream neighbors of **y** (green nodes) (color figure online)

$$N_k \sim \frac{N}{k^\nu}\,, \qquad \nu = 1 + \frac{1}{r} > 2\,. \tag{1}$$

The exponent must satisfy $\nu > 2$ for all sparse networks whose degree distribution has an algebraic tail. This bound follows from the identity $\sum_{k\geq 1} kN_k = 2L$ and the linear growth of the number of links $L$ with $N$, which is the defining property of sparse networks. For trees, in particular, $\sum_{k\geq 1} kN_k = 2L = 2(N-1)$.

However, a disconcerting feature of several complex networks [20] is that they are apparently characterized by degree distributions with tail exponent $\nu < 2$, which violates the bound in Eq. (1). Mathematically, this implies that the sum $\sum_{k\geq 1} kN_k$ grows superlinearly with $N$, which cannot occur in sparse networks with $N_k \sim N/k^\nu$. An exponent value $\nu < 2$ may arise in densifying networks [21–23], where $L$ increases superlinearly with $N$. Intriguingly, such an anomalously small exponent also occurs in undirected growing trees that are generated by complete redirection (Sect. 4.3). To be consistent with the constraint $\sum_{k\leq N} kN_k \sim N$, the amplitude of the degree distribution must grow sublinearly with $N$, namely

$$N_k \sim \frac{N^{\nu-1}}{k^\nu}\,, \qquad \nu < 2. \tag{2}$$

Networks grown by this parameter-free complete redirection mechanism: (a) are highly modular; (b) have numerous macrohubs; (c) consist almost entirely of leaves (nodes of degree 1); (d) the "core" of the network (nodes of degree $k \geq 2$) comprises a vanishingly small fraction of the network as $N \to \infty$; and (e) are non-self-averaging, namely, basic characteristics, such as $N_k$ for any $k > 1$, exhibit huge fluctuations from realization to realization. In spite of the simplicity of complete redirection, there is little analytical understanding of its intriguing consequences and these represent an appealing future challenge.

We emphasize that the redirection algorithm is extremely efficient. To build a network of $N$ nodes requires a computation time that scales linearly with $N$, with a prefactor of the order of one. Redirection also allows one to build networks with more general preferential attachment mechanisms, such as sublinear preferential attachment, with nearly the same efficiency as the original redirection algorithm (Sect. 4.2).

## 2. The random recursive tree (RRT)

We begin our discussion with the RRT, first introduced by Otter [3], in which nodes are added to the network one by one (Fig. 2). Each new node attaches to a single "target" node that is chosen uniformly at random among the already existing nodes; that is, the attachment rate $A_k = 1$, for any degree $k$. By the restriction that each new node has a single attachment point (equivalently, the out degree of every node equals 1), the resulting network is a tree. If a new node attaches to more than one pre-existing node, loops could form. The degree distribution of a network with loops is modified only in the amplitude of the degree distribution compared to growing trees. On the other hand, topological features of networks with loops are different than trees, but our focus is on the degree distribution, for which it is simplest to focus on tree networks.

The growth rules of the RRT thus are:

1. Pick one of the nodes of the RRT—defined as the target—with uniform probability.
2. Introduce a new node that links to the target node.

Starting with a single node, these two steps are repeated until the tree reaches a desired number of nodes $N$.

### 2.1. The degree distribution

We first outline how to derive the exact degree distribution and then determine the degree distribution in the limit $N \to \infty$. The degree state of any network is characterized by the vector $\mathbf{N} \equiv \{N_1, N_2, \ldots\}$, where $N_k$ denotes the number of nodes of degree $k$. When a new node is

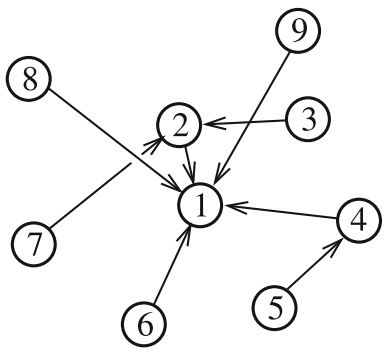


**Fig. 2** A random recursive tree of 9 nodes, showing the ordering of the nodes and each of their attachment points

introduced, the changes in the network state vector $\mathbf{N}$ are [24, 25]:

$$\begin{aligned}&\text{Attach to node of degree } 1: \quad (N_1, N_2) \to (N_1, N_2+1)\\ &\text{Attach to node of degree } k>1: \quad (N_1, N_k, N_{k+1})\\ &\to (N_1+1, N_k-1, N_{k+1}+1),\end{aligned} \tag{3}$$

while the state of all other network nodes are unchanged. Typically we are not interested in the full probability distribution $P(\mathbf{N})$, but just the average number of nodes of a given degree, $\langle N_k\rangle$, namely, the degree distribution; the angle brackets denote an average over all possible growth histories of the network.

Let us determine how the $N_k$ change when a new node is added to the network. As indicated by Eq. (3), we need to separately consider nodes of degree 1 and nodes of degree greater than 1. The number of nodes of degree 1, $N_1(N)$, i.e., the number of leaves, is a random variable that changes with each node addition according to

$$N_1(N+1) = \begin{cases} N_1(N) & \text{probability} \quad \frac{N_1}{N} \\ N_1(N)+1 & \text{probability} \quad 1-\frac{N_1}{N}. \end{cases} \tag{4}$$

These equations apply for $N \geq 2$, while the natural initial condition is $N_1(2) = 2$. This equation expresses the two possibilities when a new node joins the network: with probability $N_1/N$, the new node attaches to a node of degree 1 and the number of such nodes does not change, while with probability $(1 - N_1/N)$, the new node attaches to a node of degree $k > 1$ and $N_1$ increases by 1.

The evolution equation for the average number of leaves is therefore

$$\begin{aligned}\langle N_1(N+1)\rangle &= \left\langle N_1(N) \times \frac{N_1(N)}{N}\right\rangle + \left\langle (N_1(N)+1)\right.\\ &\quad \left.\times \left(1-\frac{N_1(N)}{N}\right)\right\rangle \\ &= 1 + \left(1-\frac{1}{N}\right)\langle N_1(N)\rangle\,.\end{aligned} \tag{5}$$

Because the relevant time-like variable that characterizes the network size is the total number of nodes $N$, we will always use $N$ as the time variable. The solution to this recursion, for $N \geq 2$, is

$$\langle N_1(N)\rangle = \frac{N}{2} + \frac{1}{N-1}\ . \tag{6}$$

The discrete approach can be used to determine higher moments of the random variable $N_1(N)$. The second moment $\langle N_1^2(N)\rangle$ is especially important as we can obtain the variance and thereby quantify degree fluctuations. From Eq. (4), we deduce the recurrence for the second moment

$$\langle N_1^2(N+1)\rangle = 1 + \left(1-\frac{2}{N}\right)\langle N_1^2(N)\rangle + \left(2-\frac{1}{N}\right)\langle N_1(N)\rangle\,,$$

whose solution is

$$\langle N_1^2(N)\rangle = \frac{N(3N+1)}{12} + \frac{N}{N-1}\ . \tag{7}$$

From the first two moments, the variance, for $N \geq 3$, is

$$\langle N_1^2(N)\rangle_c \equiv \langle N_1^2(N)\rangle - \langle N_1(N)\rangle^2 = \frac{N}{12} - \frac{1}{(N-1)^2}\ , \tag{8}$$

so the deviation of $N_1(N)$ from its average is of the order of $\sqrt{N}$. Higher cumulants of the number of leaves also grow as $\sqrt{N}$. The cumulants $\langle N_1^p(N)\rangle_c$ with arbitrary integer $p \geq 1$ are given by remarkably simple formula

$$\langle N_1^p(N)\rangle_c = p^{-1}B_p N + \frac{(-1)^{p-1}(p-1)!}{(N-1)^p} \tag{9}$$

applicable when $N \geq p+1$. Here $B_p$ are Bernoulli numbers defined [26] as the coefficients in the power series

$$\frac{z}{e^z-1} + z = \sum_{p\geq 0} B_p\,\frac{z^p}{p!}$$

Thus for large $N$, the number of nodes of degree 1 is sharply distributed about its average value. For this reason, one may ignore fluctuations and focus on the average. This same holds for all nodes of higher degrees as long as the number of such nodes is large, $N_k(N) \gg 1$. Thus we again focus on the average.

By similar reasoning as that used for $N_1$, the number of nodes of degree $k \geq 2$, evolves according to

$$N_k(N+1) = \begin{cases} N_k(N)-1 & \text{probability} \quad \frac{N_k}{N} \\ N_k(N)+1 & \text{probability} \quad \frac{N_{k-1}}{N} \\ N_k(N) & \text{probability} \quad 1-\frac{N_{k-1}+N_k}{N} \end{cases} \tag{10}$$

after each node addition. Following the same steps that led to Eq. (5), the evolution equation for $\langle N_k\rangle$ is

$$\langle N_k(N+1)\rangle = \langle N_k(N)\rangle + \left\langle \frac{N_{k-1}(N)-N_k(N)}{N}\right\rangle. \tag{11}$$

While this equation can again be solved to give the exact degree distribution for finite networks, we now restrict ourselves to the leading behavior of the degree distribution for $N \to \infty$. For simplicity, we drop the angle brackets and the argument $N$, so that we write $N_k$ for the average number of nodes of degree $k$ in a network that contains $N$ nodes. Next, we replace the discrete differences with derivatives in Eqs. (5) and (11), so that the asymptotic degree distribution evolves according to the master equation

$$\dot{N}_k \equiv \frac{dN_k}{dN} = \frac{N_{k-1} - N_k}{N} + \delta_{k,1}\ . \tag{12a}$$

The first equation is $\dot{N}_1 = -N_1/N + 1$, with solution $N_1 = N/2$. Then $\dot{N}_2 = (N_1 - N_2)/N$, with solution $N_2 = N/4$. Continuing one finds that all the $N_k$ are proportional to $N$. Thus we write $n_k \equiv N_k/N$ and reduce Eq. (12a) to

$$n_k = n_{k-1} - n_k + \delta_{k,1} \tag{12b}$$

leading to the exponential degree distribution $n_k = 2^{-k}$.

## 3. Preferential attachment

In preferential attachment, the rate $A_k$ at which a node attaches to a pre-existing node of degree $k$ is an increasing function of $k$. A ubiquitous feature of preferential attachment networks is that their degree distributions have broad tails, a fact that sparked much interest in this class of networks over the past two decades. We now derive this scale-free degree distribution using the approach of in Ref. [5].

### 3.1. Master equation

The evolution of the degree distribution for a network whose growth is governed by an attachment rate $A_k$ is (compare with Eq. (12a) for the RRT):

$$\dot{N}_k = \frac{A_{k-1}N_{k-1} - A_k N_k}{A} + \delta_{k1}. \tag{13}$$

The first term on the right accounts for the new node connecting to a pre-existing node that already has $k-1$ links, thereby increasing $N_k$ by one. Since there are $N_{k-1}$ nodes of degree $k-1$, the rate at which such processes occur is proportional to $A_{k-1}N_{k-1}$. The total rate $A \equiv A(N) \equiv \sum_{j\geq 1} A_j N_j$ in the denominator means that $A_{k-1}/A$ is the probability for a node of degree $k-1$ to become a node of degree $k$. A corresponding role is played by the second term on the right. The overall amplitude of $A_k$ is immaterial, since only the ratio $A_k/A$ appears in the master equation. The last term accounts for the introduction of a new node that has one outgoing link and no incoming links. To determine the degree distribution, we need to specify the attachment rate $A_k$. We focus on power-law preferential attachment, $A_k = k^\gamma$, with $\gamma \geq 0$. We will show that different behaviors arise for sublinear ($\gamma < 1$), superlinear ($\gamma > 1$), and linear ($\gamma = 1$) attachment rates. The linear case is especially rich because the degree distribution is nonuniversal.

When confronted with determining a non-trivial distribution, it is often instructive to first deal with the simpler problem of determining low-order moments of the degree distribution $M_\alpha(N) \equiv \sum_j j^\alpha N_j$. The zeroth and first moments of this distribution have particularly simple $N$ dependences: $\dot{M}_0 = \sum_j \dot{N}_j = 1$ and $\dot{M}_1 = \sum_j j\dot{N}_j = 2$. The equation for $M_0$ states that the total number of nodes (of any degree) increases by 1 each time a new node is introduced. Similarly, the equation for $M_1$ states the total degree of the network, $\sum jN_j$, increases by two when the single link associated with the new node is added to the network. Since both the zeroth and first moments of the degree distribution increase linearly with $N$, the total rate $A = \sum_j j^\gamma N_j$ also grows linearly with $N$, because $A$ is intermediate to the zeroth and first moments for $0 \leq \gamma \leq 1$. Asymptotically, $A \simeq \mu N$, with the as yet-undetermined amplitude $\mu$ that must range between 1 and 2 as $\gamma$ increases from 0 to 1.

Solving for the first few $N_k$ from Eq. (13), it becomes clear that each $N_k$ is also proportional to $N$. This fact suggests substituting $N_k(N) = n_k N$ and $A \simeq \mu N$ into these master equations. With this step, the overall $N$ dependence cancels, leaving behind the recursion relations $n_k = (A_{k-1}n_{k-1} - A_k n_k)/\mu$ for $k > 1$ and $n_1 = 1 - A_1 n_1/\mu$. After straightforward algebra, the degree distribution is

$$n_k = \frac{\mu}{A_k} \prod_{1\leq j\leq k} \left(1 + \frac{\mu}{A_j}\right)^{-1}. \tag{14a}$$

Using the definition $\mu = \sum_{j\geq 1} A_j n_j$ in (14a) we obtain

$$\sum_{k\geq 1} \prod_{1\leq j\leq k} \left(1 + \frac{\mu}{A_j}\right)^{-1} = 1. \tag{14b}$$

To extract the physical meaning of the general solution (14a) with $\mu$ implicitly determined by (14b) we examine the asymptotic behavior for the three generic cases of sublinear, superlinear, and linear preferential attachment.

#### 3.1.1. *Sublinear preferential attachment*

For $A_k = k^\gamma$ with $\gamma < 1$, we rewrite the product in Eq. (14a) as the exponential of a sum of logarithms, convert the sum to an integral, and then expand the logarithm inside the integral in a Taylor series. These straightforward steps lead to

$$n_k \sim \begin{cases} k^{-\gamma} \exp\left[-\mu\left(\frac{k^{1-\gamma} - 2^{1-\gamma}}{1-\gamma}\right)\right] & \frac{1}{2} < \gamma < 1, \\ k^{(\mu^2-1)/2} \exp\left[-2\mu\sqrt{k}\right] & \gamma = \frac{1}{2}, \\ k^{-\gamma} \exp\left[-\mu\,\frac{k^{1-\gamma}}{1-\gamma} + \frac{\mu^2}{2}\,\frac{k^{1-2\gamma}}{1-2\gamma}\right] & \frac{1}{3} < \gamma < \frac{1}{2}, \end{cases} \tag{15}$$

with similar, but more complicated expressions for $n_k$ for still smaller values of $\gamma$. Each time $\gamma$ decreases through $\frac{1}{m}$, where $m$ is an integer, an additional term is generated in the exponential that is an increasing function of $k$. Nevertheless, for any value of $\gamma<1$, the leading behavior is always the universal stretched exponential decay, $\exp(-\text{const.} \times k^{1-\gamma})$.

### 3.1.2. *Superlinear preferential attachment*

For $\gamma > 1$, a gelation-like phenomenon occurs in which nearly all links attach to a single node. Let us first treat the ultra singular behavior that arises for $\gamma > 2$, for which there is a non-zero probability for a "bible" to occur—a node that links to every other node in an infinite network, while only a finite number of links exist between all other nodes. To determine the probability for a bible, suppose that a network of $N+1$ nodes contains a bible (Fig. 3). The probability that the next node links to the bible is $N^\gamma/(N+N^\gamma)$, and the probability that this pattern of connections continues indefinitely is $\mathcal{P} = \prod_{N\geq 1} N^\gamma/(N+N^\gamma)$. Using the same asymptotic analysis as above, where we write the product as the exponential of a sum of logarithms, expand the logarithm for large $N$, and approximate the sum as an integral, the asymptotic behavior of this product is $\mathcal{P}=0$ for $\gamma \leq 2$, and $\mathcal{P}>0$ for $\gamma > 2$. Strikingly, there a non-zero probability for a bible to exist in an infinite network for $\gamma > 2$!

When $1<\gamma<2$, the attachment pattern of low-degree nodes is not as simple as in Fig. 3, but there continues to be a single node whose degree is of the order of $N$. There is also an infinite sequence of transition points when $\gamma$ passes through $\frac{m}{m-1}$, with $m$ an integer greater than 2, in which number of nodes of degree $k \leq m$ grows as $N^{k-(k-1)\gamma}$, while the number of nodes of degree $k > m$ remain finite for $N \to \infty$ (Fig. 4). To understand this behavior in a simple way, it is instructive to study the governing equations for each $N_k$ one by one. For $N_1$ we have

$$\dot{N}_1 = 1 - \frac{N_1}{A}\,.$$

We now make the assumption that the total attachment rate is governed by the single highest-degree node, with degree of the order of $N$. Thus $A = \sum j^\gamma N_j \sim N^\gamma$. Since $N_1$ can at most be of the order of $N$, the second term in the above equation is negligible, so that $\dot{N}_1 \sim 1$ or $N_1 \sim N$. Similarly, the equation for $N_2$ is

$$\dot{N}_2 \simeq \frac{N_1 - 2^\gamma N_2}{N^\gamma}\,.$$

Again, neglecting the second term, gives $\dot{N}_2 \simeq N^{1-\gamma}$, from which $N_2 \sim N^{2-\gamma}$. We can then verify that the term that we dropped is indeed negligible. Continuing this self-consistent procedure for general degree $k$, we find

$$N_k \sim N^{k-(k-1)\gamma}\,, \tag{16}$$

as long as the exponent of $N_k$ is positive, while $N_k$ will be finite for $N \to \infty$ for values of $k$ for which $k-(k-1)\gamma$ is negative (Fig. 4).

Thus we predict an infinite sequence of transitions at $\gamma = \gamma_m = \frac{m}{m-1}$. For $\gamma > \gamma_m$, the number of nodes of degree $k > m$ are all of $\mathcal{O}(1)$, while nodes of degrees $k \leq m$ grows sublinearly with $N$, as $N^{k-(k-1)\gamma}$. This set of transitions becomes progressively more dense as $\gamma \to 1$ from above. At $\gamma = 1$, the network changes its character from condensed, where a hub node has degree of $\mathcal{O}(N)$, to sparse, where the number of nodes of any degree is proportional to $N$.

### 3.1.3. *Linear preferential attachment*

Here, it is important to distinguish between strictly linear preferential attachment, $A_k = k$, and asymptotically linear preferential attachment, $A_k \simeq k$. In the former case, the total attachment rate is $A = \sum_k A_k N_k = \sum_k k N_k = 2N$. Substituting this value of $\mu = 2$ into Eq. (14a) and performing some simple algebra immediately leads to the discrete power-law form of the degree distribution

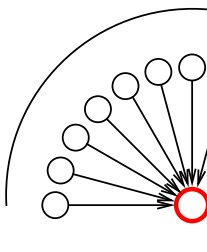


**Fig. 3** Creation of a "bible" in which each new node attaches only to the bible (red) (color figure online)

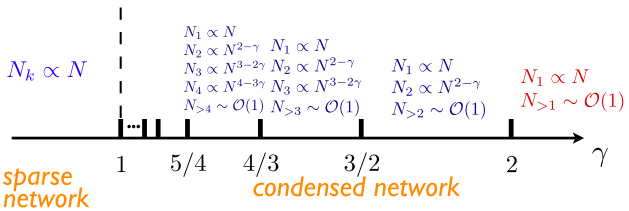


**Fig. 4** Illustration of the sequence of phase transitions that arise in superlinear preferential attachment. Starting with an ultra-condensed network for $\gamma > 2$, the network contains progressively more low-degree nodes each time $\gamma$ passes through $m/(m-1)$. The network becomes sparse when $\gamma$ reaches 1, where the number of nodes of any degree are all proportional to $N$

$$n_k = \frac{4}{k(k+1)(k+2)} = \frac{4\,\Gamma(k)}{\Gamma(k+3)}\ , \tag{17}$$

where $\Gamma$ is the Euler gamma function. From this power-law degree distribution, the mean degree $\langle k\rangle = \sum_{k\geq 1} kn_k = 2$, as it must, but the mean-square degree $\langle k^2\rangle = \infty$. Thus fluctuations in the mean degree, namely, the spread in the mean degree for different realization of large networks of $N$ nodes, diverges for $N \to \infty$.

The surprising feature of asymptotically linear preferential attachment growth is that the degree distribution exponent is non-universal. This non-universality is at odds with the common wisdom of statistical physics in which the absence of a characteristic scale leads to universal scaling properties. One natural form for an asymptotically linear attachment rate is $A_k = k + \lambda$, with $\lambda$ a constant. This modification implies that the amplitude $\mu$ in $A = \mu N$ is no longer equal to 2, but assume a wide range of values (see below). Now Eq. (14a) becomes

$$\begin{aligned} n_k = \frac{\mu}{A_k}\prod_{1\leq j\leq k}\left(1+\frac{\mu}{A_j}\right)^{-1} &\sim \frac{\mu}{k}\exp\left[-\int_1^k \ln\left(1+\frac{\mu}{j}\right)dj\right] \\ &\sim \frac{\mu}{k}\exp\left[-\mu\int_1^k \frac{dj}{j}\right] \\ &\sim k^{-(1+\mu)}\ . \end{aligned} \tag{18}$$

Thus the degree exponent $\nu = 1 + \mu$ can take any value larger than 2 merely by tuning the amplitude $\mu$.

As an explicit and surprising example, consider the attachment rate $A_k = k$ for $k \geq 2$, while $A_1 \equiv \alpha$ is arbitrary. It is now convenient to separate $A_1$ and $A_k$ for $k \geq 2$ in Eq. (14a) to recast this equation as

$$\begin{aligned} \mu &= A_1\sum_{k=2}^{\infty}\prod_{j=2}^{k}\left(1+\frac{\mu}{A_j}\right)^{-1} \\ &= \alpha\sum_{k=2}^{\infty}\Gamma(2+\mu)\,\frac{\Gamma(1+k)}{\Gamma(1+\mu+k)}, \end{aligned} \tag{19}$$

where we express the product as the ratio of gamma functions.

The sum can be evaluated by employing the identity [26]

$$\sum_{k=2}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}\ ,$$

so that Eq. (19) becomes $\mu(\mu-1) = 2\alpha$, with solution $\mu = (1+\sqrt{1+8\alpha})/2$. Thus the degree exponent $\nu = 1+\mu$ is

$$\nu = \frac{3+\sqrt{1+8\alpha}}{2}\ . \tag{20}$$

As examples, the degree distribution exponent is $\nu = 4$ for $\alpha = 3$ and $\nu = 5/2$ for $\alpha = 3/8$. For $0 < \alpha < 1$, the exponent lies in the range $2 < \nu < 3$, while for $\alpha > 1$, $\nu > 3$. While the degree distribution exponent must satisfy the lower bound $\nu > 2$, there is no upper bound for $\nu$; in particular, $\nu \to \sqrt{2\alpha}$ as $\alpha \to \infty$. We emphasize that changing just a single attachment rate leads to a global effect on the degree distribution. This global effect arises because the amplitude $\mu$ appears inside the infinite product in Eq. (14a). This multiplicative nature strongly affects the degree distribution itself and thereby the degree distribution exponent.

## 4. Network growth by redirection

We now discuss a deceptively simple modification of the RRT with profound consequences. This is the notion of redirection where a new node may attach to a pre-existing target node, or it to a neighbor of this target [4, 5].

### 4.1. Constant redirection probability

First we treat the redirection algorithm [5] that was outlined in the introduction. There is one subtlety in this algorithm because redirection requires that every node has an ancestor. To ensure this condition always holds, the initial state, for example, could consist of at least two nodes and one link, with each node defined as the ancestor of the other. Other simple starting graphs are equally suitable, such as a triangle with cyclic links.

According to the redirection algorithm, the degree distribution evolves according to

$$\dot{N}_k = \frac{1-r}{N}\Big[N_{k-1} - N_k\Big] + \frac{r}{N}\Big[(k-2)N_{k-1} - (k-1)N_k\Big] + \delta_{k,1}\ . \tag{21a}$$

The terms within the first square brackets correspond to attachment to the initially selected node, whose evolution equation is just that of the RRT (Eq. (12a)) for redirection probability $r = 0$. The terms within the second square brackets account for the change in $N_k$ due to redirection. To understand their origin, consider first the gain term. Since the initial node is chosen uniformly, if redirection does occur, then the probability that a node of degree $k-1$ receives the newly redirected link is proportional to the number of its upstream neighbors (green nodes in Fig. 1), which equals $k-2$. A parallel argument applies for the redirection-driven loss term. The crucial point is that the rate at which attachment occurs to a given node is proportional to the number of its upstream neighbors, which, in turn, is proportional its degree. Thus linear preferential attachment is implicit in this purely local redirection rule.

The redirection mechanism has an unexpected connection to the friendship paradox [27, 28], which states that the neighbors of a randomly selected node are more popular (have higher degrees), on average, than the initially selected node. As illustrated in Fig. 1, there are three distinct ways to attach node **y** by redirection from upstream nodes. The higher the degree of node **y**, the more likely attachment to it by redirection occurs. Thus we expect that node **y** will have more neighbors, on average, than the initial node **x**

By a straightforward rearrangement of terms, (21a) may be re-expressed as

$$\begin{aligned}\dot{N}_k &= \frac{r}{N}\left\{\left[k-1+\left(\frac{1}{r}-2\right)\right]N_{k-1}\right.\\ &\quad\left.-\left[k+\left(\frac{1}{r}-2\right)\right]N_k\right\}+\delta_{k,1}\\ &\equiv \frac{1}{A}\Big\{(k-1+\lambda)N_{k-1}-(k+\lambda)N_k\Big\}+\delta_{k,1}\,,\end{aligned} \tag{21b}$$

with $\lambda=\frac{1}{r}-2$ and total attachment rate $A=N/r=(2+\lambda)N$. Thus uniform attachment, in conjunction with redirection, generates shifted linear preferential attachment, with $A_k=k+\lambda$. The particular case of strictly linear preferential attachment arises for the choice $r=\frac{1}{2}$. When we now substitute attachment rate $A_k=k+\lambda$ and $\mu=2+\lambda$ into the general formula (14a) for the degree distribution, we obtain

$$\begin{aligned}n_k &= \frac{\mu}{A_k}\prod_{1\le j\le k}\left(1+\frac{\mu}{A_j}\right)^{-1}\\ &= (2+\lambda)\frac{\Gamma(3+2\lambda)}{\Gamma(1+\lambda)}\frac{\Gamma(k+\lambda)}{\Gamma(k+3+2\lambda)}\sim k^{-(3+\lambda)}\,.\end{aligned} \tag{21c}$$

Since the redirection probability lies between 0 and 1, the additive shift $\lambda$ lies between $-1$ and $\infty$. Thus the degree distribution exponent can take on any value that is greater than 2. In the extreme case of $r=1$ a star-like network arises whose detailed structure depends on the initial condition.

It is also worth mentioning the many intriguing results that emerge from simple extensions of this redirection mechanism. Starting with the RRT, each node has a genealogical tree of ancestors. It is natural to grow a network in which redirection can occur equiprobably to any node in the genealogical tree of an initial target node [29], or to all nodes in this genealogical tree [30]. The latter leads to a network that is no longer sparse, as the number of links $L$ grows as $N\ln N$. Amusingly, this redirection mechanism to all ancestors is isomorphic to a basic hypergraph model, known as the random recursive hypergraph [31].

Finally, we wish to emphasize the extreme simplicity of this redirection algorithm. Each node addition requires only two elemental operations: (i) select a target node, and (ii) choose to attach either to this target or to its ancestor. This algorithm allows one to generate a network of $N$ nodes in roughly $2N$ algorithmic steps. It is therefore possible to quickly generate very large networks. Crucially, a purely local rule—tracking the ancestor of each node—is equivalent to the global rule that underlies preferential attachment. Ostensibly, one needs to know the degrees of all the nodes in the network to implement preferential attachment. As the redirection algorithm shows, this global information is not needed.

### 4.2. Degree-based redirection

To illustrate the utility and generality of redirection, we exploit the local information that is readily available—the degree $a$ of the initial target node and the degree $b$ of the ancestor—to efficiently generate sublinear preferential attachment networks. In degree-based redirection [32], we merely define the redirection probability $r$ to be a suitably chosen function of these two degrees $a$ and $b$; that is $r=r(a,b)$ (see Fig. 5).

To show how sublinear preferential attachment can be achieved from this still-local information, we define $f_k$ as the total probability that an incoming link is redirected from a randomly selected target node of degree $k$ to the parent of this target. Similarly, we define $t_k$ as the total probability that an incoming link is redirected to a parent node of degree $k$ after the incoming node initially selected one of the child nodes of this parent. Formally, these probabilities are defined in terms of the redirection probabilities by

$$f_k=\sum_{b\ge 1}\frac{r(k,b)N(k,b)}{N_k}\,,\qquad t_k=\sum_{a\ge 1}\frac{r(a,k)N(a,k)}{(k-1)N_k}\,, \tag{22}$$

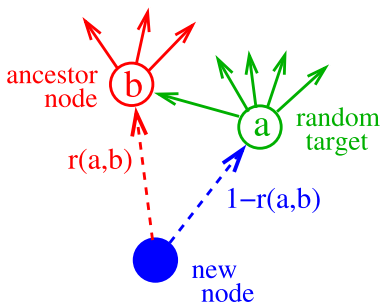


**Fig. 5** Illustration of degree-based redirection. A new node (blue) attaches to a random target of degree $a$ with probability $1-r(a,b)$ and attaches to the ancestor node (degree $b$) of the target with probability $r(a, b)$ (color figure online)

where $N_k = \sum_{b\geq 1} N(k,b)$ and $N(a, b)$ is the correlation function that specifies the number of nodes of degree $a$ that have a parent of degree $b$. Thus $f_k$ is the mean redirection probability averaged over all $N_k$ possible target nodes of degree $k$. Likewise, since each node of degree $k$ has $k-1$ children, there are $(k-1)N_k$ possible target nodes whose redirection probabilities are averaged to give $t_k$.

In terms of these probabilities $f_k$ and $t_k$, the master equation that governs the evolution of $N_k$ is

$$\dot{N}_k = \frac{(1-f_{k-1})N_{k-1} - (1-f_k)N_k}{N} + \frac{(k-2)t_{k-1}N_{k-1} - (k-1)t_kN_k}{N} + \delta_{k,1}. \tag{23}$$

The first ratio corresponds to instances of network growth for which the incoming node actually attaches to the initial target. For example, the term $(1-f_k)N_k/N$ gives the probability that one of the $N_k$ target nodes of degree $k$ is randomly selected and that the link from the new node is not redirected away from this target. Similarly, the second ratio corresponds to instances in which the link to the target node is redirected to the ancestor. For example, the term $(k-1)t_kN_k/N$ gives the probability that one of the $(k-1)N_k$ children of nodes of degree $k$ is chosen as the target and that the new node is redirected. Lastly, the term $\delta_{k,1}$ accounts for the newly added node of degree 1.

By rearranging terms, we express (23) in the generic form of Eq. (13), with the attachment rate

$$\frac{A_k}{A} = \frac{(k-1)t_k + 1 - f_k}{N}. \tag{24a}$$

Since the quantities $f_k$ and $t_k$ are normalized probabilities, the asymptotic behavior of the above expression is $A_k \sim k\,t_k$. Thus a redirection probability $r(a, b)$ for which $t_k$ is a decreasing function of $k$ will asymptotically correspond to sublinear preferential attachment. A natural choice for such a redirection probability is $r(a,b) = b^{\gamma-1}$, with $0<\gamma<1$, so that the redirection probability decreases as the degree of the parent node increases. Because $r$ depends only on the degree of the parent node (Fig. 5), Eq. (22) reduces to $t_k = k^{\gamma-1}$. Using this form of $t_k$ in Eq. (24a) yields

$$\frac{A_k}{A} = \frac{k^\gamma - k^{\gamma-1} + 1 - f_k}{N}, \tag{24b}$$

whose leading behavior is indeed sublinear preferential attachment, $A_k \sim k^\gamma$. This equivalence to sublinear preferential attachment allows to generate a network of $N$ nodes with a stretched exponential degree distribution in an algorithmic time that is also of the order of $N$.

What happens in the opposite case of enhanced redirection, in which the redirection probability is an increasing function of the degree of the parent node [32, 33]? This attachment rule leads to highly modular networks that contains multiple macrohubs, with most nodes having degree 1 (leaves). Furthermore, the degree distribution exhibits the anomalous scaling given in Eq. (2), with $\nu$ strictly less than 2. Similar phenomenology also occurs in the simpler example of redirection rule for undirected networks (see below).

### 4.3. Complete redirection in undirected networks

Link directionality is important in social and technological networks, but there are many situations where networks are undirected [34–36]. The influence of redirection on undirected networks is profound and there is little analytical understanding of this enigmatic case.

The growth rule for isotropic redirection is nearly the same as that given in Sect. 4 for directed networks, but with a small but profound difference [37, 38] that is embodied by the following growth rule:

1. Pick a pre-existing node x from the network uniformly at random.
2. With probability $1-r$, the new node n attaches to x.
3. Otherwise, with probability $r$, the new node n attaches to any neighbor of x, chosen uniformly at random.

Repeat these steps a until a network of a desired size is generated. The growth rules for directed and undirected redirection are illustrated in Fig. 6.

We focus on the limit of $r = 1$, which we term complete redirection, because this limiting case leads to the most striking phenomenology. Simulation data also suggest that it is only the special case of $r = 1$ that gives rise to emergent modularity. The network realizations shown in Fig. 7 for $r = 1$ are highly modular and each consists of a number of well-resolved modules. Each module contains a central macrohub whose degree is a finite fraction of the total number of nodes $N$; thus each macrohub is connected to a large number of leaves (nodes of degree 1). Typical networks consist almost entirely of leaves as $N \to \infty$; that is, the number of leaves satisfies $N_1/N \to 1$ as $N \to \infty$. Nodes with degrees $k \geq 2$ constitute what we term the

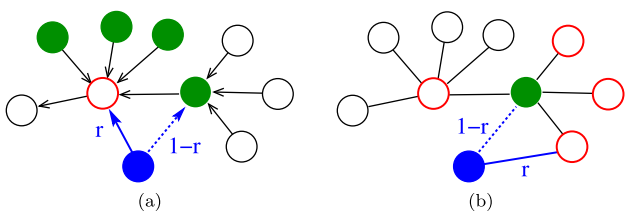


**Fig. 6** Comparison of redirection for (a) directed and (b) undirected networks. (a) The new node (blue) attaches by redirection to the unique ancestor (black) of the target (red). (b) With the same target in an undirected network, the new node attaches to any one of the red neighboring nodes (color figure online)

network "core". This core comprises an infinitesimal fraction of the network, viz., the number of core nodes $\mathcal{C} = \sum_{k\geq 2} N_k$ grows as $N^{v-1}$, with $v \approx 1.567$, as determined by numerical simulations [37].

The degree distribution for complete redirection has an algebraic tail $N_k \propto k^{-v}$ with $v \approx 1.567$. As discussed in the introduction, a degree distribution with $v<2$ cannot occur in sparse networks, which exhibit standard extensive $N_k \propto N$ scaling. However, a degree distribution with such a fat tail can arise if the amplitude grows sub-extensively with network size, that is, $N_k \sim N^{v-1}/k^v$. Thus the number of nodes of any fixed degree $k$ with $k \geq 2$ grows sublinearly in $N$, with $N_k \sim N^{v-1}$.

Several features of networks grown by complete redirection can be understood analytically [37], while others, such as the exponent $v$ currently appear to be beyond the reach of available techniques. The difficulty in making theoretical progress is that the change in the degree of a specific node depends on the degrees of all its neighbors. This inherent non-locality in the growth rule means that it is not possible to write a master equation for the degree distribution alone. Instead, the equation for the degree distribution must involve degree correlation functions between neighboring nodes, and this quantity, in turn, involves higher-order correlation functions.

## 5. Conclusions

Preferential attachment networks have been the focus of intense investigation for the past two decades. Part of the reason for this explosion of interest stemmed from the confluence of theoretical insights that were inspired by the existence of new datasets about networked systems. The network paradigm is alluring and a large number of seemingly unrelated many-body systems are now studied within the context of complex networks.

While the field has advanced significantly, some basic facts about the simplest network models seem under-appreciated. One is that the degree distribution of linear preferential attachment networks sensitively depends on microscopic details of the network growth mechanism. While the earliest theoretical studies of linear preferential attachment networks found a degree distribution exponent of $v = 3$, any exponent value with $v > 2$ can be achieved by linear preferential attachment. This non-universality is surprising because the standard lore from statistical physics suggests that exponent values should be universal and independent of the details of the network growth process.

Another important facet of complex networks that has yet to be fully exploited is that they can be generated by simple redirection algorithms. When a new node joins the network, it either attaches with a given probability to a pre-existing node that is chosen uniformly at random, or it attaches to the ancestor this target node with the complementary probability. This algorithm is simple to implement and efficient because it generates networks of $N$ nodes in an

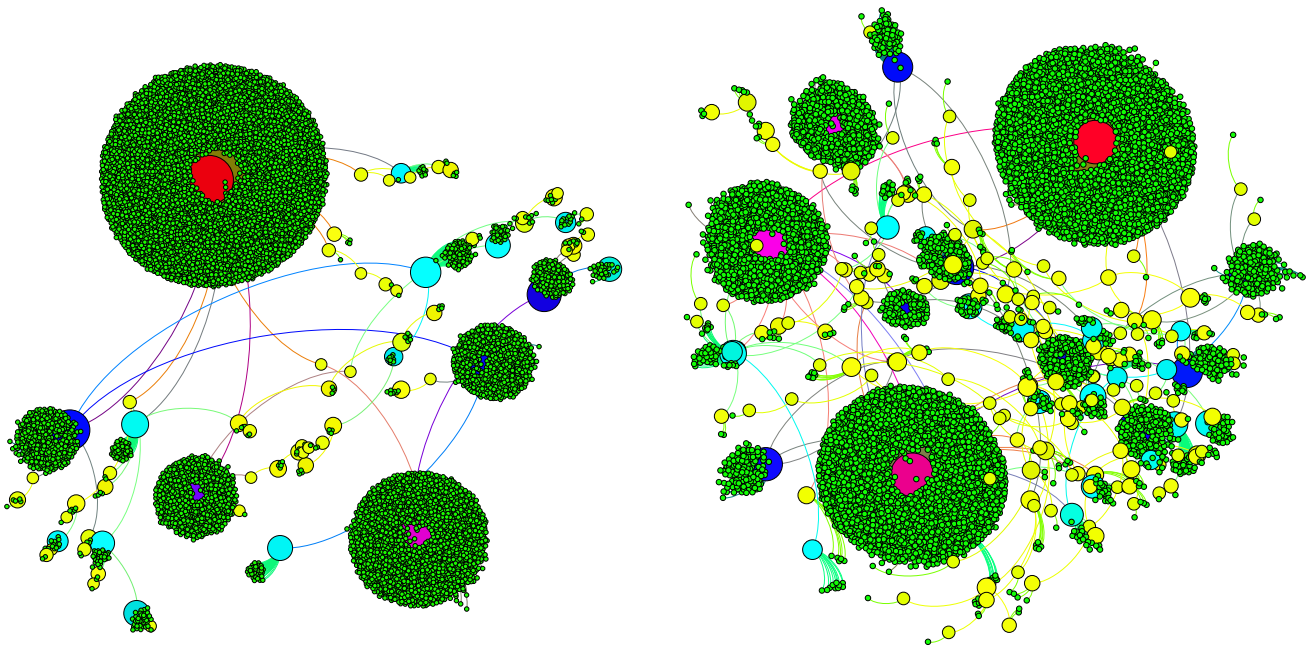

**Fig. 7** Examples of tree networks of $10^4$ nodes that are grown by complete redirection. Green: nodes of degree $k = 1$ (leaves); yellow, $2 \leq k \leq 10$; cyan, $11 \leq k \leq 99$; blue $100 \leq k \leq 500$; violet → red, $k > 501$. The node radius also indicates its degree (color figure online)

algorithmic time that is also of the order of $N$. We showed how to generate networks that are equivalent to sublinear preferential attachment and to shifted linear preferential attachment by suitable redirection rules. Our redirection perspective provides crucial insights that relate the random recursive tree to preferential attachment networks.

We also briefly discussed undirected networks that are grown by complete redirection. The resulting networks have a highly modular structure (Fig. 7): the number of core (nodes of degree $\geq 2$) scales sublinearly with the total number of nodes, as $N^{\nu-1}$, with $\nu \approx 1.567$. A natural question here is: why does this redirection mechanism lead to such singular networks? We really don't know. The master equation approach, which works so well for directed networks, is inadequate to describe the structure this class of networks. This inadequacy stems from the effective nonlocality in the growth mechanism, and different approaches seem to be needed to truly understand the behavior of this network. An even deeper reason is the lack of self averaging: the random quantities $N_k$ for any $k > 1$ exhibit huge fluctuations from realization to realization. Therefore averages $\langle N_k \rangle$ incompletely characterize each $N_k$, and, by construction, the master equation approach only gives average quantities.

The oldest and perhaps still most famous complex network is the evolving random graph or Erdős-Rényi (ER) random graph [39]. The same model appeared earlier in the work of Flory and Stockmayer [40–42]; this model turns out to be equivalent to aggregation with the product kernel [25]. The percolation transition manifested by the emergence of the giant component [43] in evolving random graphs is equivalent to gelation in aggregation [25]. The ER random graph initially consists of $N$ disjoint nodes, and it evolves by drawing randomly chosen pairs of nodes and connecting them. Thus only the number of links increases. Combining the ER graph with preferential attachment, one may postulate that nodes of degree $i$ and $j$ connect with probability proportional to $(i+\lambda)(j+\lambda)$. This evolving graph undergoes a percolation transition and later a condensation transition when the entire system condenses into a single component [44, 45] and then increment all following reference numbers by 1.

Closer to our modeling is a network that grows via two distinct mechanisms: (i) a new node is added with probability $p$, and (ii) a new link between existing nodes is created with probability $1-p$. Both of these steps can incorporate redirection in a natural way. Earlier work on similar models [46] was focused on network characteristics, such as the degree distribution. The distribution of components remains mostly unexplored and it would be interesting to analyze percolation and condensation transitions for this type of network. There are indications [46] that the percolation transition could be different from the standard Curie-type transition appearing in the ER graphs [43], viz., a Berezinskii-Kosterlitz-Thouless infinite-order transition [47–50] that often appears in growing networks.

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