CHAPTER 19

PERCOLATION AND CONDUCTION IN RANDOM RESISTOR–DIODE NETWORKS

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ABSTRACT

The geometrical properties of random networks containing non-directed bonds (resistors) and directed bonds (diodes) of *arbitrary* orientation are investigated. It is found that percolation transitions occur by varying either the bond concentration or the diode orientation, and that there is multicritical behavior when these transitions occur simultaneously. Duality and other invariance properties are exploited to derive some exact results for the location and the exponents of this multicritical transition. For the special case of directed percolation, conduction properties are also studied. In the mean-field limit, valid at five dimensions and above, the conductivity exponent of directed percolation, t_+, equals 2, compared to the mean-field limit of $t = 3$ for the conductivity exponent of the random resistor network. In two dimensions, results from the position-space renormalization group, computer simulations, and analog experiments indicate that $t_+ \cong 0.6$, compared to the estimate $t \cong 1.3$ for the random resistor network.

1. INTRODUCTION

In this article, we investigate the properties of a generalized percolation model, the *random resistor-diode network*, which contains both non-directed bonds (resistors) *and* directed bonds (diodes) of *arbitrary* orientation. The possibility of allowing for resistors and diodes in a percolation model was first mentioned in the original work of Broadbent and Hammersley.[1] However, subsequent investigations focused primarily on the conventional isotropic percolation problem, which may be thought of as a random network of resistors and vacancies.[2] The random resistor-diode network represents a logical extension of pure percolation to

encompass directionality effects. We shall attempt to convince the reader that the additional orientational degrees of freedom of the diodes gives rise to richer network behavior than that found in conventional percolation.[3-6] Therefore investigations of this model should be an interesting topic for further research.

In addition to this theoretical motivation, there also exist a number of applications of resistor-diode networks to physical phenomena in random systems where directionality effects play an important role. Several examples can be found in the field of network theory, where directed networks form an important class of models that have been considered.[7-12] A problem of general interest is the design of networks which can carry a maximum amount of "information" with a minimum degree of connectivity between the vertices of the system.[9,10] The basic results of this problem have been applied to the designing of telephone switching systems[11] and communication networks.[12] It appears that this class of problems may be an interesting area of overlap between the engineering and physics disciplines.

An important special case of networks containing diodes is *directed* percolation in which diodes of one orientation randomly occupy a lattice.[13-24] There has been considerable attention devoted to this model, mainly because of the novel, anisotropic nature of its percolation transition.[6,15-17,20,22] This anisotropy shifts the upper critical dimension, d_c, from six for isotropic percolation to five,[6,16] so that the two models are in different universality classes. Furthermore, directed percolation has connections with Reggeon field theory,[25] branching Markov processes,[26-28] and hopping conductivity in strong electric fields.[29-33] These interesting aspects of the model are reviewed in the chapter by Kinzel in this volume.[34] This article is intended to be complementary in the scope of topics that will be discussed.

In section 2, we begin by defining the resistor-diode percolation models whose properties are the focus of this paper. In section 3, we shall treat the geometric transition phenomena of the random resistor-diode network. This system is found to have four phases characterized by being either non-percolating (insulating), isotropically percolating (resistor-like), or unidirectionally percolating (diode-like). These phases are separated by percolation thresholds of qualitatively different characters. Moreover, novel multicritical behavior is predicted where concentration and orientational degrees of freedom are simultaneously critical. The position-space renormalization group (PSRG) and symmetry considerations will be used to elucidate these transitions and derive several, possibly exact exponent relations.

In section 4, we describe several conduction properties of resistor-diode

networks.[23,35-38] For simplicity, we shall primarily consider the conductivity in the geometrical model of directed percolation. This represents a simple generalization of the random resistor network to allow for a preferred orientation for current flow. In the mean-field limit, valid for spatial dimension $d \geq d_c = 5$, we find a directed conductivity exponent of $t_+ = 2$, in contrast to the mean-field limit of $t = 3$ for the conductivity exponent of the random resistor network. In two dimensions, the combined results of the PSRG, computer simulations, and analog experiments indicate that $t_+ \cong 0.6$, compared to recent estimates[39,40] of $t \cong 1.3$. In two dimensions, the value for t_+ is quite close to the estimate for the directed backbone exponent, and the two exponents are equal in the mean-field limit, suggesting that they may be equal in general. Finally, in section 5, we briefly summarize the principal results and several unresolved questions raised in this work.

2. RESISTOR-DIODE PERCOLATION MODELS

To make our discussion concrete, we define three resistor-diode percolation models which will be studied in this article:

(i) Fully directed percolation. Each lattice bond between nearest-neighbor sites may be occupied with probability p_+ by a diode which can point only in the +x, +y, +z, etc., directions. In this model, the diode "polarization" of the lattice points along the diagonal (1,1,. . .,1). Clusters grow preferentially along this axis so that they acquire an asymptotically anisotropic shape near the threshold.

Fig. 1. The bond elements of (a) fully and (b) partially directed percolation on the square lattice. The arrow corresponds to a diode, the zig-zag to a resistor.

(ii) Partially directed percolation. Along the x-axis, lattice edges are randomly occupied by positively oriented diodes so that the polarization is parallel to (1,0,. . .,0). The system is isotropic along the remaining (d-1) axes, and this may be accomplished by occupying these axes by resistors at the same probability as the diodes. More general models can be considered in which the polarization points along a diagonal of the form (1,. . .,1,0,. . .,0), and in which the isotropic portion of the network is formed by randomly oriented diodes.

(iii) Random resistor-diode network. This is the generalization of fully directed percolation which we shall be primarily concerned with in this article. In addition to positive diodes, negative diodes and resistors occuring with probability p_- and p respectively, are incorporated into the model. Each bond may now be vacant, or insulating, with probability $q = 1 - p - p_+ - p_-$. An important special case, whose conductivity properties will be treated in section 4, is the *oriented resistor-diode network*, in which negative diodes are absent.

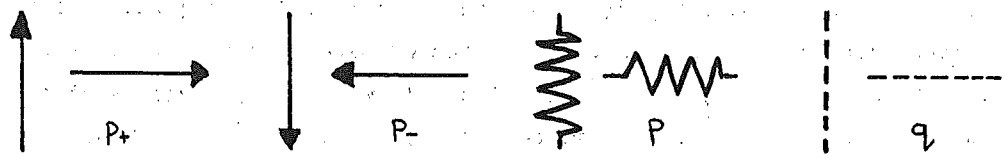

Fig. 1(c). The bond elements of the random resistor-diode network on the square lattice.

3. PERCOLATION IN THE RANDOM RESISTOR-DIODE NETWORK

A. Symmetries of the model. If we restrict ourselves to the square lattice, the random resistor-diode network possesses two symmetries which permit the locations of several of the percolation transitions to be related. There is, in addition, a lattice independent invariance of the pair-connectedness function from which exponents at different critical points can be related. It is therefore worthwhile to discuss these properties briefly.

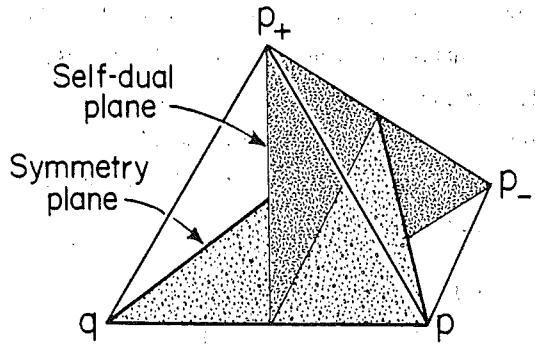

Fig. 2. The composition tetrahedron of the four component random resistor-diode network. This figure represents the intersection of region $p + p_+ + p_- + q = 1$, with the half-spaces p, p_+, p_-, and $q \leq 1$ in the four-dimensional space of bond concentrations. For any point in the tetrahedron, the perpendicular distance to any face gives the relative concentration of the bond species labelled at the opposite apex. The self-dual ($p = q$) and reflection ($p_+ = p_-$) symmetry planes are shaded.

One relatively simple feature is the reflection symmetry of the network under the interchange $p_+ \longleftrightarrow p_-$. In addition, the system obeys a dual symmetry defined by the transformation $p \longleftrightarrow q$, with p_+ and p_- remaining unchanged.[4,6,41] Under this mapping, it can be readily shown that every non-percolating configuration maps to an isotropically percolating configuration and vice-versa, while a directed percolating configuration maps to another such configuration of the same orientation. Since the square lattice is self-dual, the locus defined by $p = q$ and

$p_+ = p_-$ is therefore a line of critical points along which the critical concentration of resistors is given by $p_c = 1/2 - p_+$. This is a generalized isotropic percolation threshold in which randomly oriented diodes help form the infinite cluster. Multicritical behavior occurs along this line because the four phases of the network become simultaneously critical. New exponents may be defined to describe these phenomena, as we shall describe in the next section.

The invariance property of the network is somewhat more subtle. It is the observation that the probability $P(i \rightarrow j)$, that a directed path joins site i to site j, independent of the existence of a directed path from j to i, is a function of the variables $A_{\pm} \equiv p + p_{\pm}$ only.[42] It then follows that $P(i \rightarrow j)$ is constant on the multicritical line, since it is defined by the conditions $A_+ = A_- = 1/2$. In terms of this pair-connectedness function, A_+ and A_- are the natural variables of the problem. The distinction between p and p_+ is important, however, if it is desired to distinguish between two-way and one-way connected paths. We shall use this invariance, together with information from the PSRG, to derive exponent relations along the multicritical line in section 3.C.

B. Position-space renormalization group. Our approach is based on a simple generalization of the method introduced by Reynolds *et al* for bond percolation.[43-45] In their method, a self-dual bond cell[46] of the form shown in Fig. 3 is rescaled by a factor of b. The probability of spanning a cell defines the recursion relation for the rescaled probability. When directed bonds are introduced, this transformation can be modified to encompass either 2-way or 1-way spanning: configurations which span a cell isotropically rescale to a resistor, while configurations which span in only one direction rescale to a diode whose orientation is defined by the direction of spanning. These criteria define recursion relations for p' and $p'_{\pm}$ respectively, the renormalized probabilities of spanning a cell isotropically or in only one direction.[3-6,47]

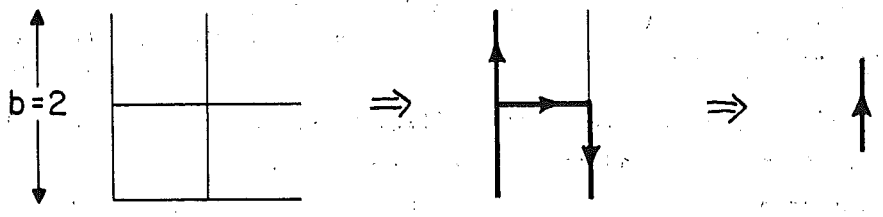

Fig. 3. Rescaling of a 2 x 2 bond cell. The middle part of the figure shows the portion of the cell needed to calculate the probability of traversing the cell vertically. A typical configuration of resistors and diodes is shown on this cell, and its rescaled counterpart is shown on the extreme right.

For the square lattice, we obtain the dual and reflection symmetric phase diagram shown in Fig. 4 in the composition tetrahedron defined by the probabilities p, p_+, p_-, and q. There are two second-order surfaces emanating upward from the multicritical line to form a wedge-shaped region. Within this volume, renormalization flow is attracted to the p_+ fixed point, thereby defining the region of the positive diode phase. An identical structure occurs below the reflection symmetry plane so that the tetrahedron is divided into four regions. In each region, the renormalization flow is attracted by one of the p_+, p_-, p, and q fixed points. These are the positive diode, negative diode, resistor, and insulating phases respectively. On the simple cubic lattice, a qualitatively similar phase diagram is obtained, except that dual symmetry no longer holds. A relatively larger region of the tetrahedron corresponds to the resistor phase, while a much smaller region corresponds to the insulating phase.

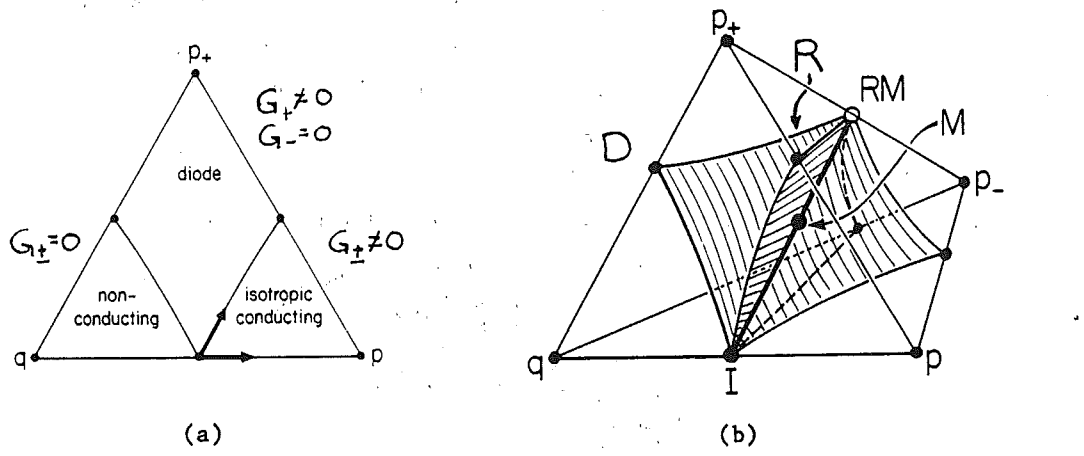

Fig. 4. (a) Composition triangle of the oriented resistor-diode network within the three-dimensional concentration space of p_+, p, and q. Fixed points of the renormalization transformation are indicated by the heavy dots. Two second-order lines divide the figure into three phases characterized by whether the forward or reverse conductances, G_+ and G_- respectively, are zero or non-zero. The second-order lines meet at the isotropic percolation point where both conductances vanish simultaneously. (b) Phase diagram of the full random resistor-diode network. The triangle in part (a) corresponds to the front and bottom faces of the tetrahedron. The shaded second-order surfaces divide the figure into the four regions indicated. These surfaces meet along a multicritical line defined by simultaneous reflection and dual symmetry. The non-trivial fixed points are: directed (D), reverse (R), isotropic (I), and "mixed" (M). The open circle indicates "random Manhattan" (RM), a lattice completely occupied by randomly oriented diodes.

In addition to these trivial fixed points, there are several non-trivial fixed points where interesting transition phenomena occur. These are:

(i) Directed percolation. There are two directed percolation fixed points which are attractors for all points on the second-order surfaces which separate the non-percolating from the positive (or negative) diode phase (Fig. 4(b)). The directed percolation problem corresponds to the p_+-q (or p_--q) axis of the tetrahedron. These axes are mapped into themselves upon renormalizing so that the model can be treated by a simple one-parameter renormalization.[14] As stated in the introduction, directed percolation is the primary focus of the chapter by Kinzel in this volume.[34] Therefore I will merely describe selected features of this transition.

Just below the transition, two independent diverging lengths, one parallel and one perpendicular to the diode polarization, are required to characterize cluster shapes (Fig. 5).[6,17,20-23] Above the threshold, percolation is confined within a narrow cone whose opening angle scales as $\xi_\perp/\xi_\parallel$.[15] Because of this anisotropy, the validity of the PSRG approach based upon an *isotropic* rescaling is open to some question. On physical grounds, a transformation based on a "connectivity"[43-45] rule is probably sensitive to $\xi_\parallel$ only, and it is not clear how to incorporate $\xi_\perp$ within this scheme. Accordingly the eigenvalue λ of the recursion relation is assumed to give the exponent $\nu_\parallel$ through $\nu_\parallel = \ln b/\ln\lambda$. Alternative renormalization schemes have been suggested, but the results so far are not promising.[48] This remains an interesting unresolved question.

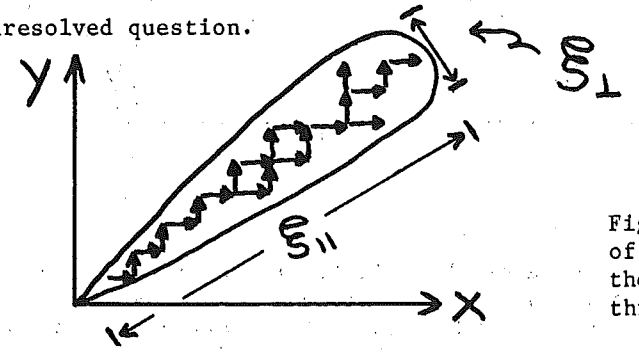

Fig. 5. Schematic picture of a typical cluster near the directed percolation threshold.

On the square lattice, rescaling by a factor of b = 2 gives $p_+^* = 1 - q^* = 0.5550$, $p_-^* = p^* = 0$, and $\nu_\parallel = 1.543$, compared to accurate numerical estimates of $p_{+c} = 0.6447$, $\nu_\parallel = 1.74$, and $\nu_\perp = 1.10$ obtained from Monte Carlo,[15] phenomenological renormalization,[17] and series expansions.[22] On the simple cubic lattice, the b = 2 rescaling gives $p_+^* = 0.239$ and $\nu_\parallel = 1.117$, compared to low-density series estimates of $p_c = 0.383$,[13] and $\nu_\parallel = 1.271$, $\nu_\perp = 0.725$ from a strong coupling expansion of a quantum spin model which is the same universality class as Reggeon field theory.[49]

The PSRG can be extended in a straightforward way to larger cells. A 3 x 3

rescaling[5] gives $p^* = 0.5795$ and $\nu_{\parallel} = 1.522$. For isotropic percolation, continuing this process to the large-cell limit provides a systematic way to improve estimates for critical parameters. However, a similar method does not work for directed percolation because the rescaling is not anisotropic.[14,36] Nevertheless, the cell PSRG provides reasonable numerical estimates at the small-cell level, and correctly shows that isotropic percolation is unstable with respect to the addition of a small fraction of directed bonds, so that the transition is in the universality class of directed percolation.

In addition to the two and three-dimensional results cited above, a theory can be constructed for directed percolation[50] in the spirit of the Flory treatment of excluded-volume effects for linear polymers.[51] It predicts the correct upper critical dimension of $d_c = 5$, and gives the analytic dimension-dependent expressions of $\tilde{\nu}_{\parallel} = (d + 9)/4(d + 2)$ and $\tilde{\nu}_{\perp} = 7/4(d + 2)$. It must be emphasized that the exponents in these expressions describe the divergence of the correlation lengths on N, the average number of bonds in a cluster, but *not* on $(p_+ - p_{+_c})$. Since N varies as $(p_+ - p_{+_c})^{-\beta-\gamma}$, we can obtain the exponents $\nu_{\parallel}$ and $\nu_{\perp}$ in two and three dimensions where numerical estimates for β and γ are available.[13,25,49] The Flory values are found to be remarkably accurate, and it would be interesting to obtain numerical estimates for β and γ in four dimensions in order to check the remaining Flory prediction. An $\varepsilon = 5 - d$ expansion[16] gives $\nu_{\parallel} = 1 + \varepsilon/12$ and $\nu_{\perp} = 1/2 + \varepsilon/16$, and the Flory theory disagrees with this already first order in ε as is the case in other models.

Finally, it should be mentioned that a large number of interesting direction dependent properties and new critical quantities have been elucidated, both for directed percolation,[52,53] and for a more general model with anisotropic occupation probabilities.[19,24,54]

(ii) Reverse percolation. There are two reverse percolation fixed points which are attractors for all points on the second order surfaces which separate the resistor from the positive (or negative) diode phase (Fig. 4(b)). This transition occurs as a lattice completely occupied by one species of diode is gradually diluted by resistors. As the concentration of resistors increases to a critical value, an infinite "backflow" path forms.[6,23,41] Because low-order backflow paths can be enumerated systematically, accurate upper bounds to the critical probability of the model can be obtained.[52,53] Furthermore, since reverse percolation on the square lattice is self-dual to directed percolation, a very accurate lower bound to the critical probability of the latter model is obtained as well. Since this duality is obeyed by the cell PSRG, the value of p^* for reverse percolation equals the value of

q* for directed percolation.

Reverse percolation may be characterized by a cone within which lattice sites are "wetted" by a fluid source at the origin. With no resistors present, one quadrant of the lattice is wetted (Fig. 6). As the concentration of resistors increases from zero, the angle of this percolating region increases from 90° to 180°. By duality, the approach of this cone angle to 180° just below the reverse transition is isomorphic to the vanishing of the percolating cone angle just above the directed transition. A peculiar feature of the reverse transition is a discontinuity in the number of wetted sites. Just below the transition, the cone angle is slightly less than 180°, and approximately half the lattice is wetted. Above the transition, the cone opens beyond 180° and an instability results in which all sites are wetted.

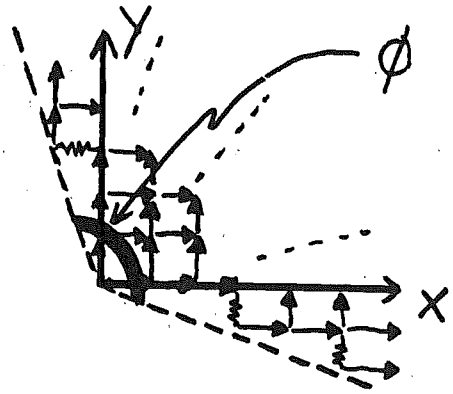

Fig. 6. Schematic picture of the percolating cone of reverse percolation.

(iii) Isotropic percolation point. This fixed point occurs at $p^* = q^* = 1/2$, and $p_{\ddagger} = p_{\pm} = 0$, the exact location for this transition. In the parameter space of the random resistor-diode network, the isotropic percolation threshold now becomes a higher-order critical point where the four phases of the network are simultaneously critical. There exist three relevant fields and three conjugate correlation length exponents. One field may be chosen to be $g \equiv p - q$, which measures the perpendicular distance to the self-dual plane. The associated eigenvalue is given by b^{1/ν_p}, with b the rescaling factor and ν_p the correlation length exponent of pure percolation, leading to a correlation length divergence of $g^{-\nu_p}$. A second field may be chosen to be the diode polarization, $h \equiv p_+ - p_-$. As $h \rightarrow 0$, the critical point is approached from one of the diode phases, and length of backflow paths opposite to the diode polarization diverges as $|h|^{-\nu_+}$, thereby defining a new independent "field-like" exponent.

Finally, there is an additional divergence as the fixed point is approached by varying $i \equiv (p + q) - (p_+ + p_-)$. This variable is a measure of the directedness of the network. It equals -1 along the p_+-p_- axis of the phase diagram (where all

bonds are directed), and it equals +1 along the p-q axis (where all bonds are either non-directed or empty). As the fixed point is approached along the i axis, an isotropic correlation length diverges as $i^{\nu}x$.

A striking feature of the small-cell PSRG transformation is that ν_p and ν_x are equal. We shall show that this holds exactly for all lattices in the next section. As a result, the linearized transformation is diagonal within the reflection symmetry plane. Thus approaching the isotropic fixed point from any direction in the reflection symmetry plane, the correlation length diverges with an exponent ν_p.

(iv) Mixed percolation. This is another higher-order critical point which is located very close to the center of the phase diagram. For the b = 2 and b = 3 rescalings, p* (= q*) is equal to 0.2543 and 0.2563 respectively, with corresponding values of $p_+^* = p_-^* = 0.2457$ and 0.2437. This is a percolation transition of an isotropic character, but mediated by a finite fraction of randomly oriented diodes. The PSRG indicates that this fixed point is the domain of attraction for the entire multicritical line. Thus given any reflection and dual symmetric network, the remaining degree of freedom, i, is irrelevant. Therefore, the critical exponents at any point on the multicritical line are the same as those of the mixed fixed point.

A second, very remarkable point is that the eigenvalues of the fields g and h are equal at the mixed and isotropic fixed points. In fact, the two eigenvalues are constant all along the multicritical line as we shall demonstrate in the next section. In addition, we shall argue that the mixed fixed point necessarily exists, and it is the domain of attraction for all points on the multicritical line.

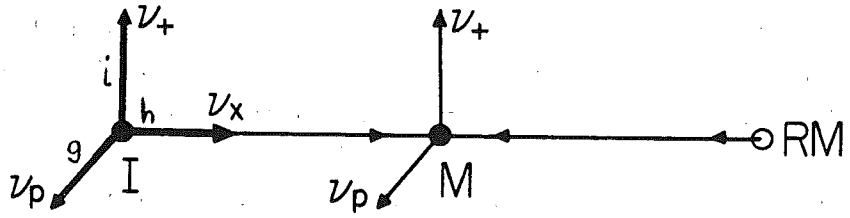

Fig. 7. The multicritical line in the phase diagram of Fig. 4(b). Moving in the vertical direction corresponds to varying the diode polarization, while the out-of-plane direction corresponds to exchanging resistors with vacancies. At the isotropic and mixed fixed points, the arrows indicate the relevant directions which are labelled by the corresponding exponent.

One special case that deserves emphasis is "random Manhattan" -- a lattice completely occupied by randomly oriented diodes. This network is obtained if all the one-way signs in downtown Manhattan were randomized at every intersection. The PSRG indicates that this system is at its critical point. If one approaches the

transition by varying the diode polarization, the correlation length opposite to the polarization diverges according to the exponent ν_+. On the other hand, if the transition is approached by increasing the diode concentration until random Manhattan is completely filled, an isotropic correlation length diverges with an exponent ν_p.

These considerations suggest a rather unusual phase structure along the multicritical line. At both the isotropic and mixed fixed points, all four phases are simultaneously critical with the same numerical values for ν_p and ν_+. However, the PSRG predicts a different nature for the critical fluctuations at the two fixed points. At the isotropic point, fluctuations involve the non-percolating and resistor phases only. However, the critical behavior on the rest of the multicritical line is described by the mixed fixed point, with non-zero values for $p^{\ddagger}$, $p^{\pm}$, p^*, and q^*. Therefore critical fluctuations should be characterized by a non-zero fraction of all four phases of the network.

C. Analytic exponent relations. The numerical results of the small-cell PSRG are suggestive of general exponent relations along the multicritical line. These can be derived[55] with the help of the properties outlined in section 3.A. The choice of the co-ordinates $g = p - q = A_+ + A_- - 1$, $h = p_+ - p_- = (A_+ - A_-)/2$, and $i = 2(p + q) - 1 = 1 - 2(p_+ + p_-)$ are especially convenient for the derivation.

First, notice that the quantities $g' = p' - q'$ and $h' = p'_+ - p'_-$, and the derivatives $\partial g'/\partial g$ and $\partial h'/\partial h$ are functions of A_+ and A_- only. As a result, all these functions are constant on the multicritical line. On the other hand, all cross derivatives vanish either by reflection or dual symmetry. The derivative $\partial i'/\partial i$ is also non-zero, although it is not a constant. Thus in the g, h, i co-ordinate system, the linearized transformation matrix is diagonal on the multicritical line with two constant elements, leading to constant values for ν_p and ν_+.

Furthermore, we can show that $\partial i'/\partial i$ equals $\partial g'/\partial g$ at the isotropic percolation point by an argument that is valid for all lattices. This argument is based on a simple geometric interpretation for these derivatives in terms of the cutting bonds in a percolating cluster.[56,57] (A cutting bond is defined as one which, if cut, renders the configuration disconnected.) From this geometric equivalence, the equality we wish to show will become almost self-evident.

First consider $\partial g'/\partial g$, which at the isotropic fixed point is the same as $\partial p'/\partial p$. The quantity p' is the pair-connectedness between opposite edges of the cell, and the derivative with respect to p is the decrease in p' when a small fraction of resistors are removed. A decrease in p' occurs only if the removed bond

is a cutting bond. Therefore the derivative may be written as the number of cutting bonds times the fractional decrease in the bond concentration.[56] This simple result can be used to derive exact relations which help characterize cluster structure near the percolation threshold.

When diodes are introduced, we can generalize these ideas to express $\partial i'/\partial i$ in terms of directed cutting bonds. For example, consider $\partial i'/\partial i$, which by the chain rule equals $\partial p_+'/\partial p_+ + \partial p_+'/\partial p_- - \partial p_+'/\partial p$. Following the arguments of the previous paragraph, these terms are, respectively, the number of cutting bonds which are positively directed, negatively directed, and isotropic, in configurations which span a cell in only one direction, times the fractional decrease in the total bond concentration (Fig. 8). At the isotropic threshold, where $p_+^* = 0$, the third term is zero. Furthermore, configurations contributing to the first two terms must have only a single directed cutting bond with all the remaining bonds being resistors. Thus for $\partial i'/\partial i$, cutting bonds are counted separately according to whether they are positively or negatively directed. Since the set of configurations entering into $\partial g'/\partial g$ and $\partial i'/\partial i$ are identical, the sum of directed cutting bonds of either orientation is equal to the total number of isotropic cutting bonds.

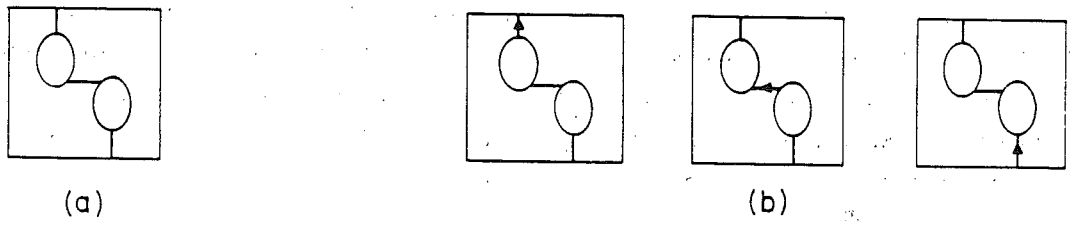

Fig. 7. Pictorial representation of the derivatives $\partial g'/\partial g$ and $\partial i'/\partial i$. In (a), an isotropic percolating configuration is shown which contains "blobs" of multiply-connected bonds and 3 cutting bonds. In (b), closely related directed configurations are shown which contain a single directed bond. Taking $\partial i'/\partial i$ counts the cutting bonds of either orientation which also equals 3. The desired result is obtained by averaging over all appropriate configurations of type (a) and (b).

Since ν_p is equal to 4/3,[58] the eigenvalue $\partial g'/\partial g$ is always relevant. Thus $\partial i'/\partial i$ is relevant as well, so that the isotropic fixed point is unstable in the i direction. At the other end of the multicritical line (random Manhattan), the eigenvalue in the i direction should also be relevant on physical grounds. Rescaling a set of random Manhattans should lead to a finite fraction of isotropically-percolating and non-percolating configurations. Thus random Manhattan is unstable in the i direction, leading to the phase diagram of Fig. 8.

The simple result given above for directed cutting bonds should have ramifications beyond deriving exponent relations. In isotropic percolation, the

transition can be described in terms of an anomaly in the number of cutting bonds.[56,59] Thus it may be that a similar description in terms of directed cutting bonds will be useful for describing the percolation transitions in the random resistor-diode network.

4. CONDUCTION IN RESISTOR-DIODE NETWORKS

Interesting directionality effects can arise experimentally in hopping conduction processes under sufficiently strong applied fields so that charge carriers cannot hop against the field.[29-33] Such a diode-like behavior has been modelled by directed percolation.[32,33] This strong-field regime is also characterized by a non-ohmic behavior in which the current depends exponentially on the field.[29-31] Other anomalous features, such as a negative resistance for a certain range of field strengths, have also been predicted.[32]

On the other hand, in order to study the conductivity of a directed system, it is most convenient to consider a simple classical network model with a well-defined conductivity. This may be accomplished by requiring a current-voltage relationship for each diode which is linear for the range of voltages at which the conductivity is measured. As a simple case of such a response, we consider networks consisting of diodes with an "ohmic" character. Under a forward-bias voltage, an ohmic diode behaves like an ideal resistor, while under a back-bias voltage, the diode is non-conducting. Experimentally, this may be achieved in an approximate way by a series combination of a real diode and resistor.

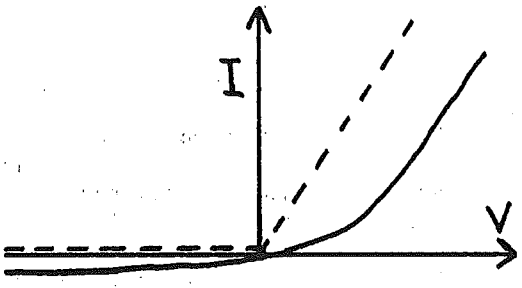

Fig. 9. Current-voltage relation of the series resistor-diode combination used in the analog experiment (full curve), and the ideal "ohmic" diode response used in the computer simulations (dashed). The finite voltage threshold for turning on the diode does not appear to play an important role in the analog experiments.

A network consisting of ohmic diodes represents a generalization of the random resistor network to include directionality effects. We will be primarily interested in the critical behavior in such networks. Although numerical estimates for one critical conductivity exponent have now been obtained, it is still an open question to relate the directed conductivity and directed diffusion[60] problems to strong-field hopping conductivity processes.

In the following sections, we shall primarily consider directed percolation for simplicity, i.e., a random network of oriented diodes and vacancies. Since the

system is anisotropic near the directed percolation threshold, the conductivity is now a tensor quantity. We shall consider this problem in a co-ordinate system where the conductivity is diagonal by applying the electric field along the diode polarization. We investigate only the largest eigenvalue of this tensor, the conductivity parallel to the polarization, which we shall refer to as the "directed" conductivity.

There are a variety of approaches available to treat the critical behavior of the directed conductivity, and we shall outline some of the primary results of each method. Before doing so, it is worth mentioning an additional interesting aspect of diode networks. That is the problem of predicting which diodes are forward and back biased in a particular network (the "state" of the system). The state is not known, *a priori*, and it must be determined self-consistently together with the voltages in the network. Thus far, there is no theoretical method available to solve this problem, and we shall discuss an approximate numerical approach for the directed percolation network in section 4.E. In this case, the problem is relatively simple because almost every diode is forward biased due to the orientation of the network. However, for a more general network, e.g., with randomly oriented diodes, the prediction of the correct state appears to be an interesting open problem that deserves further study.

A. Effective medium theory (EMT). The basic idea of this method[39,61] is to first replace a random network, in which each diode has forward conductivity g with probability p_+ and conductivity zero otherwise, by a homogeneous effective network in which each bond has the same (unknown) conductivity, g_m. The value of g_m is then calculated by a self-consistent procedure reminiscent of the effective-field approximation in magnetism. To accomplish this, one bond in the effective network is assigned the conductivity distribution of the actual random network. The value of g_m is determined by requiring that the voltage fluctuation across the special bond within the effective medium, when averaged over the correct conductivity distribution, is zero.

This approach has been applied to both partially and fully directed percolation.[35] In the former case, the calculation closely resembles the EMT for the random resistor network, except that only a $1 \times \infty$ strip of the entire lattice can carry the current which enters into the voltage fluctuation across the special bond (Fig. 10).

It is found that the directed conductivity depends linearly on p_+, and vanishes at a threshold of $p_{+_c} = 1/\sqrt{3} = 0.577\ldots$, in good agreement with the low-density series result[62] of $p_{+_c} = 0.555 \pm 0.002$. For fully directed percolation,

we must use a somewhat unconventional geometry in which the voltage fluctuation between two next-nearest-neighbor sites is evaluated (Fig. 10(b)). The orientational constraint for the diodes is also more restrictive, and only a 4-bond cluster in the lattice is relevant for the EMT. For one of the paths joining sites a and b, both bonds are assigned the correct conductivity distribution, while the two bonds on the other path both have conductivity g_m. In this case, the conductivity also varies linearly with p_+ and vanishes at $p_{+_c} = 1/\sqrt{2} = 0.707...$, compared to $p_{+_c} = 0.6447$ obtained numerically.[15,17]

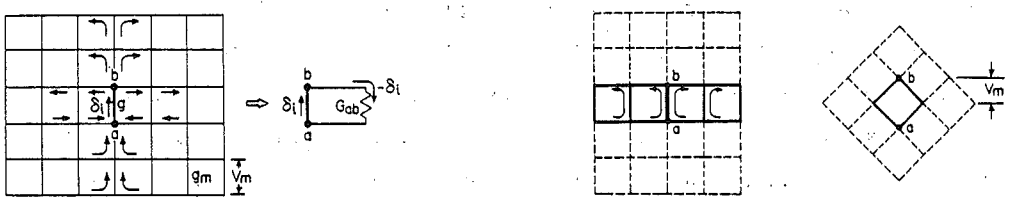

Fig. 10. Comparison of the effective media for the random resistor network, (a), with (b) fully, and (c) partially directed percolation. The special lattice bond across which the voltage fluctuation is evaluated is indicated by the thick line in (a) and (b), and the resulting current perturbation is indicated schematically by the arrows. The superimposed uniform current is not shown. In (a), an equivalent network used for the EMT calculation is shown to the right. In (b) and (c), the irrelevant portion of the lattice is shown dashed.

The results of EMT are somewhat unsatisfactory as a transition is predicted, but the linear dependence of the conductivity is built into the theory. It is not possible to detect any qualitative differences between the critical behavior for the random resistor network and for directed percolation.

An interesting final point is that EMT for fully directed percolation breaks down in certain geometries. This seems to be related to the general problem of understanding the direction dependence of the conductivity. For example, when the field is applied along the +x-direction, it is not possible to average the voltage fluctuation self-consistently. In the course of attempting this calculation, it is necessary to understand what happens when the conductivity of a single diode along the field direction is assigned a conductivity different than g_m. This is an interesting non-trivial problem in itself. All diodes transverse to the field have exactly zero voltage across them in the uniform network, and a small perturbation of one bond may have a rather strong effect on the network state. A numerical simulation indicates that if a single vertical diode is removed, two half-columns of

horizontal bonds become back-biased so that the lattice breaks into two (electrically) disconnected pieces. This peculiar phenomenon shows the important role of directionality constraints in diode networks.

B. Mean-field theory. To obtain the mean-field limit for the directed conductivity exponent, we apply the de Gennes-Skal-Shklovskii model,[63] which provides an idealized picture for the percolating cluster. For pure percolation, this infinite cluster can be represented by a regular array of nodes of mean separation ξ, joined by links whose conductances vanish as $(p - p_c)^{\zeta}$, when $p \dashrightarrow p_c$ from above (Fig. 11). The conductance of a d-dimensional network of linear dimension L can then be obtained by superposing the $(L/\xi)^{d-1}$ chains, each containing L/ξ links. This leads to a network conductance that varies as $L^{d-2}(p - p_c)^t$, with $t = \zeta + (d - 2)\nu$.

For directed percolation, the node lattice becomes anisotropic, with spacings of ξ and ξ respectively, parallel and perpendicular to the anisotropy axes of the system (Fig. 11). In a system of linear dimensions $L_{\|}$ and $L_{\perp}$, parallel and perpendicular to the field respectively, this modification leads to a network conductance that varies as $L_{\perp}^{d-1}L_{\|}^{-1}(p - p_c)^{t_+}$, with $t_+ = \zeta_+ + (d - 1)\nu_{\perp} - \nu_{\|}$. Employing the values $\zeta_+ = 1$,[64] and $\nu_{\|} = 1$, $\nu_{\perp} = 1/2$[6,16,23] valid for $d \geq d_c = 5$, we obtain $t_+ = 2$, compared to $t = 3$ for the mean-field limit of the random resistor network,[65-67] valid at six dimensions and above.[68] The exponent inequality $t_+ < t$ indicates that the directed conductivity should have a much sharper variation with bond concentration near the percolation threshold. This is intuitively plausible as the long tortuous paths that cause a very small conductivity in the random resistor network near the transition, cannot occur in directed percolation. This suggests that the inequality between these two conductivity exponents holds in all dimensions, a prediction which is confirmed numerically in two dimensions.

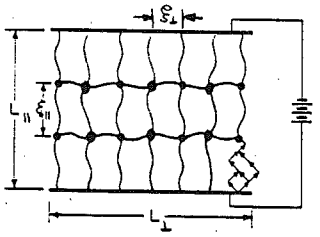

Fig. 11. Schematic node picture of a directed percolating network. Notice the non-colinearity of the node structure with the underlying lattice (lower-right).

The mean-field limit can also be obtained by considering directed percolation on a Cayley tree.[23] Furthermore, by this approach it is possible to treat a mixture of diodes with finite and infinite conductivities in the forward direction. This is the directed analog of a conducting-superconducting transition. As the fraction of

superconductor approaches a threshold from below, the directed conductivity diverges as $(p_+ - p_{+c})^{-s_+}$. For the Cayley tree, $s_+ = 1$, compared to the mean-field limit $s = 0$ for the analogous isotropic transition.[65,66]

C. Position-space renormalization group. A cell PSRG approach for directed conductivity can be formulated which closely parallels the treatment for the random resistor network.[69-73] Together with rescaling the bond occupation probabilities, the bond conductivities may be rescaled as well, leading to coupled recursion relations for these two quantities. One advantage of this treatment is that it can be extended in a straightforward way to treat both superconducting bond elements and also more complicated geometries such as the random resistor-diode network. The conductivity processes in this latter network are rather complicated, so we will primarily discuss results for the simpler case of the oriented resistor diode network (resistors, vacancies, and one species of ohmic diodes).

Before discussing the results, it should be mentioned that there are some problems in applying the PSRG to directed conductivity. We have already noted that the cell PSRG does not account for the two correlation lengths properly. However for purposes of obtaining phase diagrams and approximate exponents, a renormalization based on an isotropic rescaling should be adequate. In addition, for fully directed percolation, the directions of the lattice bonds are not co-linear with the axis along which the conductivity is measured. Presumably this problem could be overcome if the complete direction dependence of the conductivity were treated adequately.

The oriented resistor diode network has three phases, depending on whether the forward or reverse conductances, G_+ and G_- respectively, are zero or non-zero (Fig. 4(a)). In the diode phase, G_+ is non-zero, and it vanishes according to the exponent t_+ as the boundary with the non-conducting phase is approached. On the other hand, in the resistor phase, both G_+ and G_- are non-zero, and only G_- vanishes as the boundary with the diode phase is approached. The vanishing of G_- may be written as δp^{t_-}, where t_- is a "reverse" conductivity exponent, and δp is the distance from the diode phase boundary. Finally, at the isotropic percolation point, both G_+ and G_- vanish. Therefore we have a tricritical point[74] with two independent conductivity exponents. One is simply the isotropic conductivity exponent t which describes how G_+ and G_- simultaneously vanish as the transition is approached from the resistor phase. There is a second exponent, $\bar{t}$, which describes how G_+ only vanishes as the transition is approached from the diode phase.

From a b = 2 rescaling, we find $t_+ = 0.70$, $t_- = 0.89$, and $\bar{t} = 1.51$. In particular, notice that t_+ is considerably less than the estimate of $t \cong 1.3$ for

the random resistor network in two dimensions. This numerical result is consistent with the expectations stated in the earlier discussion on mean-field theory.

In addition, a similar PSRG calculation can be used to find the conductivity divergence of an ohmic diode-superconducting diode mixture. We find that the forward conductivity diverges with an exponent $s_+ = 1.90$ as the concentration of superconducting diodes approaches the transition from below. This value should be compared to the value $s = t \cong 1.30$ of the corresponding isotropic transition.

When diodes with a superconducting response are introduced, the PSRG requires that the parameter space be enlarged to describe the network self-consistently. Most generally, circuit elements should have different conductivities in the forward and backward directions. In terms of this description, the duality of the random resistor-diode network on the square lattice can be extended to directed conductivity as well. This has been stated most elegantly by Dhar *et al*[41] for a network consisting of n species of circuit elements, the i^{th} species occuring with probability p_i, and having conductivites σ_{if} and σ_{ib} in forward and backward directions respectively. In the dual network, the i^{th} species will have forward and backward conductivities $1/\sigma_{if}$ and $1/\sigma_{ib}$ respectively, and according to duality, the bulk resistances of the two systems are equal. This generalizes the duality statement for an isotropic network which relates the conductor-insulator to the conductor-superconductor transition, and gives the equality $t = s$ in two dimensions.

D. Analog Experiments. As a more direct approach to studying directed conductivity, we have constructed directed percolation networks, and measured the conductivity as the bonds (resistors and diodes in series) were cut at random.[37] For small networks, we obtained data that is fairly linear away from the transition, in accordance with the expectations of effective-medium theory. Near the transition, however, the data bends downward suggesting that $t_+ < 1$.

The results on a larger lattice of size 19 x 52 are much more interesting. At first sight, it does not appear possible to extract information about the directed conductivity exponent because of the large jumps[75] in the conductivity as a function of the number of bonds cut (Fig. 12). These jumps are strongly correlated with the size of the underlying directed backbone, however, and we can use this information to obtain an estimate for t_+. The source of these jumps are the topological constraints of directed pecolation, where the removal of a single bond in Fig. 11(b) disconnects a large portion of the network. In contrast, if the backbone in this figure consisted only of resistors, removing the arrowed bond would entail the removal of only four additional bonds to obtain the new backbone.

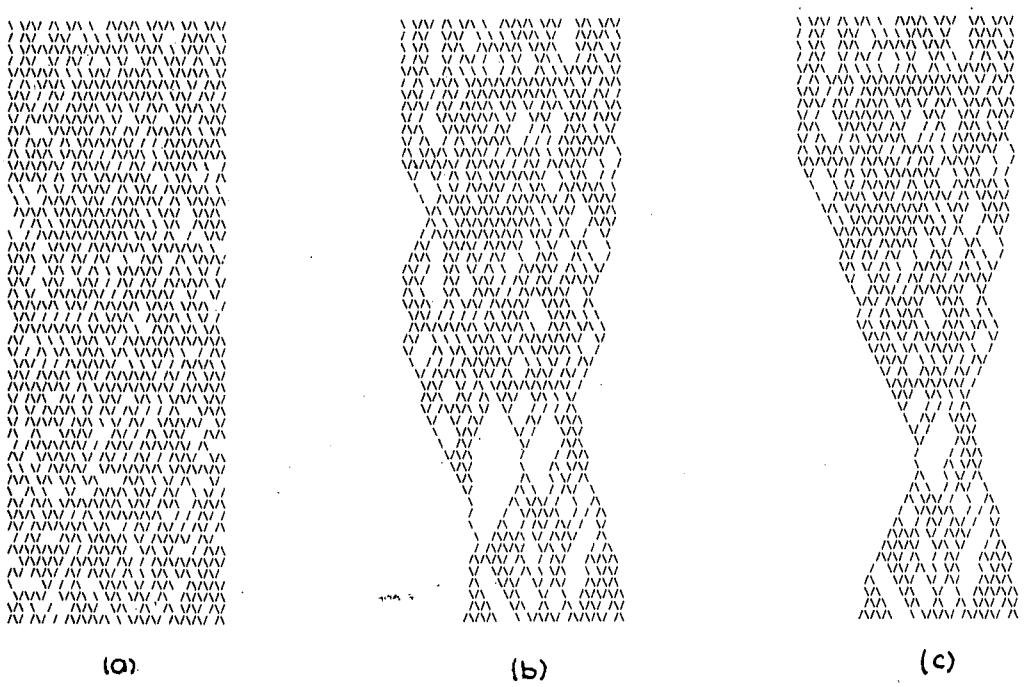

Fig. 12. (a) The 19 x 52 directed network analog experiment after 465 bonds are removed. (b) Shows the underlying directed backbone of 1128 bonds, and the arrow indicates the next bond that will be removed. (c) The directed backbone of 987 bonds after the arrowed bond in (b) is removed.

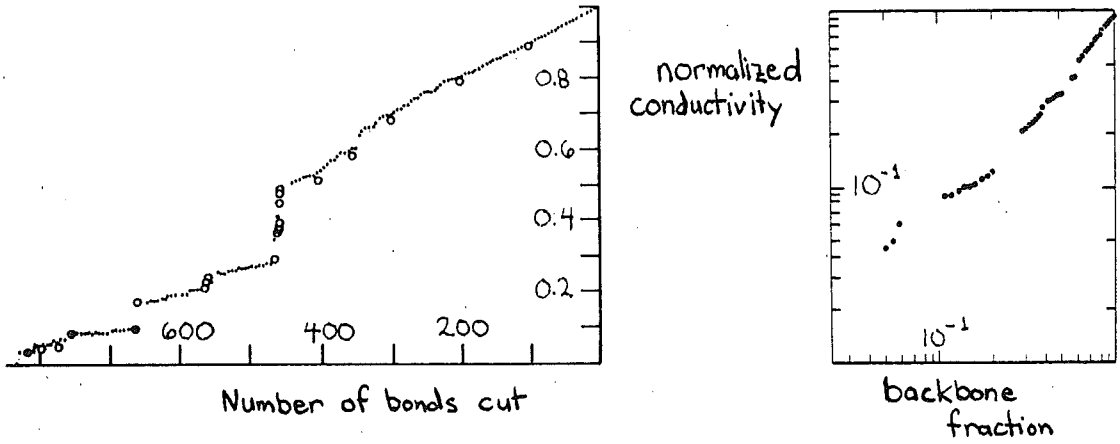

Fig. 13. (a) Plot of the conductivity versus the number of bonds cut from the network. The experimental data are shown by the dots, while the computer simulation results on the same experiment are shown by the open circles. (b) Plot of the conductivity in (a) as a function of the fraction of bonds in the backbone.

The strong correlation between the backbone size and the conductivity is illustrated in Fig. 13, where it appears that the two quantities are nearly proportional. On the other hand, the directed backbone exponent β' can be related[38] to the exponent β of the directed percolation probability, P_∞. For a bond to be in

the directed backbone, it must lie on a directed path which traverses the network from top to bottom, *and* on a directed path which traverses from bottom to top. Therefore, the backbone fraction varies as P_∞^2, or $\beta' = 2\beta$. Coupled with the estimate $\beta = 0.28$ from low-density series,[13] the data suggest that $t_+ \simeq 0.5 - 0.6$, in rough agreement with the PSRG result.

A final interesting point is that the conductivity jumps of Fig. 13(a) are typical. Repeated simulations of a 19 x 52 network shows that jumps of $\geq$ 100 bonds in the backbone size occur at least once in almost every simulation, and that the distribution of jumps sizes is quite erratic. These large fluctuations are somewhat puzzling since they seem to be contrary to the expectations of the Ginzburg criterion.[76] Below the upper critical dimension, d_c, this criterion gives the range of bond concentration within which fluctuations can be ignored. Since d_c is lower for directed percolation than for isotropic percolation, the former system is "closer" to mean-field behavior, and its critical region should be correspondingly smaller. Thus it is intriguing that the analog experiments show such large fluctuations.

E. Computer Simulations. This method provides an accurate tool for studying the directed conductivity, and also illustrates the interesting features associated with finding the correct "state" of the network as mentioned in the introduction to this section.

For directed percolation, the overall orientation of the network causes only a very small fraction of the diodes to be back-biased. Thus an approximate solution is to assume that all diodes in the directed backbone are forward-biased so that they can be treated as resistors.[38] However, other networks (e.g., with diodes of random orientation) may have a larger fraction of back-biased diodes so that finding the correct state becomes important. With this in mind, we describe a method already applied to fully directed percolation,[36] which may be applicable to non-oriented diode networks as well. It is a generalization of numerical relaxation to iteratively search for the correct state, as well as the voltages in the network, in a self-consistent way.

To illustrate the method, we briefly outline the basic idea of conventional relaxation for a resistor network.[39] Local current conservation is used to generate an improved estimate for a particular node voltage from the voltages at the neighboring nodes. Each node of the network is sequentially updated in this fashion until a voltage equilibrium is reached after many scans through the lattice. Because of the linear nature of resistor networks, this method necessarily converges, independent of the assumed initial voltages. The basic limitation of the

method is its slow convergence rate for large systems near the percolation threshold, although certain modifications, such as over-relaxation,[77] partially alleviates this problem.

For a diode network, an obvious extension of relaxation is to allow the state of the system to change at each iteration step. Back-biased diodes should turn off, while previously turned-off diodes should turn on again if the voltage across them becomes forward biased. Thus the state of the network should "float" during the calculation. This procedure leads to prohibitively slow convergence in many cases, and an apparent limit cycle behavior in some pathological cases. The reason for this oscillatory behavior stems from the potential for negative feedback between the states of certain "critical" diodes which are nearly balanced (Fig. 14). This coupling can extend over large distances so that information about the state of one critical diode propagates to another critical diode only after many iteration steps have transpired. This time delay, together with the potential for negative feedback appears to be the source of the non-convergence. This feature is an artifact of the method as a unique state of the network exists except in certain exceptional cases.[78]

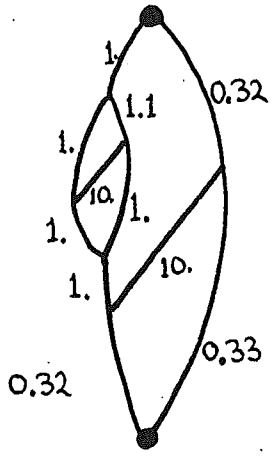

Fig. 14. Two nested Wheatstone bridges where damped oscillations in the state of the network occur if it is relaxed according to the procedure described above. The numbers refer to the conductivities of the various branches of the graph, which are chosen to make both bridges nearly balanced. Much longer-lived oscillations occur when three or more bridges are nested in this fashion.

To overcome this problem, we use a more "gradual" relaxation method in which the conductivity of each back-biased diode is reduced by a certain factor at every iteration step. After many iterations, such a diode will have a vanishingly small conductivity. It has been relaxed from the network gradually. On the other hand, if an apparently back-biased diode (with an already reduced conductivity) becomes forward-biased, its conductivity is increased by a certain factor. The conductivity of such a diode continues to be increased gradually, as long as it remains forward-biased, until the conductivity is restored to its initial value. This procedure effectively smooths out the break in the I-V response of each diode at V = 0. With this method, the oscillations in the states of the pathological nested

Wheatstone bridges are greatly reduced, and much more rapid convergence to the correct conductivity is obtained.

The simulation of the conductivity in a random directed percolation network is greatly simplified by the recursive nature of the problem. In particular, the configuration of the N^{th} row of the lattice depends only on the configuration of the previous row. As a result, much larger lattices (effectively up to 3×10^6 in linear dimension) can be simulated in directed percolation compared to isotropic percolation with the same amount of computer time.[41] For the conductivity problem, it is also relatively simple to reduce all series and non-nested parallel configurations to single bonds before starting numerical relaxation. Such a reduction saves considerable computing time in this final step of the calculation.

To estimate the conductivity exponent, we obtain Monte Carlo data for the forward conductance $G_+ = \sigma_+ L_\perp / L_\parallel$ at the critical point and employ an anisotropic finite-size scaling method.[36] Here σ_+ is the forward conductivity, and $L_\parallel$ and $L_\perp$ are the linear dimensions of the system parallel and perpendicular to the anisotropy respectively. Due to this anisotropy, these dimensions must scale up according to $L_\parallel^{1/\nu_\parallel} \simeq L_\perp^{1/\nu_\perp}$. Very approximately, if the width of the system doubles, the length must triple. Accordingly, we study a sequence of lattices beginning at a small size such as L x L = 1 x 2, 1 x 1, or 2 x 1, and scaling up to 32 x 478, 48 x 453, or 48 x 152 respectively. Only under these conditions will the conductivity scale as $L_\parallel^{t_+/\nu_\parallel}$. The Monte Carlo data is reasonably linear on a double logarithmic scale (Fig. 15), although there is a systematic curvature to a smaller asymptotic slope for large L . Based on these data, we estimate a directed conductivity exponent of $t_+ = 0.60 \pm 0.10$. The agreement between this estimate and the results of the PSRG and the analog experiment is surprisingly good in view of the approximations in the latter methods.

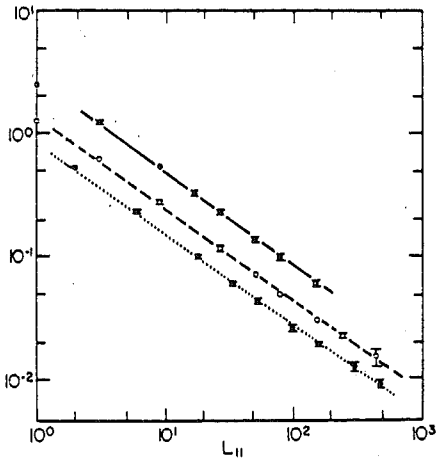

Fig. 15. Double logarithmic plot of the conductance versus $L_\parallel$ for three sequences of lattice sizes starting at 1 x 2, 1 x 1, and 2 x 1 (bottom to top respectively).

5. SUMMARY AND DISCUSSION

In summary, we have studied some of the interesting phenomena that occur when directed bonds of arbitrary orientation are incorporated into the percolation problem. A model with resistors, vacancies, and arbitrarily oriented diodes, the random resistor-diode network, was treated in detail. It was found that percolation transitions may be driven by varying either the bond concentration or the bond orientation, and there is novel multicritical network behavior when these transitions occur simultaneously. The phase diagram for the system was mapped out and its critical behavior elucidated. On the square lattice, the network possesses a number of symmetry and invariance properties from which certain critical points can be located exactly, and exact exponent relations can be derived. While some progress has been made in understanding the model, there is much more yet to be discovered. In particular, we have discussed only the correlation length exponents of the system, which are related to the "thermal" scaling powers. The order parameter and the mean-size functions and their exponents have not been considered thus far. It would be worthwhile to study these quantities in order to better characterize the nature and the universality classes of the various percolation transitions. Furthermore, studies of lattices with randomly oriented diodes would be useful for visualizing the geometry of the network, and for testing some of the exponent predictions made in this paper. Given the wealth of interesting information that has been obtained from considering the special case of directed percolation, further studies of the more general random resistor-diode network should prove to be quite fruitful.

I would also like to mention some very recent progress on the problem of directed lattice animals.[50,79-86] The geometrical properties of the model closely resemble those of directed percolation, except only a single directed cluster is allowed to grow, with each element of the lattice animal weighted by the same fugacity. In this sense, the directed animal problem is closest to the branching Markov processes mentioned in the introduction. Although this topic is somewhat outside the scope of this review, the problem deserves mention because of interesting results obtained within the past few months. This model appears to be exactly soluble in two dimensions,[81,82] and furthermore, it appears that the tantalizing connection of the isotropic lattice animal problem in (d+1) dimensions with the Yang-Lee edge singularity problem in (d-1) dimensions[83] can be modified and extended to directed animals in d dimensions.[80,85,86] Recent work also indicates that directed lattice animals can be mapped into an anisotropic hard-square lattice gas model with a complex chemical potential,[84] a feature suggestive of the Yang-Lee

edge singularity problem. This very interesting problem deserves further attention.

We have also discussed the critical behavior of the conductivity on a directed percolating network. In general, it appears that the directed conductivity exponent t_+ is smaller than the conductivity exponent t for a random resistor network in the same spatial dimension. In the mean-field limit of the directed problem, valid at five dimensions and above, t_+ was found to equal 2. In contrast, the corresponding limit for the isotropic problem, valid at six dimensions and above, is $t = 3$. In two dimensions, a variety of approximation methods indicate that $t_+ \cong 0.6$ compared to the recently quoted value of $t \cong 1.3$. The difference between these two exponents can be understood intuitively by comparing the structure of the current-carrying paths in the two problems. For directed percolation, long tortuous paths cannot occur, so that the conductivity should have a stronger concentration dependence than in isotropic percolation. Interestingly, the value of t_+ in two dimensions is quite close to the estimated value for the directed backbone exponent β', and these two exponents are equal in the mean-field limit. It would be worthwhile to extend calculations of directed conductivity to three- and four-dimensional systems in order to test whether there is a general relation between the two exponents.

In studying the conductivity of resistor-diode networks, there is a wealth of interesting problems that have not yet been examined. We have considered primarily the case of systems with a preferred direction so that the conductivity is anisotropic, and the critical behavior of one component of the conductivity tensor has been studied. However, such a network should also exhibit interesting direction dependent conductivity properties. Furthermore, more complicated geometries may be considered, such as networks containing diodes of both positive and negative orientations. Finally, we have assumed an idealized current-voltage response for each circuit element. However, the renormalization group requires the introduction of more general circuit elements with different conductivities of arbitrary values in the positive and negative directions. It may also be worthwhile to consider circuit elements whose current-voltage response does not pass through the origin. These topics of direction dependence, different network geometries, and more general I-V characteristics appear to be interesting new areas for future study.

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