Multiscaling approach in random resistor and random superconducting networks

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We report on a variety of novel features for the distribution of voltage drops across the bonds of a random resistor network. To describe this distribution analytically, we introduce a simple geometrical model, with a hierarchical structure of links and blobs, which appears to capture the basic features of random networks near the percolation threshold. On this model, we find that the voltage distribution is a log binomial, and that an infinite hierarchy of exponents is required to characterize the moments of this distribution. On general grounds, we argue that this exponent hierarchy emerges naturally from an underlying distribution which, at the percolation threshold, can be written in the form, $L^{\phi(\ln V/\ln V_{\max})}$, where L is the linear size of the system, V is the voltage drop, and $V_{\max}$ is the maximum value of this voltage drop. The nonconstancy of $\phi(y)$ as a function of y is an unconventional feature in the context of a scaling approach, and a variety of novel properties result. These are tested by numerical simulations of the voltage distribution for square-lattice networks at the percolation threshold. In particular, the moments of this distribution are found to scale independently, with the exponents of the positive moments in excellent agreement with those of the hierarchical model. We also discuss some intriguing properties associated with the voltage distribution above the percolation threshold, most notably, that the higher moments of the distribution are nonmonotonic functions of the bond concentration. Finally, we exploit duality arguments to investigate the voltage distribution of a random superconducting network.

I. INTRODUCTION

The critical behavior of the random resistor network is a classical percolation problem whose properties have recently been found to be considerably richer and more complex than previously imagined.[1-6] Many of these new features emerge naturally upon studying the underlying distribution of voltage drops across the bonds in the network. In a previous paper,[1] we reported some preliminary results for this voltage distribution, both from analytic calculations on a simple hierarchical lattice model, and from numerical simulations of square-lattice random resistor networks at the percolation threshold. In this paper, we showed that a conventional scaling approach does not provide a complete description of the random resistor network, as an infinite hierarchy of exponents is needed to characterize the moments of the voltage distribution. The same result was also found independently by studying the properties of resistance noise in random networks by Rammal *et al.*[2] An analogous behavior was subsequently found in various growth phenomena, such as diffusion-limited aggregation,[7-11] as well as in the localization problem,[12] and in dynamical systems.[13] The presence of an infinite set of exponents was first discovered in the context of turbulence by Mandelbrot.[14] Further work for the specific problem of turbulence can be found in Refs. 15 and 16 In this paper, our goal is to understand the properties of the voltage distribution in greater detail, and to present a general formalism to account for its anomalous scaling properties.

To construct the voltage distribution, we consider a random network of bonds in which a unit resistance is associated with each bond, and a unit voltage is applied at the opposite edges of the random network. Each bond can then be characterized by the absolute value of the voltage drop V across it, or equivalently, by the absolute value of the current I flowing through it. Alternatively, a bond can be characterized by its power dissipated, V^2. The voltage distribution is built up by recording the absolute value of the voltage drops in all the bonds in the network, and then averaging over a sufficiently large number of realizations of the network. Our primary interest is in the quantitative features of this distribution of voltage drops when the network is strongly disordered, e.g., when the system is near the percolation threshold.

A few basic properties of the voltage distribution can be obtained through elementary geometric considerations. At the percolation threshold, the maximum voltage drop, $V_{\max}$, occurs in the links, or singly connected bonds.[17,18] These are bonds which would render the entire network disconnected if they were to be cut. Therefore, these bonds carry the total current passing through the network. If the applied potential and the bond conductances are both defined to have value unity, then $V_{\max}=I=G$, where G is the conductance of the network. From this connection, the dependence of $V_{\max}$ on the system size can be inferred. On the other hand, the minimum voltage drop, $V_{\min}$, will be found in the nearly balanced bonds embedded within blobs at the smallest possible length scale.

Complete information about the nature of the voltage distribution is contained in the moments

$$M(k)\equiv\langle V^k\rangle=\sum_{\ln V} n(V)V^k\,, \tag{1}$$

where $n(V)$ is defined as the number of bonds characterized by an absolute value of the voltage drop V, and where the sum is over all values of $\ln V$, which turns out to be the natural variable in the problem. Some of these moments have immediate physical interpretations. For example, $\langle V^0 \rangle$ is the average number of bonds in the backbone,[19] and $\langle V^2 \rangle$ is the average conductance. In addition, other positive moments of the distribution, which may be less familiar, can be shown to have physical relevance, as well as providing information on the high-voltage tail of the distribution. Thus the quantity $\langle V^4 \rangle$ is related to the magnitude of noise in the network,[2] while, as $k \to \infty$, $\langle V^k \rangle$ weights the high-current-carrying or "hottest" bonds of the network most strongly, so that this moment is simply related to the number of links. On the other hand, negative moments weight the low-current-carrying or "colder" bonds in the network more strongly, and thus provide information on the low-voltage tail of the distribution whose features are very different from that of the high-voltage tail. For example, $\langle V^{-1} \rangle$ can be shown to be related to the coefficient of hydrodynamic dispersion. This phenomenon arises when passive tracer particles are convected through a random network and are dispersed due to the multiplicity of paths available for the tracer in which each path has a different transit time.[20,21] If molecular diffusion is neglected, then the time for a tracer particle to traverse a given bond is proportional to the inverse of the current in the bond, while the probability for the tracer to enter a particular bond at a junction is proportional to the current entering the bond. As a result, the kth moment of the transit time distribution will vary as $V^{-(k-1)}$, and the quantity $\langle t^2 \rangle - \langle t \rangle^2$, which quantifies the magnitude of the dispersion, will be related to $\langle V^{-1} \rangle$. In contrast to the conductance, which is limited by the highest-current-carrying bonds of the network, dispersion is limited by the lowest-current-carrying bonds in the network, where the tracer essentially gets "stuck."

To gain some qualitative insights into the voltage distribution, we have previously introduced[1] a simple hierarchical model which accurately describes the geometrical features of random networks near the percolation threshold. On the hierarchical model, we found that the voltage distribution is a log binomial, which leads naturally to an infinite set of exponents being required to describe the moments of this distribution. Consequently, the voltage distribution cannot be characterized by a unique typical voltage scale. This was confirmed in numerical simulations,[1,2] and by series and the ϵ expansion,[3] where it has been found that an infinite set of independent exponents was required to account for the scaling behavior of the voltage moments, $M(k)$.

These results are in stark contrast with the conventional scaling properties of many distributions in statistical mechanics. For example, consider the cluster-size distribution in percolation, $n(s)$, which is the number of clusters containing s sites. For a finite-size system of linear dimension L at the percolation threshold, this distribution can be written in a scaling form,[22]

$$n(s) \sim s^{-\tau} f(s/L^{d_f}) , \tag{2}$$

with two unknown exponents; τ, the cluster number exponent, and d_f, the fractal dimension of the critical cluster. From this functional form, the moments of the distribution can easily be shown to scale in a simple way,

$$\sum_s s^k n(s) \sim L^{kd_f} L^{d_f(1-\tau)} . \tag{3}$$

The linear dependence of the kth moment of the cluster-size distribution on k stems from the fact that there is a *single* dominant critical cluster size, $s^* \sim L^{d_f}$, that is responsible for the critical behavior.

Although novel scaling properties are found for the specific example of the voltage distribution, a much wider range of applicability has been discovered recently. For example, the properties of the growth site probability distribution in diffusion-limited aggregation are rather similar to those of the voltage distribution. This is the first-passage probability for a random walker to land at a given surface site of an aggregate at some intermediate stage of its growth. The growth probabilities at the fastest growing tips and the "deepest fjords" of the aggregate are found to scale quite independently as a function of the aggregate size. In fact, due to the analogy between diffusion-limited aggregation and dielectric breakdown,[23] the growth probabilities can be directly related to the voltage distribution on the surface of an equipotential aggregate with appropriate boundary conditions.[7,8,10] Moreover, the novel scaling behavior of the voltage distribution can be interpreted and reformulated on the basis of the recent description of measures on fractal sets.[13] This characterization again stresses the necessity of considering a spectrum of exponents to describe the scaling properties of fractal systems, rather than considering only a single exponent, such as the fractal dimension, which provides information that is rather limited in scope.

The outline of the remainder of this paper is as follows. In Sec. II, we present a general formalism to describe the voltage distribution of random resistor networks, and discuss the anomalous scaling properties which follow. We contrast these new features with those inherent in conventionally scaling distributions, such as the cluster-size distribution in percolation. In Sec. III, we calculate the voltage distribution on the hierarchical lattice and show that it is a log binomial. Although the exact discrete form of this distribution is essentially trivial, its continuum version is quite rich, and we shall use it to illustrate many of the novel scaling features introduced in Sec. II. From the hierarchical lattice, we can also infer the functional form of the voltage distribution of a random resistor network above the percolation threshold, p_c. In addition, a generalization of the model is introduced which suggests that the log binomial form is a characteristic feature of the voltage distribution on any hierarchical structure. A similar logarithmic voltage dependence is found numerically on percolation clusters. In Sec. IV, we present numerical data for the voltage distribution. Both exact data from small lattice sizes, and approximate Monte Carlo data from larger lattice sizes at p_c is considered. We find that the moments of various voltage moments at and above p_c do indeed scale independently. A scaling analysis of the voltage distribution is performed in order to test some of the more striking features that emerge from our theoreti-

cal formulation of the problem. In Sec. V, we discuss some intriguing physical features associated with the properties of the voltage distribution above the percolation threshold. In Sec. VI, we consider the voltage distribution of a random superconducting networks and show, by duality, that it is related to that of the random resistor network in two dimensions. Finally, in Sec. VII, we give some general conclusions.

II. VOLTAGE DISTRIBUTION IN RANDOM NETWORKS

We wish to discuss, in very general terms, some of the salient and anomalous features of the voltage distribution in random resistor networks, and contrast these features with those of distributions which scale in a conventional manner. Consider an $L\times L$ square lattice in which the bonds are either unit resistances with probability p, or insulators with probability $1-p$. If a unit potential, $\Delta V=1$, is applied across the opposite edges of the network, then the second moment of the voltage distribution coincides with the conductance G. At the percolation threshold, p_c, this moment vanishes as $L^{-\zeta/\nu}$, where ζ is the conductance exponent and ν is the correlation length exponent. Since other moments provide useful additional information about the nature of the voltage distribution, we are led to consider the statistical properties of the general kth moment of the distribution,

$$M(k)=\sum_{\ln V} n(V)V^k \sim L^{-p(k)/\nu}, \tag{4a}$$

where the asymptotic relation serves to define the moment exponents $p(k)$.

Alternatively, we may define the distribution of currents, or equivalently, the voltage distribution under the condition that the total current I flowing through the network is equal to 1, rather than imposing a unit external potential drop. From the relation $\Delta V G=I$, the condition $I=1$ corresponds to imposing the boundary condition $\Delta V=G^{-1}$ for the voltage drop across the opposite edges of a given configuration of the network. In this case, each bond will be characterized by a voltage drop $v=V/G$, if V is the voltage drop across the bond when a unit external potential is imposed. We now define the following moments $\mathscr{L}(k)$ and the associated exponents $\tilde{\zeta}(k)$ through

$$\mathscr{L}(k)=\sum_{\ln v} n(v)v^k \sim L^{\tilde{\zeta}(k)}, \tag{4b}$$

with $\tilde{\zeta}\equiv\zeta/\nu$. The moments in (4a) and (4b) are simply related by $\mathscr{L}(k)=G^{-k}M(k)$ and the moment exponents are therefore related by $\zeta(k)=kp(2)-p(k)$. At p_c, we also have $v=V/V_{\max}$ since $G=V_{\max}$. Therefore $n(v)$ is the number of bonds in which a fraction v of the total current in the network flows. In analogy with the moments, $M(k)$, we deduce that $\mathscr{L}(0)$ coincides with the total number of bonds in the backbone, $\mathscr{L}(2)$ scales as the resistance, $\mathscr{L}(4)$ is related to the amplitude of the noise,[2] and $\mathscr{L}(\infty)$ coincides with the number of links. In general, *each* $\mathscr{L}(k)$ can be regarded as a length measure of the backbone, which ranges from the total number of bonds in the backbone, for $\mathscr{L}(0)$, to the number of links, for $\mathscr{L}(\infty)$. As k increases, $\mathscr{L}(k)$ is sensitive to progressively finer geometrical aspects of the backbone, and each $\mathscr{L}(k)$ has a distinct scaling behavior (see also Ref. 2). For the discussion in the remainder of the paper, we shall primarily consider the moments defined by (4a).

If $n(V)$ were a scaling function of V, then the difference $p(k)-p(k-1)$ would be a constant proportional to the "gap" exponent. This standard behavior is what would be obtained if the voltage distribution had a scaling form analogous to the one given for the cluster-size distribution in Eq. (3). Let us now study those conditions under which $p(k)-p(k-1)$ is not a constant, and the ramifications that such a property would have for the voltage distribution. We rewrite $M(k)$ as

$$M(k)=\sum_{\ln V} e^{\ln n(V)+k\ln V}\equiv\sum_{\ln V} e^{F(V,k)}, \tag{5}$$

with

$$F(V,k)=\ln(n(V))+k\ln V. \tag{6}$$

From (4a), $F(V,k)$ is expected to diverge logarithmically for large L, so that we may attempt to evaluate the sum in Eq. (5) by steepest descents. The value of V for which the summand is a maximum is given by the condition

$$\frac{d}{dV}[\ln(n(V))+k\ln V]=0, \tag{7}$$

and this yields for the maximizing value of V

$$V^*=-k\frac{n(V^*)}{n'(V^*)}. \tag{8}$$

In general, then, for each value of k, there is a corresponding distinct value of $V^*=V(k)$ which locates the peak value of the product $n(V)V^k$. In the context of a finite-size scaling approach, let us now make the following scaling ansatz for $V(k)$, and a corresponding one for $n(V(k))$:

$$V(k)=A(k)L^{-\alpha(k)}, \tag{9a}$$

$$n(V(k))=B(k)L^{f(k)}. \tag{9b}$$

For a conventionally scaling distribution, the exponents α and f would be independent of k and the amplitudes $A(k)$ and $B(k)$ would also be smooth functions of k, so that there is effectively a unique typical value V^* that characterizes the distribution. However, consider the situation of the voltage distribution, where the exponents α and f turn out to be dependent on k. This novel behavior will be justified, both by analytical calculations on the hierarchical lattice, and by numerical simulations on the random resistor network, in later sections. However, let us now investigate the quantitative consequences of the scaling ansatz (9) for the voltage moments. From the steepest-descents approach, we retain only the largest term in (5), and find

$$M(k)\sim e^{F(V^*,k)}=n(V(k))[V(k)]^k\sim L^{f(k)-k\alpha(k)} \tag{10}$$

and therefore, by comparing with (4), we identify

$$\frac{p(k)}{\nu}=k\alpha(k)-f(k). \tag{11}$$

Thus the exponent $p(k)$ characterizing the kth moment of the voltage distribution decomposes in a natural way into two factors. From the second line of Eq. (10), we see that the exponent $f(k)$ represents the dependence of $n(V(k))$ on L, i.e., the fractal dimension of the set of bonds characterized by the voltage drop $V(k)$. A novel feature of the voltage distribution is that there is a distinct fractal dimension for the subset of the bonds that have a voltage drop corresponding to a given fixed value of k. Similarly, $\alpha(k)$ represents the manner with which $V(k)$ scales with the system size L, and the independent scaling of each $V(k)$ is indicative of the fact there is no unique typical voltage scale that characterizes the distribution.

We also note, from (8), that since there is a unique value $V(k)$ for each value of k, V^* can be considered as an independent variable V. Using the fact that $\partial/\partial V=(\partial k/\partial V)(\partial/\partial k)$, then we can maximize (7) with respect to k rather than with respect to V. We thus obtain

$$\frac{\partial k'}{\partial V}\left[\frac{\partial}{\partial k'}\ln N(V(k'))+\frac{\partial}{\partial k'}\ln V(k')\right]_{k'=k}=0 . \tag{12}$$

Then from (9) we have, upon neglecting terms of order $1/\ln L$,

$$\frac{\partial k'}{\partial V}\left[\frac{\partial f}{\partial k'}-k\frac{\partial \alpha}{\partial k'}\right]_{k'=k}=0 , \tag{13}$$

which leads to

$$\frac{\partial f}{\partial k}=k\frac{\partial \alpha}{\partial k} . \tag{14}$$

With this result we find, upon differentiating (11),

$$\frac{1}{\nu}\frac{\partial}{\partial k}p(k)=\alpha(k) . \tag{15}$$

Therefore, if we know the exponent $p(k)$, then we can extract $\alpha(k)$, and subsequently $f(k)$, thereby performing the decomposition of $p(k)$ suggested by (11) in a direct manner. A similar approach leading to (11) and (15) has been developed in Ref. 13.

Using some very rudimentary knowledge of $\alpha(k)$ and $f(k)$, we can now write $n(V)$ in a scaling form. We know that as $k\to\infty$, $V(k)\to V_{\max}$, the maximum voltage drop in the entire network, a drop which will occur across the links in the percolating backbone. At the percolation threshold, this maximum voltage drop coincides with the conductance G. Since the latter quantity scales as $L^{-t/\nu}$, with $t/\nu=\tilde{\xi}$ in two dimensions, we therefore conclude that $\alpha(\infty)=\tilde{\xi}$. Furthermore, since the number of links is known to scale as $L^{1/\nu}$, we also know that $f(\infty)=1/\nu$.

We now define a quantity $x\equiv\ln V(k)/\ln V_{\max}$ which turns out to be a fundamental way of quantitatively characterizing all the bonds in the network. We shall show that the voltage distribution is a scaling function of this basic variable. From (9a), we have

$$\begin{aligned}\ln V(k)&=\ln A(k)-\alpha(k)\ln L ,\\ \ln V(\infty)&=\ln A(\infty)-\alpha(\infty)\ln L .\end{aligned} \tag{16}$$

Therefore for very large values of L we may write

$$x=\frac{\alpha(k)}{\alpha(\infty)} , \tag{17}$$

with correction terms of order $1/\ln L$. Thus in locating the maximum of $n(V)V^k$ for large L, i.e., fixing k, the ratio $\ln V/\ln V_{\max}$ also remains constant, as indicated by (17). Assuming that $\alpha(k)$ is a monotonic function of k, as is reasonable physically, we can uniquely invert (17) and obtain $k=k(x)$. Hence from (9b) we find

$$n(V)\sim C(x)L^{\phi(x)} , \tag{18a}$$

where

$$\phi(x)=f(k(x)),\quad C(x)=B(k(x)) ,$$

and

$$x=\frac{\ln V}{\ln V_{\max}} . \tag{18b}$$

Alternatively, this distribution can be written in terms of V only. By first taking the logarithm of $n(V)$ and eliminating L through the relation $\ln V_{\max}=-\tilde{\xi}\ln L$, and finally reexponentiating, we obtain,

$$n(V)\sim V^{-\psi(x)} , \tag{18c}$$

where $\psi(x)=\phi(x)/x\tilde{\xi}$. Thus we see that the voltage distribution can be written as a power law in V, but with an exponent that is also voltage dependent. Equations (18) represent one of the primary results of this paper.[24] They express the fact that the bonds in the network are divided into different sets characterized by the value of $x=\ln V/\ln V_{\max}$. Each set has an independent fractal dimension $\phi(x)$, and an independent voltage singularity exponent $\alpha=x\tilde{\xi}$. As a result of the voltage-dependent exponent in (18), an infinite hierarchy of exponents is required to describe the moments of the voltage distribution. Similar scaling behavior was also found in growth phenomena, where the quantity analogous to the voltage is the probability that a perimeter site becomes part of an aggregating cluster.[10,11] Note that from (17),

$$\phi(x)=f\left[k\left[\frac{\alpha}{\alpha(\infty)}\right]\right] .$$

Thus $\phi(x)$ gives the functional dependence of f on $\alpha/\alpha(\infty)$. A similar function f as a function of α was introduced in Ref. 13.

To appreciate the difference between the novel scaling of the voltage distribution and the conventional behavior characterized by a small number of exponents, consider the cluster-size distribution in the percolation problem. From the functional form of this distribution given in (2), maximizing the product $s^k n(s)$ gives, in analogy with Eq. (7),

$$\frac{(k-\tau)}{s}+\frac{f'(s/L^{d_r})}{f(s/L^{d_f})}\frac{1}{L^{d_f}}=0 . \tag{19}$$

Since $f(x)$ is a rapidly decreasing function of x, the second term in (19) is less than 0. Now there are two possibilities: For $k<\tau$, there is no maximizing value of s, but rather an extremum at the minimum possible value of s. In this case, clusters of the smallest size dominate in

the summand for the kth moment, leading to a finite value for this quantity. However for $k > \tau$, a maximum in the summand occurs for a typical size s^* given by

$$s^* \sim A(k) L^{d_f} , \tag{20}$$

where $A(k)$ is a nonsingular function of k. Thus when $k > \tau$, there is a unique typical size, in a scaling sense, which varies as L^{d_f}, and this size dominates in the moments of the cluster-size distribution. To compute the moments, note that it is not sufficient to merely keep the largest term in $\sum_s s^k n(s)$, as was done in Eq. (10) for the voltage distribution. In the voltage distribution, if one were to expand about the maximum point to second order and perform the resulting Gaussian integral, one would find that integrating over the width of the peak would lead to logarithmic corrections to the voltage moments. On the other hand, for the cluster-size distribution, this same procedure gives a power-law correction, which represents the necessary shift in the exponent needed to reproduce the correct scaling of the moments of the cluster-size distribution given in (3).

In the following section, we explicitly calculate the voltage distribution on the hierarchical lattice in order to illustrate its novel scaling features.

III. VOLTAGE DISTRIBUTION FOR THE HIERARCHICAL MODEL

A. Two-dimensional version of the model

To describe the geometry of the percolating backbone and the properties of the voltage distribution, consider the hierarchical model shown in Fig. 1. To obtain the structure at the Nth level of iteration, each bond in the $(N-1)$th level is replaced by the first-order structure, or "unit cell." The resulting model is self-similar on all length scales up to the scale of the entire system, and the hierarchical embedding of the links and blobs is a crucial feature which appears to provide an accurate representation of the geometric features of a percolating backbone in all spatial dimensions. One should think of the iteration index as a parameter which controls the effective linear dimension of the hierarchical lattice. The backbone of a finite-size network may therefore be described by a hierarchical lattice which is iterated to a level where its length scale matches that of the backbone. For the model shown in Fig. 1, the critical behavior that results provides an excellent approximation for the exponents of two-dimensional percolation.

If a unit potential is applied at the opposite ends of an Nth-order hierarchical lattice, then it is easy to show that the voltage distribution is

$$n(V(j)) = 2^N \begin{bmatrix} N \\ j \end{bmatrix} , \tag{21}$$

where the voltage $V(j)$ takes on the values $2^j/5^N$, with the integer index j running from 0 to N, so that the maximum value of V, $V_{\max}$, equals $(\frac{2}{5})^N$, while the minimum value of V, $V_{\min}$, equals $(\frac{1}{5})^N$. The distribution is therefore a simple binomial in j, but with j varying logarithmically in V. Thus we conclude that the voltage distribution is log binomial (Fig. 2). From this simple discrete form of the distribution, it is straightforward to calculate the voltage moments, $M(k)$. Notice that in calculating the moments, one sums over the index j, which is logarithmic in the voltage. We find

$$M(k) = \left[\frac{2(1+2^k)}{5^k} \right]^N . \tag{22}$$

Notice, in particular, that the average value of the voltage, $M(1)/M(0) \equiv V_{\text{av}}$ equals $(\frac{3}{2}/5)^N$, which is very different from the most probable value of the voltage, $V_{\text{MP}} = (\sqrt{2}/5)^N$, as $N \to \infty$.

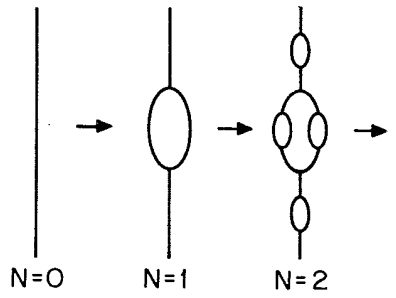

FIG. 1. The first few levels of iteration in the hierarchical lattice model. The $N=1$ level is the "unit cell" which is substituted for each bond at a given level to generate the next level. Shown is the case where λ, the ratio of the number of bonds in the blobs to the number of links in the first order structure, N_B/N_L, is equal to unity.

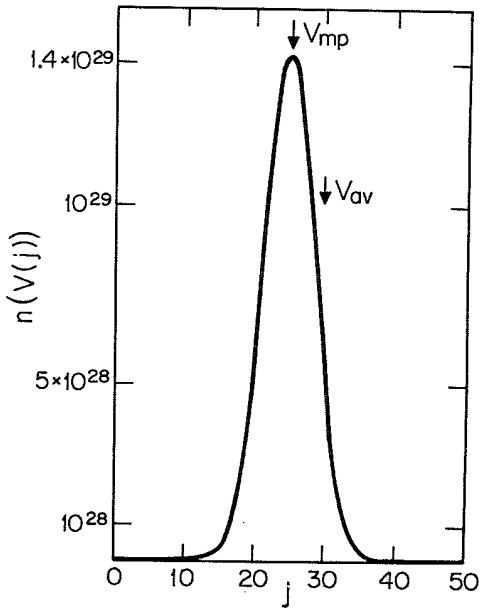

FIG. 2. The voltage distribution for the two-dimensional version of the hierarchical lattice [Eq. (21)] for the case $N=50$. The distribution is plotted as a function of $j = (\ln V + N \ln 5)/\ln 2$. The position of the average value of the voltage, $V_{\text{av}} = (\frac{3}{2}/5)^N$, and the most probable value, $V_{\text{MP}} = (\sqrt{2}/5)^N$, are indicated.

To calculate critical exponents, we need to write a quantitative relationship between the iteration index N and a physical length scale L. A very natural way to accomplish this goal is to note that for a finite-size system of linear dimension L, the number of links varies exactly[18] as $L^{1/\nu}$. Since the number of links also equals 2^N in the Nth-order hierarchical lattice, we can therefore infer an effective linear dimension, L equal to $2^{N\nu}$. Using this fundamental relation in (22), we find that the exponent $p(k)$, defined in Eq. (4), equals

$$p(k)=k-1+[k\ln(\tfrac{5}{4})-\ln(1+2^{-k})]/\ln 2 . \tag{23a}$$

Thus an *infinite* set of independent exponents is required to describe the moments of the voltage distribution, as the difference, $p(k)-p(k-1)$, depends on k. In Table I, we give the numerical values of $p(k)$ for a representative sample of k values, together with the corresponding estimates from the numerical simulations. The agreement between the two sets of estimates is quite good for $k>0$, but there is an increasing discrepancy between the two sets of exponents as k becomes more negative. This stems from the pronounced low-voltage tail in the lattice voltage distribution (Fig. 4), a feature which is absent in the present version of the hierarchical model. We shall attempt to rectify this shortcoming below.

From (11) and (15), we can easily calculate the exponents which describe the scaling of $V^*(k)$ and $n(V^*(k))$. We find

$$\nu\alpha(k)=\frac{\partial}{\partial k}p(k)=\frac{\ln 5}{\ln 2}-\frac{1}{1+2^{-k}} , \tag{23b}$$

$$\nu f(k)=1+\frac{k2^{-k}}{1+2^{-k}}+\frac{\ln(1+2^{-k})}{\ln 2} . \tag{23c}$$

To find the scaling function for $n(V)$, we now take the continuum limit of the voltage distribution in the limit $N\to\infty$. To accomplish this, we make use of the relations $L=2^{N\nu}$ and $V_{\max}=(\frac{2}{5})^N$, in order to write N in terms of physically relevant parameters. Now, using the lowest-order form of Stirling's approximation, $\ln k!=k\ln k-k$, it is straightforward to show that $n(V)$ can be cast in the form

$$n(V)=L^{\phi(x)} , \tag{24a}$$

or, by eliminating the length scale in favor of the voltage, as in (18c)

$$n(V)=V^{-\psi(x)} , \tag{24b}$$

with $x=\ln V/\ln V_{\max}$. In (24), the exponents ϕ and ψ are given by

$$\phi(x)=1-[(1-y)\ln(1-y)+y\ln y]/\ln 2 ,$$
$$\psi(x)=\frac{\phi(x)}{x\tilde{\zeta}} , \tag{25}$$

where the parameter $y=j/N$ is related to x by

$$y=[\ln 5-x\ln(\tfrac{2}{5})]/\ln 2 . \tag{26}$$

Notice that $\phi(y)$ is a peaked function that is symmetric about $y=\frac{1}{2}$, as is expected in forming a continuum approximation to the symmetric binomial. If one were to keep only the first two terms in the power series representation for $\phi(y)$ about $y=\frac{1}{2}$, one would obtain the familiar Gaussian approximation, leading to a log-normal form for the distribution. This feature is a general consequence of the scaling form (24a), since, as $L\to\infty$, only a small neighborhood in the vicinity of the maximum of $\phi(x)$ will dominate in the distribution.

However, the Gaussian approximation, though familiar, is inadequate, since the tail of the distribution dominates in the calculation of the higher moments. This stems from the fact that voltage moments are actually exponential moments when all quantities are written in terms of the fundamental variable $x=\ln V/\ln V_{\max}$. That is,

$$\sum_{\ln V} V^k n(V)=\sum e^{k\ln V} n(V)$$
$$=\sum \exp[-k\alpha(\infty)x+\phi(x)]\ln L ,$$

where we have used $\ln V_{\max}=-\alpha(\infty)\ln L$. This shows that for each value of k, the maximum of the combination $\phi(x)-k\alpha(\infty)x$, which is located at $x_{\max}(k)$ dominates in the moments, rather than the maximum of $\phi(x)$ itself, which is located at $x_{\max}(0)$. Thus the Gaussian approximation of expanding $\phi(x)$ about its maximum value,

$$\phi(x)\simeq\phi(x_{\max}(0))+\tfrac{1}{2}[x-x_{\max}(0)]^2\phi''(x)+\cdots ,$$

is clearly inadequate, since the dominant contribution arises from $x_{\max}(k)$.

Equations (24) show that the functional form for $n(V)$ in the hierarchical model is as predicted from the general arguments of Sec. II. We now want to demonstrate how one can obtain the exponents $p(k)$, $\alpha(k)$, and $f(k)$, by first taking the continuum approximation for $n(V)$ and then following the approach outlined in Sec. II. The exponents obtained in this way, will coincide with the exact expressions already obtained in (23).

In terms of the continuum form of the voltage distribution, we compute the moments, $M(k)$, by replacing the

TABLE I. The exponents $p(k)$ from the two-dimensional version of the hierarchical model together with the numerical estimates of these exponents from simulations on the random resistor network at the percolation threshold. For the cases $k=0$ and $k=2$, the best numerical estimates for the exponents $p(k)/\nu$ are -1.58 and -0.973, respectively.

	$p(k)/\nu$	
k	Data	Model
−1.0	−5.70 ±0.30	−2.930
−0.8	−4.61 ±0.15	−2.634
−0.6	−3.34 ±0.25	−2.343
−0.4	−2.47 ±0.07	−2.057
−0.2	−2.00 ±0.05	−1.776
0.0	−1.58 ±0.04	−1.500
1.0	−0.22 ±0.05	−0.197
2.0	0.976±0.01	0.991
3.0	2.07 ±0.02	2.097
4.0	3.12 ±0.02	3.150
5.0	4.14 ±0.02	4.174
6.0	5.15 ±0.02	5.182

sum for $M(k)$ by its maximum value, as in Sec. II. That is,

$$M(k)=\sum n(V)V^k \simeq n(V^*)V^{*k}, \tag{27}$$

where V^* satisfies

$$\left.\frac{\partial \ln(n(V))}{\partial \ln V}\right|_{V=V^*} = -k . \tag{28}$$

From (25) and (28), we find that V^* is given by choosing the value $y^*=y(k)$ with

$$y^*=\frac{1}{1+2^{-k}} . \tag{29}$$

Substituting this value of y^* in (26), we finally obtain

$$V^* \sim L^{-\alpha(k)}, \tag{30}$$

with

$$\nu\alpha(k)=\frac{\ln 5}{\ln 2}-\frac{1}{1+2^{-k}} . \tag{31}$$

In addition, by substituting the value of y^* in (24), we find

$$n(V^*) \sim L^{f(k)} \tag{32}$$

with

$$\nu f(k)=1+\frac{k2^{-k}}{1+2^{-k}}+\frac{\ln(1+2^{-k})}{\ln 2} . \tag{33}$$

Now by combining (29) and (31), we find

$$p(k)=(k-1)+[k\ln(\tfrac{5}{4})-\ln(1+2^{-k})]/\ln 2 , \tag{34}$$

which is identical to (23a). Thus we have shown that the moment exponent $p(k)$ can be obtained correctly from the continuum form of the voltage distribution, when the general formalism of Sec. II is employed.

In Fig. 3, we plot $\phi(x)$ as a function of x. This function is identical to the functional dependence of f on $\alpha/\alpha(\infty)$, as noted in Eq. (18). The latter function is obtained by eliminating the k dependence in (33) in favor of the $\alpha/\alpha(\infty)$ dependence by using (31). The quantity $\phi(x)$ gives the fractal dimension of the set of bonds having a value of the voltage drop which is characterized by x. The symmetry of this plot indicates that the fractal sets of hottest and coldest bonds, which are symmetrically related with respect to the most probable value of the voltage, have the same critical behavior.

B. Generalization to arbitrary dimensions

The model introduced above can also be generalized in a simple way to describe the percolating backbone in general dimensions. We introduce a parameter, $\lambda=N_B/N_L$, defined to be the ratio of the number of bonds in the blobs to the number of links in the first-order hierarchical lattice. As λ decreases, the role of the blobs also decreases, a feature which also occurs as the spatial dimension increases towards 6 or decreases towards 1. We have found that the *ad hoc* choice of $\lambda=(6-d)/4$ for $2\le d\le 6$, $\lambda=d-1$ for $1\le d<2$, and $\lambda=0$ for $d>6$ provides a connection with the spatial dimension that is plausible qualitatively. Furthermore, by using the connection between the number of links and the effective linear dimension of the system, $(2/\lambda)^N=L^{1/\nu}$, as in the two-dimensional version of the model, an excellent quantitative description of the geometrical features of the percolating backbone in arbitrary dimensions is obtained.

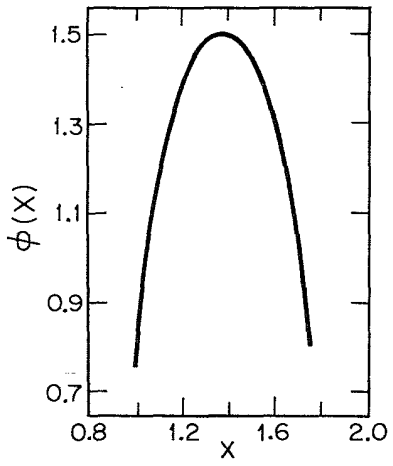

FIG. 3. The fractal dimension $\phi(x)$ of the set of bonds characterized by the value $x=\ln V/\ln V_{\max}$, plotted as a function of x. The value of x ranges from $x=1(V=V_{\max})$ to $x\simeq 1.7$ (low-voltage region).

The results obtained for $\lambda=1$ (corresponding to $d=2$) can now be straightforwardly generalized to any value of λ, yielding

$$n(V(j))=\left[\frac{2}{\lambda}\right]^N \lambda^j \begin{bmatrix} N \\ j \end{bmatrix} \tag{35}$$

with

$$V(j)=2^j\left[\frac{\lambda}{\lambda+4}\right]^N, \quad V_{\max}=\left[\frac{2\lambda}{\lambda+4}\right]^N . \tag{36}$$

In the continuum limit we find

$$n(V)=L^{\phi}\left[\frac{\ln V}{\ln V_{\max}}\right], \tag{37}$$

with

$$\phi(y)=1-[(1-y)\ln(1-y)+y\ln y-y\ln\lambda]/\ln(2/\lambda) \tag{38}$$

and

$$y=\left\{\left[\ln\left[\frac{\lambda}{2(\lambda+4)}\right]+1\right]\frac{\ln V}{\ln V_{\max}}-\ln\left[\frac{\lambda}{(\lambda+4)}\right]\right\}\frac{1}{\ln 2} . \tag{39}$$

The maximum of $n(V)V^k$ is given by $y^*=y(k)$ with

$$y^*=\frac{1}{1+2^{-k}/\lambda} . \tag{40}$$

Finally the exponents $\alpha(k)$, $f(k)$, and $p(k)$ are given by

$$\nu\alpha(k)=1+\frac{1}{\ln(2/\lambda)}\left[\ln\left[1+\frac{\lambda}{4}\right]+\frac{\lambda 2^{-k}\ln 2}{1+\lambda 2^{-k}}\right], \quad (41a)$$

$$\nu f(k)=1+\frac{1}{\ln(2/\lambda)}\left[\ln\left[1+\frac{\lambda}{2^k}\right]+\frac{k\lambda 2^{-k}\ln 2}{1+\lambda 2^{-k}}\right], \quad (41b)$$

$$p(k)=k-1+\frac{1}{\ln(2/\lambda)}\left[k\ln\left[1+\frac{\lambda}{4}\right]-\ln\left[1+\frac{\lambda}{2^k}\right]\right]. \quad (41c)$$

Thus we find, just as in the two-dimensional case, that these exponents are identical with those found from the exact discrete form of the voltage distribution. Note also that in the limit $\lambda\to 0$ ($d\to 1$ or $d\to 6$), the percolating backbone consists only of links so that we obtain

$$\nu\alpha(k)=\nu f(k)=1, \quad p(k)=k-1 . \quad (42)$$

Thus in the limit where the backbone becomes strictly one dimensional, the exponents characterizing the fractal dimension of the links and the singularity of $V=V_{\max}$ are both equal to unity, while the moment exponents display constant-gap scaling.

C. Voltage distribution of a generalized hierarchical model

In the two-dimensional version of the hierarchical lattice, the voltage distribution can be obtained formally by considering the binomial

$$[2(V_1+V_2)]^N , \quad (43)$$

with $V_1=\frac{1}{5}$ and $V_2=\frac{2}{5}$. Upon expanding this expression, we find that the coefficient of each factor of $V=V_1^j V_2^{N-j}$ ($j=0,1,2,\ldots,N$) equals the number of bonds with voltage drop equal to V in the Nth-order lattice. When the distribution is written in the form of (43), it is clear that the symmetric distribution of an Nth-order structure originates directly from the symmetry of the "microscopic" first-order distribution on the $N=1$ hierarchical structure. However, the voltage distributions of finite-size random networks at the percolation threshold display considerable asymmetry (cf. Fig. 4), and it is clear that the distribution of the hierarchical model is too simplistic. Therefore, we now introduce a very simple generalization of the hierarchical model which provides a better qualitative description of the voltage distribution of random resistor networks, as well as providing insights into the logarithmic voltage dependence of this distribution.

We consider a hierarchical lattice in which the microscopic distribution is not necessarily symmetric. Such a construction was already employed in a very simplified form in the preceding section to describe the voltage distribution in higher dimensions. Thus, suppose that there are M bonds in the unit cell, and that each bond has a voltage drop V_i across it. Then for an Nth-order iterate of the unit cell, it is straightforward to show that the generating function for the voltage distribution, analogous to (43), is the multinomial

$$\left[\sum_{i=1}^{M} V_i\right]^N . \quad (44a)$$

More generally, one can consider a microscopic voltage distribution which is generated by any set of voltages, $\{V_i\}_{i=1}^M$, and not necessarily a voltage set which are the equilibrium voltages on a particular small cell. Furthermore, one can also allow for the possibility that there is an arbitrary weight w_i associated with each elementary voltage V_i. For such an arbitrary microscopic distribution, the resulting iterated distribution will be

$$\left[\sum_{i=1}^{M} w_i V_i\right]^N . \quad (44b)$$

Because of this multinomial form, the average value of the voltage on an Nth-order structure is given by

$$V_{\rm av}=\left[\sum_{i=1}^{M} w_i V_i\right]^N \Big/ \left[\sum_{i=1}^{M} w_i\right]^N . \quad (45a)$$

On the other hand, by locating the peak in the multinomial expression, we find that the most probable value of the voltage is simply given by the geometrical mean of the bond voltages on an Nth-order structure. After a number of simple steps, this can be written simply in terms of the bond voltages on the first-order structure as

$$V_{\rm MP}=\left[\prod_{i=1}^{M} V_i^{w_i}\right]^{N^*} =\exp\left[N\sum_i w_i \ln V_i \Big/ \sum_i w_i\right], \quad (45b)$$

where $N^*=(N/\sum_i w_i)$. For a given hierarchical lattice, therefore, the mean and most probable voltages can be quite different. This difference stems from the fact that each bond voltage on an Nth-order structure is a particular Nth-order product of the original V_i's. The result that the voltage distribution has a log-multinomial form is a consequence of the fact that the bond voltages arise from a multiplicative process. This multiplicative property appears to be the mechanism responsible for making $\ln V$, rather than V, the natural variable for expressing the voltage distribution of resistor networks on percolating lattices.

With the generalized hierarchical model, we have attempted to mimic the qualitative features of the distributions from the square-lattice resistor network at the percolation threshold, by choosing an appropriate elementary set of voltages and corresponding weights. We find that it is possible to reproduce qualitative features of the lattice data reasonably well. For a given elementary set of voltages $\{V_i\}$ and weights $\{w_i\}$, it is possible to construct a distribution that is asymmetric and discontinuous for small N, but much more symmetric and continuous for large N. These features also occur in the voltage distribution on percolating lattices, lending support for the hypothesis that a hierarchical construction is a fundamental ingredient that is needed to provide a complete description of the voltage distribution.

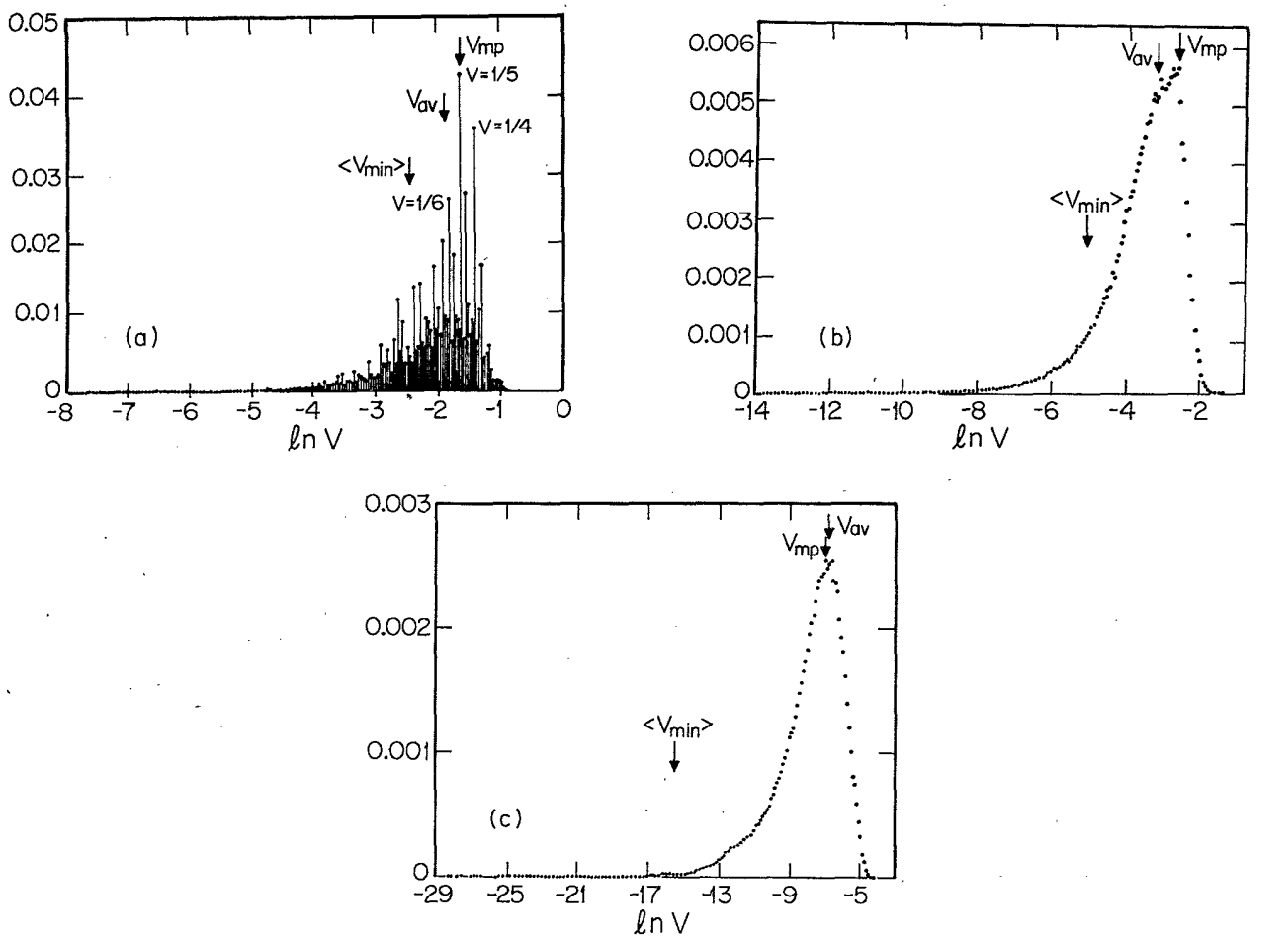

FIG. 4. The voltage distribution of an $L \times L$ square lattice random resistor network at the percolation threshold for (a) $L=4$ (exact), (b) $L=10$ (30 000 configurations), and (c) $L=130$ (950 configurations). The distribution has been normalized by the number of configurations and also by the number of bonds in the backbone. The resulting data are binned and plotted on a logarithmic scale. The horizontal scales of the examples shown have been adjusted so that the increment in the quantity $x=\ln V/\ln V_{\max}$ from one bin to the next is the same for every system size. The mean value and the most probable values of the voltage for these three distributions are indicated. For $L=4$, a number of peaks, corresponding to simple rational fractions of the external (unit) potential drop, is indicated as well.

IV. NUMERICAL ANALYSIS OF THE VOLTAGE DISTRIBUTION

A. Qualitative features

In order to test the predictions for the scaling behavior of the voltage distribution and its moments, we have performed numerical simulations on $L \times L$ square lattice random resistor networks with L ranging from 2 to 130. The distribution is defined with respect to a unit potential drop imposed across opposite edges of the system, and free boundary conditions in the transverse direction. For a given configuration, the backbone of the percolating network is first found by the algorithm introduced by Herrmann,[25] after which the voltage at each site is determined by standard numerical relaxation methods. For $L=2$, 3, and 4, we have enumerated all $2^{[L^2+(L-1)^2]}$ configurations of the network and thereby found the exact voltage distribution for any value of p between 0 and 1. For $L \geq 5$, an exact enumeration is no longer feasible, and our distribution is now approximate, being based on the averaging of 50 000 configurations for $L=5$, to 950 configurations for $L=130$.

In Fig. 4, we plot the voltage distribution on a logarithmic horizontal scale for three different system sizes. For $L=4$, the distribution has considerable discrete structure, and it is possible to identify a number of peaks which arise from a preponderance of bonds with voltage drops which are simple rational fractions of the (unit) imposed external voltage drop. Notice also that the distribution is strongly asymmetric: the distribution falls off very rapidly for values of V relatively close to $V_{\max}$, but decays slowly on the low-voltage side. As the system size increases, the distribution becomes smoother, and the support of the distribution increases by many orders of mag-

nitude. Furthermore, the distribution gradually becomes more symmetric as the system size increases as the contribution of the low-voltage end of the distribution appears to become relatively less important.

It is also worth noting that the logarithmic binning of the voltages, used in Fig. 4, is necessary if one wishes to obtain a nonsingular shape of the distribution for large lattices. If the bond voltages were binned on a linear scale, then one would find that essentially only the first few bins of the smallest voltages would contain any data. If one were to decrease the bin width in order to increase the resolution, then the same phenomenon of data residing only in the first few bins would still occur. Upon repeated rescalings of the bin width, the same qualitative situation would continue to persist until a point was reached where the resolution is so fine that a smooth distribution would no longer be found.

B. Voltage moments

From the voltage distributions, we have calculated the average moments, $M(k)$, as a function of L, for a range of k values. To demonstrate that each moment scales independently as a function of L, we consider the normalized moment $\overline{M}(k) \equiv [M(k)/M(0)]^{1/k}$. If there existed a constant-gap exponent in the scaling behavior of the successive moments, then all the $m(k)$ should scale identically. The behavior of a representative sample of $m(k)$, for both positive and negative values of k is shown as a function of L in Fig. 5. The very different slopes of the various straight-line fits to the data confirm that each moment does scale independently with L.

To obtain a more precise estimate of the moment exponents, $p(k)/\nu$, we have plotted the ratio $\ln M(k)/\ln L$ versus $1/\ln L$. If we write for the moments, $M(k)=D(k)L^{-p(k)/\nu}$, with $D(k)$ an arbitrary amplitude, then a plot of $\ln M(k)/\ln L$ versus $1/\ln L$ should asymptotically lie on a straight line of slope $\ln D(k)$ which intercepts the vertical axis at $-p(k)/\nu$. One of the striking features of these plots is that the data for any value of k appear to form a sequence of points which is horizontal asymptotically, suggesting the $D(k)=1$ for all k. From these extrapolations of the moments, we plot, in Fig. 6, the exponent $p(k)/\nu$ as a function of k. We obtain a second estimate of the exponent $p(k)/\nu$ by taking the exact data for $L=3$ and $L=4$ and performing a linear extrapolation of $\ln M(k)/\ln L$ versus $1/\ln L$. This is tantamount to performing a renormalization procedure whereby the 4×4 cell is rescaled into a 3×3 cell.[26] Quite strikingly, the two sets of exponent estimates virtually coincide for all values of k studied, indicating that the cell-to-cell renormalization procedure is exceedingly accurate.

The qualitative form of the $p(k)/\nu$ as a function of k clearly indicates that more than one typical voltage is needed to characterize the underlying distribution. As mentioned in Sec. II, if the voltage distribution scaled in a conventional manner, then there would be a unique typical value of the voltage which dominates in all the moments. Accordingly, $p(k)/\nu$ would be a linear function of k. Clearly, this is not the case in the figure, as there is considerable curvature in the data, especially in the vicinity of $k\simeq-0.5$. However, for both large positive and large negative values of k the data does become rather linear. This suggests that there is one typical voltage, at the hot end of the distribution, which dominates in all the

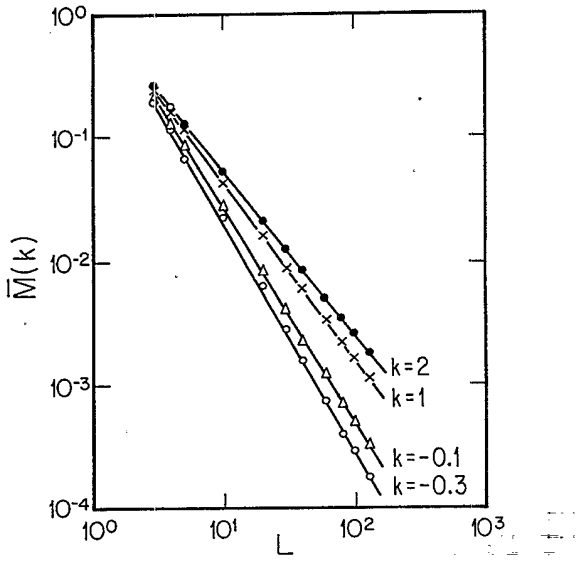

FIG. 5. Double logarithmic plot of the normalized moments of the voltage distribution, $\overline{M}(k)=[\sum n(V)V^k/\sum n(V)]^{1/k}$, versus L. Data for a representative sample of k values has been shown. The straight lines represent least-square fit to the data, and the corresponding slopes are -1.31 for $k=2(\bullet)$, -1.42 for $k=1(\times)$, -1.72 for $k=-0.1(\triangle)$, and -1.84 for $k=-0.3(\circ)$.

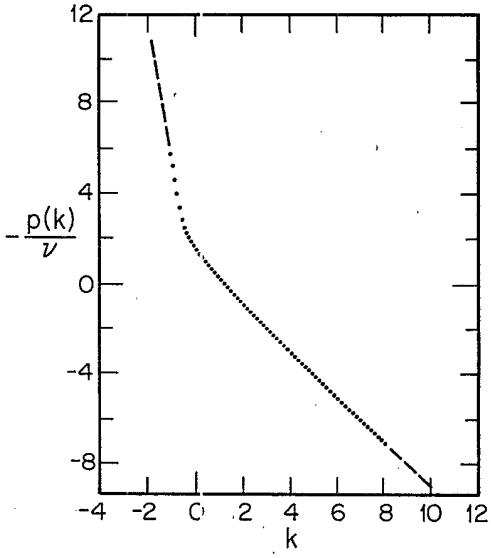

FIG. 6. The exponent $-p(k)/\nu$ plotted versus k, as obtained from an extrapolation of the voltage moments. The asymptotic linear dependences for $k\to\infty$ and for $k\to-\infty$ are indicated by dashed lines. From the asymptotic slopes, we can extract a typical hot voltage which scales very approximately as L^{-1} and a typical cold voltage which scales very approximately as $L^{-6.5}$.

positive moments, for sufficiently large values of k, and a second typical voltage, at the cold end of the distribution, which dominates in all the negative moments, for sufficiently large negative values of k. The transition that occurs when $k \simeq -0.5$ suggests that there is a relatively sudden change in the relative importance of the contribution of the hottest and coldest bonds to the moments in this regime.

At each point along the curve of $p(k)$ versus k, the slope of the tangent and the intercept of the tangent with the vertical axis are, respectively, $\alpha(k)$ and $-f(k)$, as follows from the decomposition, $p(k)/\nu = k\alpha(k) - f(k)$, given in Eq. (11). Figures 7(a) and 7(b) show these two functions as determined by a numerical evaluation of the derivative of $p(k)$ as a function of k. As mentioned earlier, $f(k)$ represents the fractal dimension of the set of bonds which have a voltage drop across them equal to V^*. Thus as $k \to \infty$, $f(k)$ gives the fractal dimension of the singly connected bonds, and $f(k)$ appears to be approaching the expected asymptotic value of $1/\nu = 0.75$. In addition, as k decreases towards zero, $f(k)$ slowly increases, indicating that the fractal dimension of the progressively lower-current-carrying bonds increases. When $k = 0$, $f(k)$ reaches a maximum which coincides with the fractal dimension of the entire backbone. Finally, for negative values of k, $f(k)$ is strongly decreasing, indicating that the lowest-current-carrying bonds have a negative fractal dimension so that they contribute negligibly to the voltage distribution as $L \to \infty$.

The behavior of $\alpha(k)$ can be interpreted similarly. As $k \to \infty$, $\alpha(k)$ approaches a value which is approximately equal to unity, and this is close to the value of the exponent ζ/ν, as expected by the equality of $V_{\max}$ at the percolation threshold, and the conductance of the network. The value of $\alpha(k)$ is slowly varying for all positive values of k, but then varies rapidly once k becomes negative. This indicates that the qualitative shape of the positive and negative sides of the voltage distribution as a function of L will be quite different. For positive k values, sets of bonds characterized by nearby values of k will have relatively large fractal dimensions and also fairly similar values of $\alpha(k)$. This is consistent with the distribution having a well-defined peak. On the other hand, for negative values of k, sets of bonds with small or negative values of the fractal dimension have very different values of $\alpha(k)$. This is consistent with a very long tail that is increasing in length while rapidly decreasing in relative importance as the system size increases.

Further evidence for this general picture of the voltage distribution can be obtained from an analysis of V^* as a function of k [Fig. 7(c)]. As previously discussed, the value of V^* locates the maximum of the function $n(V)V^k$. This typical voltage is slowly varying with k. However, as k becomes negative, V^* rapidly decreases over several orders of magnitude, and approaches the minimum value of the voltage drop over all bonds in the ensemble. The mechanism causing this sharp transition can be seen from the behavior of the voltage distribution *itself*. Upon varying the value of k, one can follow a continuous variation in the location of the maximum of $n(V)V^k$, i.e., in the location of V^*. However, as k is de-

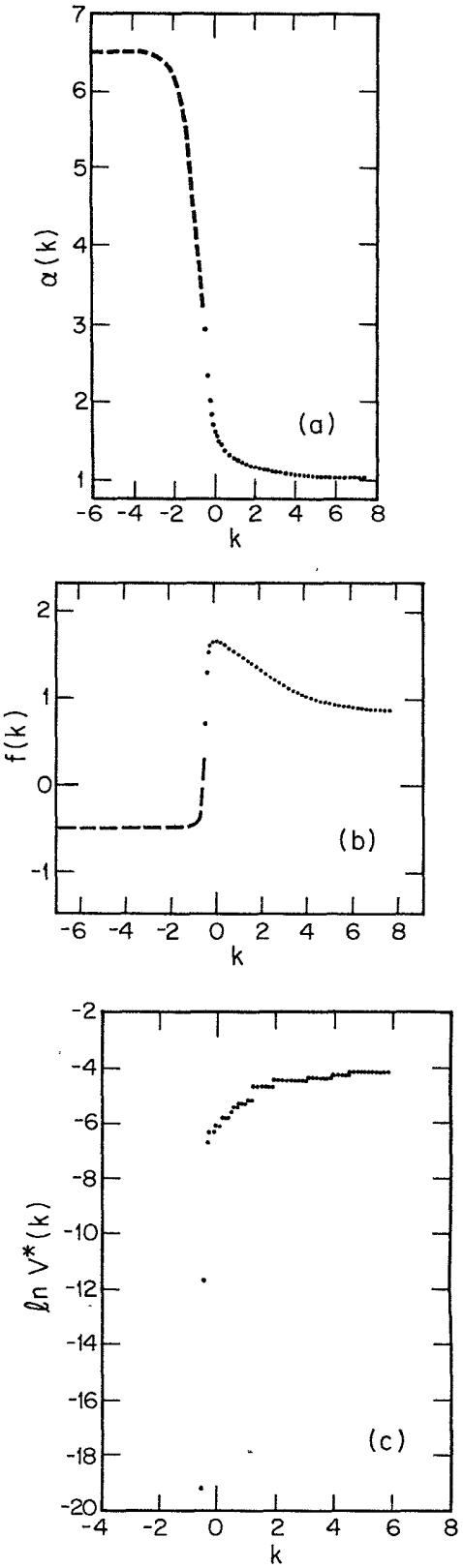

FIG. 7. The functions (a) $\alpha(k)$, (b) $f(k)$, and (c) $\ln V^*(k)$ versus k for the case $L = 80$. The three plots show that a sudden transition in behavior occurs for k in the vicinity of -0.5. The dashed line represents the expected behavior from the asymptotic behavior of $p(k)$ of Fig. 6.

creased to a value near 0, a second peak in $n(V)V^k$ begins to form which is located at a smaller value of V relative to the primary peak. Due to the presence of a zero in the second derivative of $n(V)$ on the low-voltage side, the function $n(V)V^k$ will develop two well-defined peaks for more negative values of k; this is especially apparent for k in the vicinity of -0.5. As k continues to decrease, the secondary peak gradually becomes larger than the primary peak, corresponding to a sudden jump in the value of V^*. For more negative values of k, the original primary voltage peak becomes negligible compared to the secondary peak. This apparent discontinuity in the behavior of V^* is the source of the sudden transitions in $\alpha(k)$ and $f(k)$ as a function of k.

C. Scaling analysis

Next we wish to test whether the voltage distribution is consistent with the scaling form for $n(V)$ given in Eq. (18b). The results of one fairly sensitive test are shown in Fig. 8, where we compute the exponent $\phi(x)$ in two independent ways. In one method, we start with the estimates of the moment exponents, $p(k)$, and extract the values of $\alpha(k)$ and $f(k)$, as previously discussed. Then, by eliminating the variable k, we finally plot the quantity $\phi(x)=f(k(x))$ as a function of $x=\alpha/\alpha(\infty)$. On the other hand, we can obtain $\phi(x)$ by independent means. A straightforward way to accomplish this would be to fix a value of $x=\ln V/\ln V_{\max}$, and calculate the corresponding value of $n(V)$ for various system sizes and thereby obtain $\phi(x)$ by extrapolating $\ln(n(V))/\ln L$ to $L\to\infty$. In practice, this method converges rather poorly, and we have adopted the following procedure, which becomes identical to the previous one for very large L. For a given value of k, we locate the maximum of the function $n(V)V^k$, i.e., $V^*=V(k)$, for different system sizes. The data for V^* and $n(V^*)$ for various system sizes are then extrapolated to $L\to\infty$ in order to infer the values of $f(k)$ and $\alpha(k)$ from the asymptotic values of $\ln(n(V^*))/\ln L$, and $\ln V^*/\ln V_{\max}$, respectively. From $f(k)$ and $\alpha(k)$, we then obtain $\phi(x)$. The data of $\phi(x)$ versus x that results are shown in the figure. The agreement between the two sets of independently obtained data confirms the general validity of the scaling hypothesis for the voltage distribution and its moments.

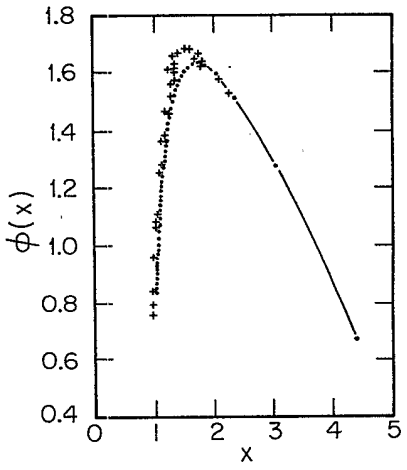

FIG. 8. The quantity $\phi(x)$, obtained by two independent methods, plotted as a function of x. (●): Values obtained by first taking the data in Fig. 6 and performing the decomposition of $p(k)/\nu$ into $f(k)$ and $\alpha(k)$, as indicated by Eq. (11), and finally eliminating the variable k to obtain $\phi(x)=f(k(x))$, with $x=\alpha/\alpha(\infty)$. (+): Values obtained by fixing a value of $x=\ln V/\ln V_{\max}$, and taking the corresponding value of $n(V)$ for a sequence of system sizes L in order to obtain $\phi(x)$ from the extrapolated value of $\ln n(V)/\ln L$ as $L\to\infty$.

A final test of the scaling hypothesis is to perform the classical data "collapsing" analysis. From Eq. (18a), the distribution $n(V)$ divided by $L^{\phi(x)}$ should be a function only of the amplitude, $C(x)$. Such a scaling plot is shown in Fig. 9. For system sizes $L=40$, 60, 80, and 130, the scaled distributions do appear to lie on a single universal curve as required by our scaling hypothesis. For smaller system sizes, there is a systematic deviation of the data from the universal curve which would arise if there were correction terms to $n(V)$ in (18a).

V. VOLTAGE DISTRIBUTION ABOVE THE PERCOLATION THRESHOLD

The behavior of the voltage moments above the percolation threshold, and the form of the voltage distribution can be easily obtained using a conventional scaling approach. First, we construct an intensive quantity for the voltage moments by defining $m(k,p)=M(k,p)/L^{d-k}$, where $M(k,p)$ are the moments defined in (5), except that they are now generalized to any value of p. At $p=1$, $V=L^{-1}$ and $n(V)=L^d$, so that $M(k,1)=L^{d-k}$ and $m(k,1)=1$ for all k. Precisely at p_c, since $M(k,p_c)\sim L^{-p(k)/\nu}$, we have $m(k,p_c)\sim L^{-p(k)/\nu-(d-k)}$. By scaling, we then have for $p>p_c$,

$$m(k,p)\sim L^{-p(k)/\nu-(d-k)}g\left[\frac{\xi}{L}\right], \tag{46}$$

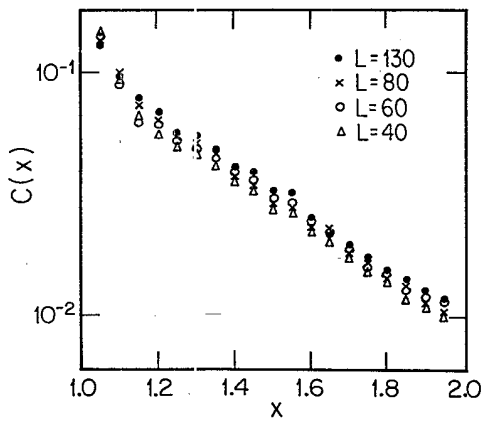

FIG. 9. Data collapsing of the amplitude function, $C(x)$ from the voltage distribution for four different system sizes, $L=40$, 60, 80, and 130. The quantity $n(V(k))/L^{\phi(x)}$ is plotted versus $x=\ln V/\ln\langle V_{\max}\rangle$ on a semilogarithmic scale to isolate the amplitude $C(x)$.

where $g(z)\sim$const for $z\ll 1$, and $g(z)\sim z^{-p(k)/\nu-(d-k)}$ for $z\gg 1$. We therefore find for the critical behavior above p_c, in the thermodynamic limit,

$$m(k,p)\sim(p-p_c)^{t(k)}, \tag{47}$$

where

$$t(k)=(d-k)\nu+p(k). \tag{48}$$

Above p_c, it will be convenient to work with quantities which are independent of L in the $L\to\infty$ thermodynamic limit. Therefore we define a new voltage variable $\overline{V}=VL$. Thus for $p>p_c$, we have $\overline{V}(k)\sim L^{1-\alpha(k)}g(\xi/L)$, where $g(x)\sim x^{1-\alpha(k)}$ for $x\ll 1$ and $g(x)\sim$const for $x\gg 1$. Then for $p>p_c$ and for $L\to\infty$, we have $\overline{V}(k)\sim\xi^{1-\alpha(k)}$. Similarly to (18b), let us now define the quantity

$$\bar{x}\equiv\frac{\ln\overline{V}}{\ln\overline{V}_{\max}}=\frac{1-\alpha(k)}{1-\alpha(\infty)}. \tag{49}$$

The voltage distribution per unit volume is given by $\rho(\overline{V},p)=n(V,p)/L^d$, where $n(V,p)$ is the number of bonds with voltage drop V at bond concentration p. Since $n(V(k),p_c)\sim L^{f(k)}$, we have for $p>p_c$

$$\rho(\overline{V},p)\sim L^{-d+\psi(\bar{x})}h\left[\frac{\xi}{L}\right], \tag{50a}$$

where $h(z)\sim$const for $z\gg 1$ and $h(z)\sim z^{-d+\psi(x)}$ for $z\ll 1$. In this equation, $\psi(\bar{x})=f(k(\bar{x}))$, and $k(\bar{x})$ is obtained by inverting (49). We therefore find for the critical behavior above p_c in the thermodynamic limit

$$\rho(V,p)\sim(p-p_c)^{\beta(\bar{x})}, \tag{50b}$$

with $\beta(\bar{x})=[d-\psi(\bar{x})]\nu$. Thus for a given value of $\overline{V}$, $\rho(\overline{V},p)$ can be regarded as the order parameter corresponding to a given fractal set of bonds which is characterized by the value $\bar{x}$, and $\beta(\bar{x})$ is the corresponding critical exponent. Thus $\beta(k=0)$ is the usual critical exponent relative to all the backbone bonds, $\beta(k=2)$ is the exponent relative to the fractal set contributing to the resistance, and $\beta(k=\infty)=d\nu-1$ is the exponent relative to the hottest bonds.

The same result can be obtained from the hierarchical lattice, in conjunction with the nodes and links picture of a random network. In the nodes and links picture, a random network has the form of a superlattice with an average spacing given by the correlation length ξ. The new feature in our representation of this picture is that the links of the superlattice are not of the form of linear chains, but rather, each link is a hierarchical lattice whose iteration index N is such that the effective linear dimension of the hierarchical lattice equals the correlation length.

Accordingly, the voltage distribution above the percolation threshold in a d-dimensional system of linear size L will be given by the voltage distribution on one hierarchical lattice, with an appropriate value of the iteration index, times the number of these lattices in the complete network. Thus we have, from (35),

$$n(V)_{\rm tot}=\left[\frac{L}{\xi}\right]^d\left\{\left[\frac{2}{\lambda}\right]^N\lambda^j\begin{bmatrix}N\\ j\end{bmatrix}\right\}, \tag{51}$$

where the parameter λ can be related to the spatial dimension d, as in Sec. III B, and the voltage $V(j)$ takes on the discrete values

$$V(j)=\frac{2^j}{(\lambda+4)^N}\left[\frac{\xi}{L}\right], \tag{52}$$

which are indexed by the integer $j=0,1,2,\ldots,N$. Since each hierarchical lattice has a linear dimension ξ, N and ξ are related by

$$\left[\frac{2}{\lambda}\right]^N=\xi^{1/\nu}. \tag{53}$$

In analogy with the scaling form given in (37) for $p=p_c$, one can use (51)–(53) to show that the scaling form of the voltage distribution for $p>p_c$ is given by (49).

From the voltage distribution given in (46), it is straightforward to show that the kth moment now varies as

$$M(k)=\left[\frac{L}{\xi}\right]^{d-k}\frac{(2/\lambda)^N}{(\lambda+4)^N}(1+2^k)^N. \tag{54}$$

Using (53) to eliminate N, we can directly write that the intensive quantity, $m(k)\equiv M(k)/L^{d-k}$ scales with ξ as in (46) and (47). Thus we see that each moment vanishes with an independent exponent as the percolation threshold is approached from above.

In addition to the novel critical behavior of the moments of the voltage distribution, we have also found the very intriguing feature that the higher moments are nonmonotonic as a function of p, with a peak located away from the percolation threshold (Fig. 10). To appreciate this phenomenon, consider diluting a network, starting from $p=1$. As p decreases, less total current will flow at constant external voltage because the conductance is decreasing. On the other hand, there will be an enhancement of the local currents flowing in those bonds which form "bottlenecks" in the network, and a sufficiently high moment of the voltage distribution will be extremely sensitive to these bottlenecks. The competition between these two effects is the apparent source of the striking nonmonotonicity in the higher-voltage moments. In the case of the second moment, the former effect dominates, and this quantity therefore decreases monotonically with p, as it must, on the basis of elementary considerations. However, there is no such restriction on the higher moments, *a priori*.

From the exact distribution for systems of linear size $L=2$, 3, and 4, we find that the normalized moments, $m(k)$, are monotonic for $k\leq 6$, but that there is a dramatic peak in the higher moments for values of p in the range 0.8–0.9. Furthermore, by examining the slopes of $m(k)$ versus p at $p=1$, we observe that both the second and fourth moments appear to decrease progressively more strongly as the system size increases. This suggests that

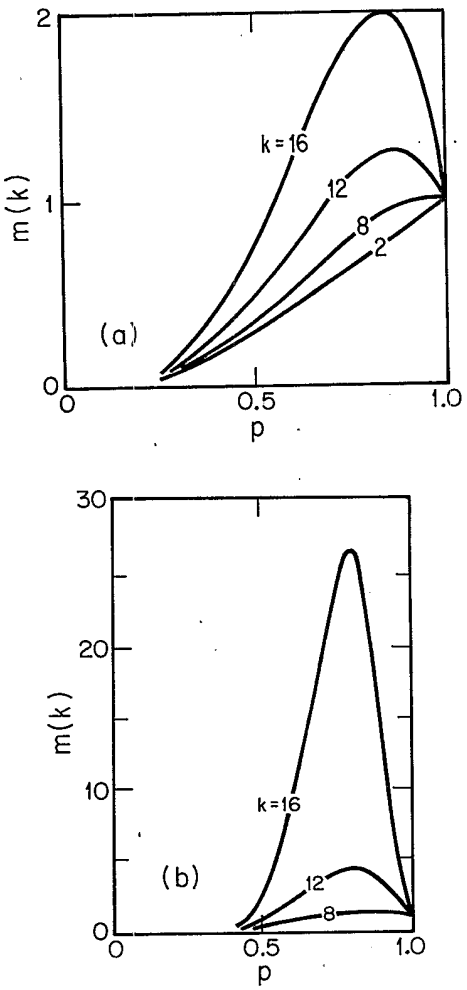

FIG. 10. Plots of the exact normalized moments of the voltage distribution, $m(k,p)=\langle V^k\rangle/L^{2-k}$, as a function of p for (a) a system of linear dimension $L=2$ and (b) $L=4$.

both functions will decrease initially at $p=1$, and that they will also be monotonic in the thermodynamic limit. However for the sixth moment, the initial slope is positive, but decreasing as L increases. This suggests the possibility that $m(6)$ will be either slowly varying or nonmonotonic as a function of p for larger values of L. Finally, as is clear from Table II, all the moments beyond the sixth are strongly nonmonotonic. Corresponding to this nonmonotonicity in the moments, we also find that $V_{\max}$, averaged over all configurations, is also strongly nonmonotonic.

This phenomenon has potentially interesting experimental ramifications for physical realizations of the random resistor network. At some value of p, which should be near to the location of the peak in the average value of $V_{\max}$, there will be a local maximum in the heating of resistors in the bottlenecks. Suppose now that each bond is a resistor and a fuse in series so that when the current limit of the fuse is exceeded, it burns out and becomes an insulator. If the applied potential is high enough, the network will be susceptible to a failure which is initiated when the highest-current-carrying bond will burn out.[27] The point of maximum susceptibility is not at the percolation threshold, as one might naively expect, but is close to the point where the average value of $V_{\max}$ attains its maximum.

VI. VOLTAGE DISTRIBUTION OF THE RANDOM SUPERCONDUCTING NETWORK

Now we consider the voltage distribution in a random superconducting network. In this problem, superconducting bonds are present with probability p, and normal bonds are present with probability $1-p$. For small values of p, there are finite superconducting clusters in a background of normal resistors. As $p\to p_c$, the conductivity diverges due to the formation of a superconducting cluster which spans across the finite-size system. For a system of linear dimension L at the percolation threshold, the nonsuperconducting, or normal, configurations are characterized by very large clusters of resistors which are almost touching each other (Fig. 11).

If a unit voltage drop is applied across the opposite faces of the cell, there will be a distribution of voltage drops across the current-carrying normal bonds in the network, while there will be no voltage drop across the superconducting bonds. The current-carrying bonds exclude those normal bonds which are entirely embedded within superconducting clusters, and also those normal bonds which happen to join two sites in the same superconducting cluster. In two dimensions, these bonds are dual to bonds which are in finite clusters and bonds which are on dangling ends of a percolating cluster, respectively. Of the current-carrying bonds, a special role is played by the "bridges." These are normal bonds between two large, but nonspanning superconducting clusters which cause the entire network to become superconducting if any one of these bonds is replaced by a superconductor. The bridges play a role analogous to the links in the random resistor network. Similar to the case of the links, it can be proved[28] that for any spatial dimension, the fractal dimension of the bridges is $1/\nu$, i.e., that the number of bridges, N_B, varies as $L^{1/\nu}$. These bridges are the highest-

TABLE II. The values of the first derivative of the normalized moments, $\langle V^k\rangle/L^{2-k}$, with respect to p at $p=1$ for various values of k for a system of linear size $L=2$, 3, and 4.

L	$k=2$	4	6	8	10
2	−1.60	−1.470	−0.744	+0.47	+2.3
3	−1.818	−1.769	−0.434	+2.17	+6.52
4	−1.916	−1.978	−0.313	+3.22	+9.47

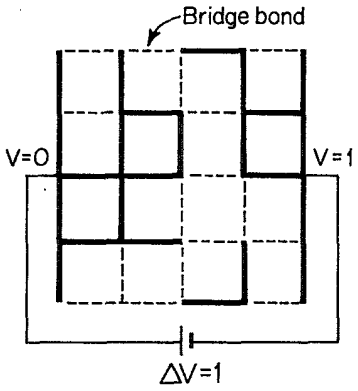

FIG. 11. Schematic picture of a random superconducting network just below the percolation threshold. A bridge bond is indicated.

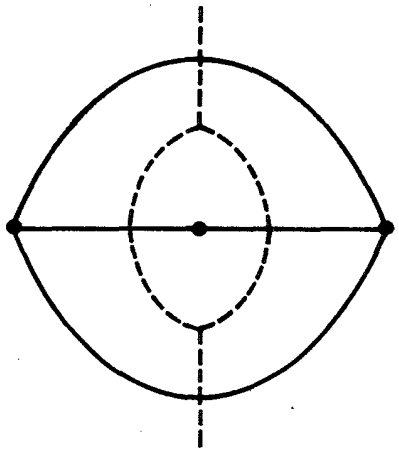

FIG. 12. The dual of the two-dimensional version of the hierarchical model (solid lines), while the original hierarchical model is indicated by the dashed lines. Shown is the particular case where the iteration index $N=1$.

current-carrying bonds in the network and they are characterized by the maximum value of the voltage drop, $V_{\max}=1$.

In close analogy with the definition given for the random resistor network in (4b), we now introduce the voltage distribution, $n(V)$ and the corresponding moments, $\mathscr{L}'(k)$, by

$$\mathscr{L}'(k)=\sum n(V)V^k \sim L^{\tilde{\zeta}'(k)} , \tag{55}$$

where the sum is over all normal bonds in the network. As in the case of the random resistor network, the second moment gives the conductance of the superconducting network. Thus $\tilde{\zeta}'(2)=\tilde{\zeta}_s$, the critical exponent related to the divergence of the conductivity, $\tilde{s}=2-d+\tilde{\zeta}_s$. In addition, $\tilde{\zeta}'(4)$ is related to the magnitude of the noise in the network, while $\tilde{\zeta}'(\infty)$ give the fractal dimension of the bridges. In the Appendix we prove that for two dimensions,

$$\tilde{\zeta}'(k)=\tilde{\zeta}(k) , \tag{56}$$

where $\tilde{\zeta}(k)$ is the moment exponent introduced in Eq. (4b). Equation (56) generalizes to all values of k the well-known result for $k=2$, namely that the conductivity exponents of a random resistor network and a random superconducting network coincide.[29] Thus the moments of the voltage distribution for the random superconducting network are also described by an infinite set of independent exponents.

It is also interesting that this qualitative picture can be obtained by considering the dual of the two-dimensional version of the hierarchical model (Fig. 12). At the Nth level of iteration, the dual of the hierarchical model is a parallel array of bridges of various lengths j, running from 1 to $N+1$, with the number of bridges of length j equal to $2^N\binom{N}{j}$. This model provides a useful intuitive picture of the geometric structure of the normal bonds which intervene between superconducting clusters just below the percolation threshold. Furthermore, the exponents on the dual model also satisfies the identity (56).

Above two dimensions, the duality relation (56) no longer holds. As a result, there appears to be no simple correspondence between the voltage distributions of the random resistor network and the random superconducting network. In particular, in the mean-field limit, the infinite exponent hierarchy for the random resistor network reduces to a single exponent because of the predominant role played by the links. On the other hand, a very different behavior occurs for the superconducting network. From the mean-field value of the superconducting exponent, $s=0$, we deduce that

$$\tilde{\zeta}'(2)=4 , \tag{57a}$$

while for the bridge exponent we have

$$\tilde{\zeta}'(\infty)=2 . \tag{57b}$$

Thus instead of having a single exponent in the mean-field limit, there is still a spectrum of exponents. This stems from the fact that the voltage distribution remains broad as the bridges do not entirely dominate.

In the random resistor network, it proved useful to use the voltage moments as a geometrical measure of the backbone, as in (4b). The insights gained from this formulation can be used to discuss conductivity and other physical processes on the percolating backbone. In the superconducting network, current flow is limited by the geometrical properties of the *surface* of a superconducting cluster. It is therefore of interest to define the voltage distribution for the normal bonds on the surface of the superconducting cluster in order to characterize its geometrical features. Thus consider the following moments:

$$\mathscr{L}''(k)=\sum{}^{*} n(V)V^k \sim L^{\tilde{\zeta}''(k)} , \tag{58}$$

where the asterisk indicates that the sum is restricted to only the normal bonds on the surface of superconducting clusters. Since $\mathscr{L}''(k)\leq\mathscr{L}'(k)$, it follows that

$$\tilde{\zeta}''(k)\leq\tilde{\zeta}'(k) . \tag{59}$$

This inequality becomes an equality, at least for $k \to \infty$, since both moments reduce to a sum over the bridge bonds in this limit.

In two dimensions, we can use the inequality (59) to obtain useful information about the surface of large percolating clusters. The external surface of a large percolating cluster has been recently studied and its fractal dimension has been estimated[30] to be $d_H \simeq 1.75$. On the other hand, in the limit $k \to 0$, Eq. (58) describes only the unscreened part of the external surface, i.e., the portion of the superconducting surface where the voltage drop is not identically equal to zero. This part of the external surface, whose fractal dimension is given by $\tilde{\zeta}''(0^+)$, excludes portions of the surface inside of the crevices, where the surface is completely screened from the electric field. Clearly we have

$$\tilde{\zeta}''(0^+) \leq d_H = 1.75 \ . \tag{60}$$

It is of interest to know whether the unscreened surface has the same fractal dimension of the external surface, namely whether (60) holds as a strict inequality, since the former quantity is more amenable to external probes. This information can be obtained from (59) in the limit $k \to 0^+$, yielding $\tilde{\zeta}''(0^+) \leq \tilde{\zeta}'(0^+)$. From (56), we also have $\tilde{\zeta}'(0^+) = \tilde{\zeta}(0^+) = d_f$, where d_f, the fractal dimension of the backbone, has been estimated to equal 1.66.[31] Therefore $\tilde{\zeta}''(0^+) \leq 1.66$ and comparison with (60) shows that $\tilde{\zeta}''(0^+)$ is strictly less than d_H. This is in agreement with recent numerical findings[32] that $\tilde{\zeta}''(0^+) \simeq 1.34$.

VII. DISCUSSION AND CONCLUSIONS

In this paper, we have presented a completely new approach for studying the classical percolation problem of the random resistor network. By focusing on the distribution of voltage drops across the bonds in the network, we have found a wealth of new phenomena. The moments of the voltage distribution all scale independently, so that an infinite hierarchy of exponents is needed to characterize the scaling of the voltage distribution. This result is a direct consequence of the fact that there is no unique, typical value of the voltage in a random resistor network. Instead, we argued that the voltage drops in a random network originate from a multiplicative process, which leads to an underlying distribution that can be naturally written in terms of the logarithm of the voltage.

As a consequence of this line of reasoning, we argued that the backbone of a percolating cluster contains an infinity of fractal sets, each with a distinct value of the fractal dimension. A particular fractal set contains the subset of all the bonds in the backbone which are characterized by a fixed value of the fundamental parameter $x = \ln V / \ln V_{\max}$. Sets characterized by different values of x are sensitive to different geometrical features of the backbone. For $x \to -\infty$, the corresponding fractal set of bonds are those embedded within blobs at the finest length scale where the voltage drop is a minimum, while as $x \to 1$, the corresponding set of bonds are the links, i.e, the highest-current-carrying bonds in the network.

A number of extensions of the voltage distribution were also considered. Above the percolation threshold, moments of the voltage distribution also scale independently, but now as a function of the scaling field $p - p_c$. We also found that sufficiently high moments are nonmonotonic functions of p, a feature that stems from the competition between a globally reduced current flow and a locally enhanced current flow in the vicinity of bottlenecks, as p decreases from unity. We also considered the voltage distribution for the random superconducting network. By duality considerations, we showed that the distributions of the random superconducting and random resistor networks are simply related on the square lattice. Thus, in close analogy with the conducting backbone of a random resistor network, an infinite hierarchy of exponents is needed to characterize the geometrical properties of the surface of large superconducting clusters.

There are a number of interesting possibilities for future work. It would be worthwhile to study the voltage distribution of the random superconducting network above two dimensions, where we can no longer exploit duality arguments to make a relation with the random resistor network. From such a study, one might hope to develop a better intuition for the relevant surface geometrical properties which strongly influence and limit current flow between superconducting clusters. It should also be interesting to consider the voltage distribution for the general two-component composite, in which each lattice bond may have a conductivity equal to σ_1 with probability p, and conductivity σ_2, with probability $1-p$. This would provide a very general description of the voltage distribution in which the random resistor and random superconducting networks are merely special cases.

Most importantly, perhaps, the voltage distribution appears to provide a valuable conceptual framework, on which further developments can be based. On the theoretical side, the "multifractal" aspect of the voltage distribution gives rise to a variety of novel scaling phenomena which have drastic quantitative consequences in a wide range of problems, such as growth phenomena, dynamical systems, and turbulence. In the context of random networks, potential applications include crack propagation, dielectric breakdown, and hydrodynamic dispersion in disordered media. The former two processes are sensitive to the high-voltage tail of the voltage distribution, while hydrodynamic dispersion is limited by the low-voltage tail. The insights gained from the general study of the voltage distribution should therefore prove useful in these more applied problems. Finally, in the context of nonequilibrium processes, the analog of the voltage distribution is the first passage probability distribution for a random walker to reach a growing fractal structure. The detailed properties of this distribution provide both deep insights into the geometry of growing fractal aggregates, as well as offering potential applications for understanding fundamental chemical processes, such as catalysis, on the surface of these structures.

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APPENDIX

We prove here the following equality for the square lattice

$$\mathscr{L}(k,p)=\mathscr{L}'(k,1-p)\ , \tag{A1}$$

where

$$\mathscr{L}(k,p)=\sum_{\langle ij\rangle}(\langle\,|V_i-V_j|/G\,\rangle)^k\sim L^{\tilde{\zeta}(k)} \tag{A2}$$

is the kth moment of the voltage distribution, normalized by the conductance G, in the random resistor network where each bond is a unit conductor with probability p or an insulator with probability $1-p$. Similarly,

$$\mathscr{L}'(k,1-p)=\sum_{\langle\alpha\beta\rangle}\langle\,|V_{\alpha'}-V_{\beta'}|^k\rangle\sim L^{\tilde{\zeta}'(k)} \tag{A3}$$

is the kth moment of the voltage distribution in a random superconducting network in which each bond has an infinite conductance with probability p or a unit conductance with probability $1-p$. In (A2) and (A3), the average is over all bond configurations when a unit voltage drop is applied across the opposite edges of the system. From (A1) to (A3), it also follows that

$$\tilde{\zeta}(k)=\tilde{\zeta}'(k)\ . \tag{A4}$$

To prove the above results, consider a square lattice A and its dual A^D (Fig. 13). Assign to each bond ij in the original lattice a conductance σ_{ij}, and apply a unit voltage drop, $\Delta V=1$ across the lattice. If V_i-V_j and I_{ij} are, respectively, the voltage drop and the current across the bond ij, then we have

$$V_i-V_j=I_{ij}\sigma_{ij}^{-1}\ . \tag{A5}$$

Due to the boundary condition $\Delta V=1$, we also have

$$I=G\ , \tag{A6}$$

where I and G are, respectively, the total current and the conductance for each bond configuration.

In the dual lattice, for each bond $\alpha\beta$ which is dual to ij, we define the following quantities:

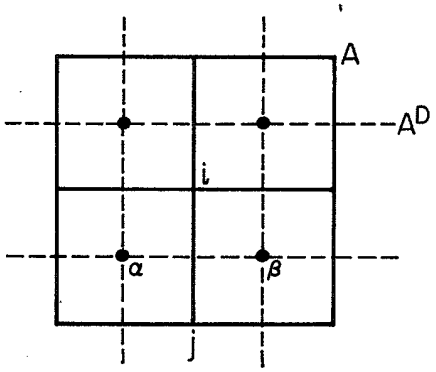

FIG. 13. The square lattice A (solid lines) and its dual A^D (dashed lines).

$$V_\alpha-V_\beta=I_{ij}\sigma_{ij}\ . \tag{A7}$$

Using (A5), this voltage difference can also be written as

$$V_\alpha-V_\beta=(V_i-V_j)\sigma_{ij}\ . \tag{A8}$$

Following Straley,[29] it is easy to show that the difference, $V_\alpha-V_\beta$, satisfies Kirchhoff's laws for the dual lattice in which each bond $\alpha\beta$ has a conductance $\sigma_{\alpha\beta}=\sigma_{ij}^{-1}$, when the external potential drop $\Delta V=I=G$. As a consequence, $V'_\alpha-V'_\beta=(V_\alpha-V_\beta)/G$ is the voltage drop across bond $\alpha\beta$ in the dual lattice when the external potential drop $\Delta V=1$.

Therefore for each configuration $\{\sigma_{\alpha\beta}\}$, the kth moment of the voltage distribution is given by

$$\mathscr{L}'(k,\{\sigma_{\alpha\beta}\};A^D)=\sum_{\langle\alpha\beta\rangle}|V'_\alpha-V'_\beta|^k=\sum_{\langle ij\rangle}\left[\left[\frac{V_i-V_j}{G}\right]\sigma_{ij}\right]^k\ . \tag{A9}$$

If $\sigma_{\alpha\beta}=1$ with probability $1-p$, and $\sigma_{\alpha\beta}=\infty$ with probability p, then by averaging (A9) over all bond configurations, we obtain (A1) for a lattice which is self-dual, such as the square lattice.

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