

Multifractal structure of the incipient infinite percolating cluster

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By analyzing the voltage distribution of a random resistor network, we show that the backbone of the percolating cluster can be partitioned into an infinity of subsets, each one characterized by a fixed value of $x \equiv \ln V / \ln V_{\max}$, where V is the voltage across each bond and V_{\max} is its maximum value. Each subset is characterized by a distinct value of the fractal dimension $\phi(x)$, and as a consequence an infinite set of order parameters is required to describe the backbone structure. A new scaling approach and a real-space renormalization-group treatment are presented to treat the novel aspects of this problem. The mechanism for multifractality based on an underlying multiplicative process is illustrated on a hierarchical model.

The percolation problem has received considerable attention, particularly for its simplicity and great utility in describing a wide variety of geometrical problems.¹ In the bond percolation, each bond of a d -dimensional lattice is occupied with probability p and vacant with probability $1-p$. As p approaches the percolation threshold p_c , the linear dimension of finite clusters, ξ , diverges as $\xi \sim (p - p_c)^{-\nu}$. The order parameter, P_∞ , i.e., the density of sites in the infinite cluster, approaches zero as $P_\infty \sim (p - p_c)^\beta$, and the k th moment of the cluster number distribution, $S(k) \equiv \sum_s s^k n(s)$, diverges as $(p - p_c)^{-\gamma(k)}$, where $n(s)$ is the number of clusters of size s per site. The exponents $\gamma(k)$ are not independent but rather obey a constant "gap" relation

$$\gamma(k)/\nu = kd_f - d, \tag{1}$$

where $d_f = d - \beta/\nu$ is the fractal dimension of the incipient infinite cluster and ν^{-1} is the fractal dimension of the singly connected bonds in the backbone.² Thus all the critical exponents may be expressed in terms of these two fractal dimensions only. The simple linear dependence of $\gamma(k)$ on k in (1) follows from the standard scaling form¹ $n(s) \sim s^{-\tau} f(s(p - p_c)^\sigma)$, with $\tau = 1 + d/d_f$ and $\sigma = 1/\nu d_f$.

The description of percolation in terms of only two exponents contrasts with what has been found recently for various physical properties of random systems at the percolation threshold. For example, in a dilute Ising model, the correlation length diverges with the critical exponent ν , while in the dilute Heisenberg model, the correlation length diverges with an exponent ξ , which is identical to the resistivity exponent of a random resistor network.² In apparently unrelated applications, the amplitude of the resistance noise has been found to diverge³ near the percolation threshold with an exponent ξ_{noise} , while the mean-square first-passage time for tracer particles to traverse through a random resistor network at the percolation threshold diverges⁴ with an exponent ξ_{FPT} . These apparently diverse problems can be unified within the framework of the voltage distribution.^{3,5-7} In this context, it has been found that the k th moments

of the voltage distribution scale with an infinite hierarchy of independent exponents, and that for $k = -1, 2, 4$, and ∞ , they are related, respectively, to ξ_{FPT} , ξ , ξ_{noise} , and $1/\nu$.

In this Brief Report, we will show that these independent exponents can be related to a novel "multifractal" structure of the percolating backbone, in which the backbone bonds can be partitioned in an infinity of subsets (see Fig. 1). Each subset can be characterized by a unique value of a parameter $x \equiv \ln V / \ln V_{\max}$ in the corresponding resistor network problem. Here V is the absolute value of the voltage drop across the bonds in the subset and V_{\max} is the maximum value of V (i.e., the voltage across the singly connected bonds). Each subset is characterized by its own fractal dimension $\phi(x)$, and different subsets play dominant roles in various physical problems. Examples of related multifractal behavior where critical behavior is governed by a hierarchy of critical exponents include, e.g., turbulence,⁸⁻¹⁰ nonlinear dynamics,¹¹ growth phenomena,¹²⁻¹⁴ localization,¹⁵ and the absorption of diffusive particles on a polymer chain.¹⁶

We will also argue, in contrast to ordinary critical phenomena, that this multifractal structure of the backbone leads naturally to the introduction of infinitely many order parameters corresponding to the infinity of fractal sets. This description in terms of infinitely many order parameters is also reminiscent of spin glasses, where an infinity of order parameters is also required to describe the system adequately.¹⁷ A new scaling approach is presented to describe the multifractality of the backbone, and a real-space renormalization-group method is also developed to treat this problem. Our predictions are supported both by analytic calculations on a hierarchical model, and also by computer simulations on a two-dimensional random resistor network at p_c .

Consider a random resistor network of linear dimension L at p_c . We characterize each bond by the absolute value of the voltage drop V across it, when a unit potential is applied across the opposite edges of the network. At the percolation threshold, the maximum voltage drop in any bond, V_{\max} , occurs in the links,^{2,18} or singly con-

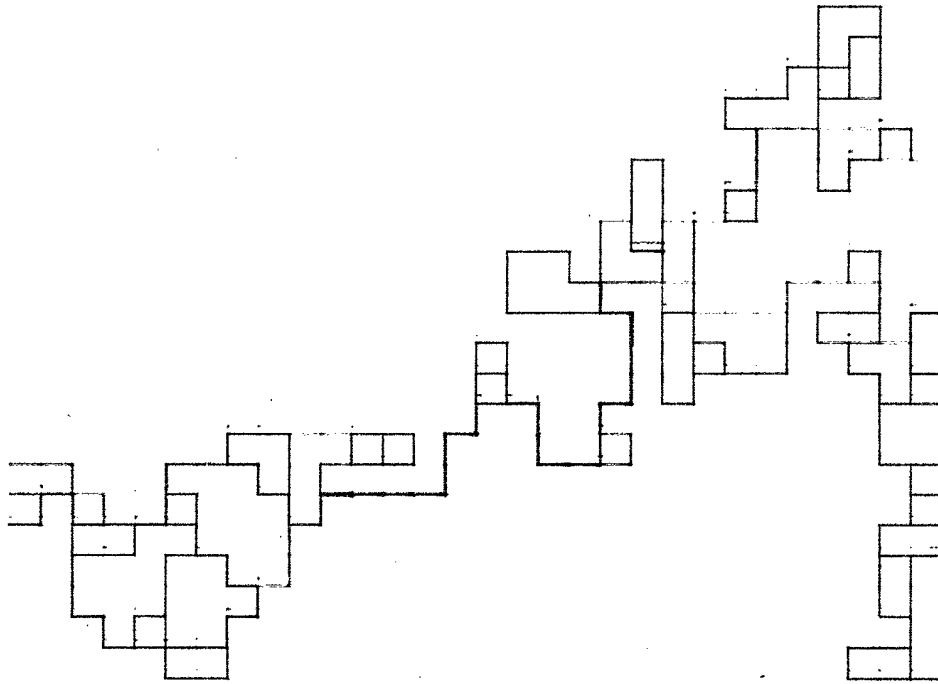


FIG. 1. Illustration of the fractal sets within the percolating backbone. The links (shown in red) carry the entire current, and are characterized by a value of $x = 1$, while bonds deep within blobs (violet) are characterized by larger values of x . The backbone bonds have been divided into several fractal sets of equal ranges of the variable $x = \ln V / \ln V_{\max}$ increasing in value in the following order of colors: red, brown, orange, yellow, light blue, dark blue, and violet.

nected bonds, which are the bonds which could render the network disconnected if they are cut. Therefore, these bonds carry the total current I passing through the network so that $V_{\max} = I = G$, where G is the conductance of the network. Since G scales as $L^{-\zeta}$, with ζ the conductance exponent, $V_{\max} \sim L^{-\zeta}$. Moreover, the number of links, N_{links} , scales as $L^{1/\nu}$. Thus, the subset of bonds characterized by the value $x = 1$ has a fractal dimension $1/\nu$ and a ‘‘singularity exponent’’ ζ .

Next, we partition all the other backbone bonds into different subsets, each with its own fractal dimension and singularity exponent. A natural way of accomplishing this task is suggested by considering the following moments and their critical behavior at p_c :

$$M(k) = \sum_V n(V)V^k \sim L^{-p(k)/\nu}, \tag{2}$$

where $n(V)$ is the number of bonds with an absolute value of the voltage drop V across them. In contrast to conventional scaling, it has been found that the $p(k)$ form an infinite independent set of exponents.^{3,5-7} This leads us to a new type of scaling ansatz to describe the scaling of the voltage distribution. If $V^* = V(k)$ is the value of V which maximizes the summand in (2) for each value of k , we write

$$V(k) \sim A(k)L^{-\alpha(k)}, \tag{3}$$

$$n(V(k)) \sim B(k)L^{f(k)}, \tag{4}$$

where $A(k)$ and $B(k)$ are amplitudes which depend on k . From (2) we obtain

$$p(k)/\nu = k\alpha(k) - f(k), \tag{5}$$

while the condition that V^* is the maximum implies that

$$\frac{d}{dk} \frac{p(k)}{\nu} = \alpha(k) \tag{6a}$$

and

$$\frac{d\alpha(k)}{dk} < 0. \tag{6b}$$

Therefore, given $p(k)$, one can calculate $\alpha(k)$ and $f(k)$, and vice versa. Similar approaches leading to (5) and (6) were first developed by Halsey *et al.*¹¹ We now show that the voltage distribution $n(V)$ is a scaling function of the basic variable $x = \ln V / \ln V_{\max}$. From (3) we have

$$\ln V(k) = \ln A(k) - \alpha(k) \ln L. \tag{7}$$

Therefore as $L \rightarrow \infty$ we may write

$$x = \alpha(k)/\alpha(\infty). \tag{8}$$

Since $\alpha(k)$ is monotonic, from (6b), we can invert (8) and we obtain $k = k(x)$, so that (4) becomes

$$n(V) = C(x)L^{\phi(x)}, \tag{9}$$

where $\phi(x) = f(k(x))$ and $C(x) = B(k(x))$. Alternatively, we can express the $n(V)$ in terms of V alone, by eliminating L through the relation $\ln V_{\max} = -\zeta \ln L$ to obtain

$$n(V) \sim V^{-\psi(x)}C(x), \tag{10}$$

with $\psi(x)=\phi(x)/x^{\xi}$. Thus, we see that the voltage distribution can be written as a power law in V , but with an exponent that is also voltage dependent.¹⁹ Equation (9) expresses the fact that the bonds in the network divide naturally into different sets characterized by a given value of x . Each set has an independent fractal dimension $\phi(x)$, directly related to the moment exponents $p(k)$.

Above p_c , it is convenient to work with quantities which are independent of L in the $L \rightarrow \infty$ thermodynamic limit. Therefore we define a new intensive voltage variable $\bar{V}=LV$. By standard scaling, $\bar{V}(k) \sim \xi^{1-\alpha(k)}$ as $L \rightarrow \infty$, where ξ is the correlation length. Corresponding to the infinity of fractal sets, there are infinitely many conjugate order parameters, defined as the density of bonds, $\rho(\bar{V}, p)$, characterized by the value of the quantity $y = \ln \bar{V} / \ln V_{\max} = [1 - \alpha(k)] / [1 - \alpha(\infty)]$. Using scaling, it is easy to show that $\rho(\bar{V}, p) \sim (p - p_c)^{\beta(y)}$, where the order parameter critical exponent, $\beta(y) = d - f(k(y))$, and $k(y)$ are defined by inverting the relation between y and k .

An important question that multifractal scaling raises is how can one adapt renormalization-group ideas to calculate the fractal dimensions of all the bond sets that comprise the backbone. One simple way to renormalize geometrical quantities is by mapping a $b \times b$ cell into a single bond.²⁰ For example, to find the fractal dimension d_f of the backbone, one may evaluate the mass rescaling of the backbone, $m' = \lambda m_0$, where λ is the average number of bonds in the backbone of the spanning cluster in the $b \times b$ cell at p_c . The fractal dimension is then obtained from the eigenvalue $\lambda = dm' / dm_0$, by $d_f = \ln \lambda / \ln L$. This same procedure cannot be applied directly to each fractal subset of bonds in order to calculate the spectrum of fractal dimensions, since there are only a small number of bonds in a small cell, and this cannot be representative of the infinity of fractal sets. Thus it appears that a direct renormalization procedure is inadequate for understanding multifractal behavior.

A very simple alternative is to renormalize the moments in Eq. (2), which in fact can be considered as a partition function of the problem. If the voltage across the cell to be rescaled is V_{ext} , the voltage across the renormalized bond is still V_{ext} ; therefore the renormalized moments are $M'(k) = (V_{\text{ext}})^k$. These are related to the moments $M(k)$ of the voltage distribution in the cell by $M'(k) = \lambda M(k)$, with the eigenvalue $\lambda = b^{p(k)}$. In practice, by choosing $V_{\text{ext}} = 1$, one finds $-p(k) = \ln M(k) / \ln b$, and ultimately the fractal dimension of all the fractal sets by using (5). Therefore, the small cell apparently contains all the information of the large system, but instead of renormalizing $n(V)$ directly, one considers the infinite set of relevant variables $M(k)$.

The voltage distribution can be calculated exactly on the hierarchical model⁵ of the percolating backbone shown in Fig. 2. On this model, we found that the moments scale with an infinite set of exponents $p(k)$. Once the decomposition of $p(k)$ into $\alpha(k)$ and $f(k)$ is performed via (5) and (6), we also find that $\alpha(k)$ and $f(k)$ coincide with the scaling exponents defined in (3) and (4), and obtained independently by the method of steepest

descents. Finally, the voltage distribution can be cast in the scaling from (9) and (10), thus supporting our general predictions. Instead of presenting the details of the calculations here (they can be found in Ref. 19) we shall illustrate on the hierarchical model the basic mechanism which produces multifractality. The hierarchical model is obtained by successive iterations, substituting each bond of the previous iteration with the unit cell. If $\Delta V = 1$ is the voltage drop across the cell, the voltage across the two bonds in the blob is $V_1 = \frac{1}{5}$, while the voltage drop across the links is $V_2 = \frac{2}{5}$. From the invariance of Kirchhoff's laws under the transformation $V \rightarrow \lambda V$, it follows that if the voltage across the system is $\Delta V = \lambda$, the voltages across the individual bonds are λV_1 and λV_2 . At the next iteration, $N = 2$, the network is made of four unit cells. Due to self-similarity, the voltage across the two cells generated by the links is V_1 . Therefore, within each cell one obtains two bonds with voltage $V_1 V_1$, and two bonds with voltage $V_1 V_2$, and similarly for the other two cells as shown in Fig. 2. Due to this multiplicative process, the k th moments at the N th iteration are given by

$$M(k) = (2V_1^k + 2V_2^k)^N. \tag{11}$$

Since $V_1 = \frac{1}{5}$ and $V_2 = \frac{2}{5}$, we find the infinite set of critical exponents

$$p(k) = \frac{\ln M(k)}{N \ln 2} = 1 + \ln [(\frac{1}{5})^k + (\frac{2}{5})^k] / \ln 2, \tag{12}$$

where we have used the relation $L = 2^{N\nu}$, with L the

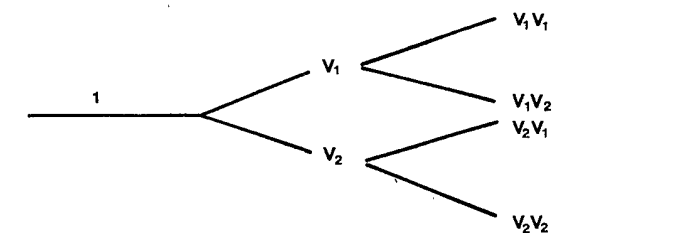
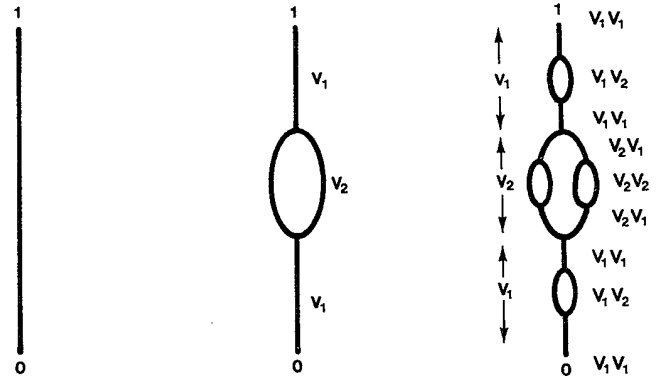


FIG. 2. The first few levels of iteration in the hierarchical lattice model and schematic representation of the multiplicative process (from Ref. 5). The $N = 1$ level is the “unit cell” which is rescaled into a single bond in our renormalization procedure.

linear dimension and 2^N the number of links. For a more general hierarchical model, obtained by successive iterations of a unit cell with n bonds, with voltage drops V_1, \dots, V_n , the moments are given by

$$M(k) = (V_1^k + \dots + V_n^k)^N.$$

In conclusion, we see that multifractality arises in self-similar structures when the field (in our case V) associated with each element satisfies a multiplicative rule. The rule is satisfied when the equation obeyed by the field is invariant under the scaling transformation $\phi \rightarrow \lambda\phi$. The importance of the multiplicative process was also stressed in Refs. 11 and 21.

Next we illustrate our renormalization approach for the voltage moments by rescaling the unit cell of the hierarchical model (Fig. 2) into a single bond. If the voltage across the cell is unity, the moments $M(k)$ are

$$M(k) = 2\left[\left(\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)^k\right] = 2^p(k), \quad (13)$$

where for iteration index $N=1$, $L^{1/\nu}=2$. This result for $p(k)$ coincides with the result found by the exact calculation in (11). Notice also that by renormalizing the moments in the unit cell we were able to obtain the fractal dimensions of all the fractal sets, although in the unit cell only two fractal sets, the hottest and coldest bonds, are present originally.

The theoretical picture of multifractality has also been confirmed by numerical simulations on the square lattice at p_c for various system sizes. From the voltage moments found numerically, one finds that the $p(k)$ are indeed independent. Alternatively, we have evaluated the $p(k)$ by an exact rescaling of a 4×4 into a 3×3 cell. The two sets of results for $p(k)$ agree extremely well, lending support for our renormalization-group approach. In addition, from $p(k)$ one can calculate $\alpha(k)$ and $f(k)$, and finally the fractal dimension $\phi(x)=f(k(x))$ (Fig. 3). We have also calculated the fractal dimensions of the

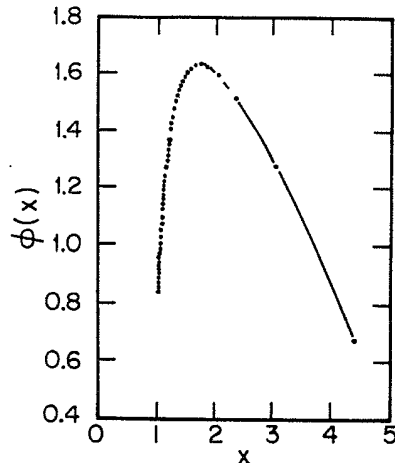


FIG. 3. Numerical estimates of the fractal dimension $\phi(x)$ obtained from simulations of the square-lattice resistor network at the percolation threshold.

various fractal sets directly, by computing the number of bonds corresponding to a given value of x on large cells. The agreement of the fractal dimension obtained by these two methods provides additional confirmation of our scaling theory.

In conclusion, we have shown that the bonds in the percolating cluster can be partitioned into an infinity of fractal subsets, each with a distinct value of the fractal dimension. This partitioning offers a visually appealing and a conceptually novel description of the structure of the percolating backbone.

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