Rapid Communications

Nonuniversal opinion dynamics driven by opposing external influences

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(Received 31 July 2019; published 11 November 2019)

We study the opinion dynamics of a generalized voter model in which N voters are additionally influenced by two opposing news sources whose effect is to promote political polarization. As the influence of these news sources is increased, the mean time to reach consensus scales nonuniversally as N^{α}. The parameter α quantifies the influence of the news sources and increases without bound as the news sources become increasingly influential. The time to reach a politically polarized state, in which roughly equal fractions of the populations are in each opinion state, is generally short, and the steady-state opinion distribution exhibits a transition from near consensus to a politically polarized state as a function of α.

DOI: 10.1103/PhysRevE.100.050301

A disheartening feature of current social discourse is its high degree of political polarization, particularly in the US and Europe (see, e.g., [1–7]). In recent decades, this polarization has increased to the point where, for example, parents affiliated with one US political party are loathe to have their children wed someone affiliated with the other major party [8]. We have no illusions of being able to explain the societal forces that have led to such polarization. To understand this phenomenon would require characterizing the decision-making processes of each individual, the pattern of social connections, and the role of external influences. However, such an engineering approach has little chance of revealing the mechanisms that underlie potential new dynamics.

To help unravel the governing features of social polarization, we introduce a simple two-state voterlike [9–13] opinion formation model that is augmented by the presence of two competing news sources (Fig. 1). The latter accounts for the tendencies of consumers to consult sources that align with their political persuasion, and that some news sources promulgate fixed political viewpoints [14–16] ("zealots" in a voter model framework [17–20]). While many variants of the voter model—inspired by real decision making—have been investigated (see, e.g., [21–32]), the role of external influences, such as news media, has only recently been considered [33–35].

In this work, we show that the influence of news sources significantly hinders the approach to consensus, but in an unexpected no-universal way. We also introduce the notion of the *polarization time*, namely, the time required to reach a politically polarized state, in which there are equal fractions of the population in each opinion state, from any initial condition. This polarization time offers a useful way to characterize the opinion dynamics of social systems. A crucial ingredient in our modeling is that we drastically idealize the underlying social networks. We anticipate that simulations of our model on more realistic networks would reveal phenomenology similar to what we find analytically. However, such an approach would provide little understanding of the mechanisms that lead to the slow consensus or the nonuniversal properties that we find.

In our model, each individual has two possible opinion states, denoted as $+$ and $-$. Individual opinions are updated according to voter model dynamics: a randomly selected voter adopts the opinion of a randomly selected neighbor. This update step is repeated *ad infinitum*. We account for the different propensities of news media and neighboring voters to influence a given voter as follows: for a voter linked to one news source and k other voters, the news source is picked with probability p/R and a neighboring voter is picked with probability $(1-p)k/R$, where $R = p + k(1-p)$ is the total rate of picking any neighbor. The parameter p thus quantifies the relative influence of a news source and a neighboring voter. [If a voter is connected to both news sources, then $R = 2p + k(1-p)$.] Once an interaction partner is selected, the voter adopts the opinion of this partner.

For simplicity and analytical tractability, we treat two types of social networks (Fig. 1): (a) A complete graph of N voters, with L_+ (L_-) connections between voters and the $+$ ($-$) news source; the news sources connect either to random voters or to disjoint voters. (b) More realistically, a two-clique graph with N voters in each clique, with L_+ connections between the $+$ news source and random voters on clique C_+ (and correspondingly for C_-), and $L_0 = N^{\beta}$ links between nodes in different cliques. In both cases, $0 < L_{\pm} \leqslant N$, with corresponding link densities $\ell_{\pm} = L_{\pm}/N$.

We focus on four characteristics of the collective opinion state: (i) the consensus time $T_{\rm con}$, defined as the average time to reach either $+$ or $-$ unanimity; (ii) the polarization time $T_{\rm pol}$, defined as the average time to go from a state with unequal fractions of $+$ and $-$ voters to the politically polarized state with equal fractions of $+$ and $-$ voters; (iii) the exit probability, defined as the probability to eventually reach $+$ consensus when the initial density of $+$ voters equals x, and (iv) the steady-state opinion distribution.

Our main results are as follows: (i) $T_{\rm con}$ typically grows algebraically with N, with a nonuniversal exponent that can be arbitrarily large. Based on an annealed-link approximation (see below), we find, for the complete graph,

$$T_{\rm con} \sim \begin{cases} N, & 0 \leqslant \alpha < 1 \\ N \ln N, & \alpha = 1 \\ N^{\alpha}, & \alpha > 1, \end{cases} \tag{1a}$$

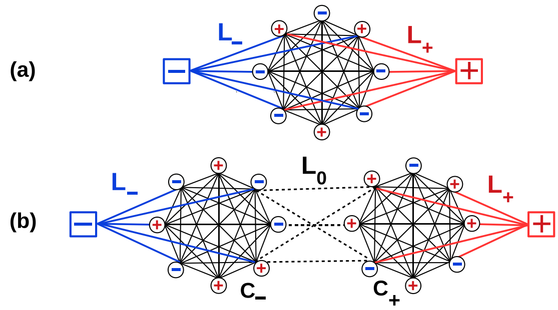

FIG. 1. Two opposing news sources (squares) that influence voters (circles) on (a) a complete graph, or (b) a two-clique graph. The news sources have $L_\pm$ links to individuals. For the two-clique graph, there are $L_0 = N^\beta$ interclique links.

where $\alpha = \min(\alpha_+, \alpha_-)$ and $\alpha_\pm = p\ell_\pm/(1-p)$. For voters on the two-clique graph, in which the news sources have equal link densities ($\ell_+ = \ell_- \equiv \ell$) and the cliques are sparsely interconnected ($\beta < 1$),

$$T_{\rm con} \sim \begin{cases} N^{2-\beta}, & 0 \leqslant \alpha < 1 \\ N^{2-\beta}\ln N, & \alpha = 1 \\ N^{\alpha+1-\beta}, & \alpha > 1. \end{cases} \tag{1b}$$

For $p \to 1$, i.e., influential news sources, the exponent of the consensus time becomes arbitrarily large. That is, competing and well-connected news sources hinder the approach to consensus. Our results for $T_{\rm con}$ for $p \to 0$ on the two-clique graph are consistent with a previous study of the voter model on this graph [36].

(ii) When the two news sources are equally connected to the population, the polarization time $T_{\rm pol}$ scales as

$$T_{\rm pol} \sim \frac{N}{\alpha}. \tag{2}$$

Hence, political polarization occurs quickly (i.e., linearly with N) when voters are well connected to competing news sources. (iii) The exit probability has an antisigmoidal shape (Fig. 2) because the competing news sources tend to drive the population to a politically polarized state. (iv) For the complete and the two-clique graphs, the opinion distribution undergoes a transition from a homogeneous to a polarized state as the influences of news sources become stronger.

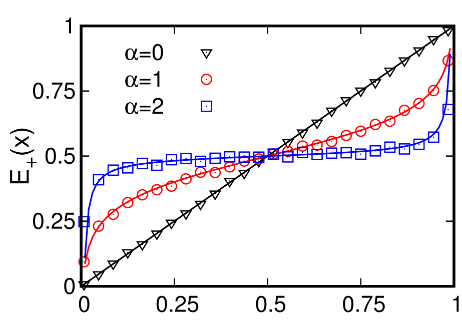

FIG. 2. Exit probability versus initial fraction of + voters for the complete graph of 128 voters. The solid curves represent Eq. (12) and symbols give simulation results from 10^4 realizations. For $\alpha = 1$, we choose $\ell = 1$ and $p = 1/2$, while for $\alpha = 2$ we choose $\ell = 1$ and $p = 2/3$.

We now outline the calculations that underlie our results; a closely related approach is given in Ref. [37]. Suppose that we know $r_\pm(x)$, the rates for x, the fraction of voters with + opinion, to change by $\pm\frac{1}{N} \equiv \pm\delta x$. Let $P(x,t)\delta x$ be the probability that the fraction of + voters lies between x and $x + \delta x$. The Fokker-Planck equation for P is

$$\frac{\partial P}{\partial t} = \mathcal{L}P, \qquad \mathcal{L} = -\frac{\partial}{\partial x}V(x) + \frac{\partial^2}{\partial x^2}D(x), \tag{3}$$

with drift velocity $V(x) = [r_+(x) - r_-(x)]\delta x$ and diffusion coefficient $D(x) = [r_+(x) + r_-(x)]\delta x^2/2$. We can view the instantaneous opinion x as undergoing biased diffusion in the interval [0,1] in the presence of the effective potential

$$\phi(x) = -\int^x \frac{V(x')}{D(x')}dx'. \tag{4}$$

The exit probability $E_+(x)$ satisfies the backward equation $\mathcal{L}^\dagger E_+(x) = 0$ [38–40], where the adjoint operator is

$$\mathcal{L}^\dagger \equiv V(x)\frac{\partial}{\partial x} + D(x)\frac{\partial^2}{\partial x^2}, \tag{5}$$

subject to the boundary conditions $E_+(0) = 0$, $E_+(1) = 1$. The formal solution for $E_+(x)$ is

$$E_+(x) = \frac{\int_0^x \exp[\phi(x')]dx'}{\int_0^1 \exp[\phi(x')]dx'}. \tag{6}$$

Similarly, the consensus and polarization times satisfy the backward equation $\mathcal{L}^\dagger T(x) = -1$ [38–40]. The boundary conditions for $T_{\rm con}$ are $T(0) = T(1) = 0$, while the boundary conditions for $T_{\rm pol}$ are $\frac{\partial T}{\partial x}|_{x=0} = 0$ and $T(\frac{1}{2}) = 0$. The formal solutions are [41]

$$\begin{aligned} T_{\rm con}(x) &= E_+(x)I(x,1) - [1 - E_+(x)]I(0,x), \\ T_{\rm pol}(x) &= I(x, 1/2), \end{aligned} \tag{7}$$

where $I(a,b) = \int_a^b dx' \int_0^{x'} dx'' \exp[\phi(x') - \phi(x'')]/D(x'')$.

We now apply an annealed-link approximation to this formalism to determine $E_+(x)$, $T_{\rm con}$, and $T_{\rm pol}$ for voters on the complete and on the two-clique graphs (Fig. 1). In this approximation, we replace the true transition rates for each voter on a given fixed-link network realization by the average transition rate, in which a link is present with probability proportional to its density.

Complete graph. By straightforward enumeration of all relevant events, the transition rates $r_\pm(x)$ for voters on the complete graph are [41]

$$\begin{aligned} r_+(x) &= \tfrac{1}{2}NAx(1-x) + B_+(1-x), \\ r_-(x) &= \tfrac{1}{2}NAx(1-x) + B_-x. \end{aligned} \tag{8}$$

The first term in $r_\pm$ accounts for a voter that adopts the opinion of a neighboring voter and the second term accounts for adopting the opinion of the news source. The coefficients

A and $B\pm$ are

$$A=\frac{(1-\ell_+)(1-\ell_-)}{1-(1/N)}+\frac{(1-p)(\ell_++\ell_--2\ell_+\ell_-)}{(1-p)+(2p-1)/N}+\frac{(1-p)\ell_+\ell_-}{(1-p)+(3p-1)/N}, \tag{9a}$$

$$B_\pm=\frac{p\ell_\pm}{2}\left[\frac{1-\ell_\mp}{(1-p)+(2p-1)/N}+\frac{\ell_\mp}{(1-p)+(3p-1)/N}\right]. \tag{9b}$$

Using (8) and (9) in the definitions of $V(x)$ and $D(x)$, their ratio is

$$\frac{V(x)}{D(x)}=\frac{2\left[B_+(1-x)-B_-x\right]}{Ax(1-x)+(1/N)\left[B_+(1-x)+B_-x\right]}. \tag{10}$$

Importantly, V/D is of order 1, except when x is of order $\frac{1}{N}$ from the boundaries at 0 and 1. Within these boundary layers, the second term in the denominator of V/D ensures its finiteness even when $x=0,1$. Considerable simplification arises by excluding these thin boundary layers and consequently dropping this second term. This approximation has a vanishingly small effect on the consensus time for large N. We find the positions of the resulting slightly smaller interval $[a_-,1-a_+]$ by equating the two terms of the denominator of V/D. This gives $a_\pm=B_\mp/AN$. In this truncated interval, we may drop the term of order $\frac{1}{N}$ in (10) and thereby find that the effective potential (4) has the logarithmic form

$$\phi(x)=-\ln[x^{\alpha_+}(1-x)^{\alpha_-}], \tag{11}$$

where $\alpha_\pm=2B_\pm/A$. We can also explicitly evaluate the integrals in Eqs. (6) and (7) for specific values of $\alpha_\pm$. For simplicity, we specialize to the symmetric case of equally connected news sources ($\alpha_+=\alpha_-=\alpha$) so that $a_+=a_-\equiv a=\alpha/(2N)$. Performing the integral in Eq. (4) when the $\frac{1}{N}$ term in V/D in Eq. (10), is dropped, the exit probabilities for $\alpha=1$ and $\alpha=2$ are (Fig. 2)

$$E_+(x)=\frac{1}{2}\left[1-\frac{G_\alpha(x)}{G_\alpha(a)}\right], \tag{12}$$

where

$$G_1(x)=\ln(x^{-1}-1),$$
$$G_2(x)=x^{-1}-(1-x)^{-1}+\ln(x^{-1}-1)^2.$$

The antisigmoidal shape of $E_+(x)$ arises because the effective potential (11) tends to drive the population to the politically polarized state of $x=\frac{1}{2}$.

Using the same approximation in the first of Eqs. (7), the consensus time is

$$T_{\rm con}(x)=N[H_\alpha(a)-H_\alpha(x)], \tag{13}$$

where, for simple rational values of α, H_α is

$$H_{1/2}(x)=-4\sin^{-1}\sqrt{x}\,\sin^{-1}\sqrt{1-x},$$
$$H_1(x)=-\ln[x(1-x)],$$
$$H_{3/2}(x)=(x-\tfrac{1}{2})[\sin^{-1}\sqrt{x}-\sin^{-1}\sqrt{1-x}]/\sqrt{x(1-x)},$$
$$H_2(x)=\tfrac{1}{6}\{x^{-1}(1-x)^{-1}-2\ln[x(1-x)]\},$$

which lead to the forms for $T_{\rm con}$ given in Eq. (1a).

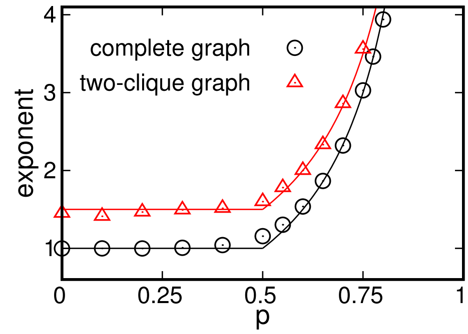

FIG. 3. Consensus time exponents versus p on the complete graph for news-source link density $\ell=1$ and for the two-clique graph, with interclique link density exponent $\beta=\frac{1}{2}$. Symbols represent simulations, while the curves represent the annealed-link approximation.

We can understand the N dependence of $T_{\rm con}$ for arbitrary α in terms of the logarithmic effective potential (11). According to Kramers' theory [42], the time to reach the boundaries at a and at $1-a$ are proportional to $\exp[\phi(a)]$ and to $\exp[\phi(1-a)]$, respectively. Because the potential scales logarithmically in N as $x\to a$ or $x\to 1-a$, there is an algebraic, rather than an exponential, dependence of $T_{\rm con}$ on N. This behavior contrasts with various nonconserved voter models [43,44], where the effective potential is linear in N, leading to a consensus time that grows exponentially with N. For $\alpha<1$, the effect of the logarithmic potential is subdominant with respect to fluctuations [45] and the latter drive the system to consensus, leading to $T_{\rm con}\sim N$. These predictions agree with our simulation results (Fig. 3). When $\ell_+\neq\ell_-$, the lowest barrier height in the potential determines the exponent; therefore $\alpha=\min(\alpha_+,\alpha_-)$ as in Eq. (1a). We also numerically verified that there is negligible difference in the consensus time when connections between the two news sources and the population are random or disjoint, with the same density of links.

To determine $T_{\rm pol}$ in a simple way, consider the extreme case where each news source has a single link to the complete graph. This weak connectivity leads to the longest possible polarization time. Suppose that the system starts in the $-$ consensus state. At some point, an "informed" voter, one that is linked to the $+$ news source, changes its opinion from $-$ to $+$. When this happens, the informed voter now disagrees with all its $N-1$ neighbors, but agrees with the news source. Because the former are much more numerous they dominate the subsequent opinion changes of the informed voter.

The state space of this reduced system may be schematically represented as in Fig. 4. Here $|0\rangle$ denotes the $-$ consensus state, $|1\rangle$ denotes the excited state, where only the informed voter has changed opinion, and $|P\rangle$ denotes the

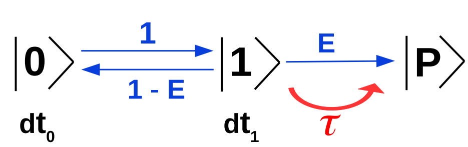

FIG. 4. State space of the reduced system.

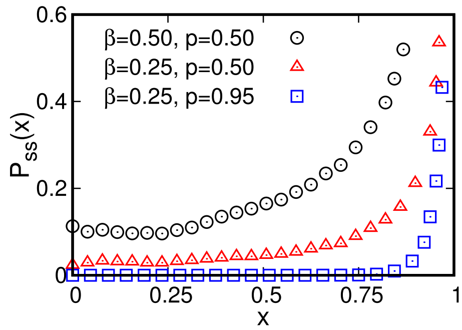

FIG. 5. Distribution of fraction x of + opinion voters on clique C_+ of 128 voters on the two-clique graph, with $\ell = 1$. On C_- the opinion distribution is a mirror image about $x = \frac{1}{2}$.

polarized state, in which the fraction of + and − voters is equal. Additionally, E is the exit probability to reach $|P\rangle$, which equals $\frac{2}{N}$ [41]. The polarization time satisfies the backward equations

$$T_0 = dt_0 + T_1, \qquad T_1 = (1-E)(dt_1 + T_0) + E\tau. \tag{14}$$

Here $T_0 \equiv T_{\rm pol}$ and T_1 are the times to reach the polarized state starting from $|0\rangle$ and $|1\rangle$, respectively, $dt_0 = [r_+(0)+r_-(0)]^{-1} \sim \frac{2}{\alpha}$ is the time to leave $|0\rangle$, $dt_1 \approx 1$ is the time to leave $|1\rangle$, and $\tau = 2N(1-\ln 2)$ is the conditional time to reach $|P\rangle$ from $|1\rangle$ by voter model dynamics [41]. Solving these equations gives Eq. (2), when subdominant terms are dropped. For well-connected news sources, $T_{\rm pol}$ is less than the consensus time because for $T_{\rm pol}$ the system state is driven towards the minimum of the effective potential, while to reach consensus the system has to surmount a potential barrier.

We obtain the steady-state opinion distribution, $P_{\rm ss}(x) \equiv P(x, t \to \infty)$, by setting $\frac{\partial P}{\partial t} = 0$ in Eq. (3). We need to apply reflecting boundary conditions because for all $\alpha > 0$, the endpoints are not fixed points of the stochastic dynamics. Imposing normalization, we find

$$P_{\rm ss}(x) = \frac{x^{\alpha_+-1}(1-x)^{\alpha_--1}}{B[1-a_+;\alpha_+,\alpha_-] - B[a_-;\alpha_+,\alpha_-]}, \tag{15}$$

where $B(x;y,z)$ is the incomplete beta function [46]. For $\ell_+ = \ell_- = \ell$, $P_{\rm ss}(x) \propto [x(1-x)]^{\alpha-1}$. This distribution undergoes a transition from bimodality (i.e., near consensus states) to unimodality (i.e., politically polarized states) as α increases through 1.

Two-clique graph. We may adapt the above argument for the polarization time on the complete graph to obtain both $T_{\rm con}$ and $T_{\rm pol}$ on sparsely interconnected two-clique graphs ($\beta \to 0$). Here the polarized state corresponds to the conventional picture of equal fractions of voters in the + and − states, with + voters in one clique and − voters in the other clique. Let x, y be the fraction of + voters on cliques C_+ and C_-, respectively (Fig. 1), and denote the state of the system by (x, y). It is convenient to take the initial condition as the maximally polarized (MP) state (1,0). The population tends to remain close to the MP state because news sources tend to drive opinions to this state, and the transition time out of this state, $dt_0 \sim N^{1-\beta}$, is large for $\beta \to 0$ (Fig. 5). This figure shows the increasing opinion polarization as the number of interclique links is reduced or the interactions with news sources become stronger.

For small β, intraclique links are dominant compared to intraclique links. For the clique with nonunanimity, the opinion dynamics thus is controlled by intraclique links. The dynamics of the two-clique graph therefore reduces to that of two *isolated* cliques that are additionally influenced by news sources [41]. For an isolated clique connected to a single news source, we obtain the exit probability by setting $\ell_- = \ell$, $\ell_+ = 0$ in the expression for V/D in Eq. (10). Using this in Eqs. (4) and (6) gives

$$E_+(y) = \begin{cases} \frac{(\alpha+2N)^{1-\alpha} - [\alpha+2N(1-y)]^{1-\alpha}}{(\alpha+2N)^{1-\alpha} - \alpha^{1-\alpha}}, & \alpha \neq 1 \\ \frac{\ln(2N+1) - \ln[2N(1-y)+1]}{\ln(2N+1)}, & \alpha = 1. \end{cases} \tag{16}$$

We can now compute $T_{\rm con}$ by using Eq. (14). In the present case, the MP state, the MP state with one opinion change, and + consensus correspond to $|0\rangle$, $|1\rangle$, and $|P\rangle$ in Fig. 4, respectively. The quantity $E_+(\frac{1}{N})$ in (16) corresponds to E in Fig. 4. Additionally, the terms that involve dt_1 and τ give subdominant contributions [41], and we thereby obtain Eq. (1b). The same type of reasoning also gives $T_{\rm pol}$ in Eq. (2).

To summarize, the presence of two opposing news sources promotes political polarization in the voter model. The news sources give rise to an effective logarithmic potential, which leads to an anomalously long consensus time and a short time to reach a politically polarized state. Our modeling has the advantages that it is analytically tractable and has predictive power to elucidate the anomalous scaling properties of these two timescales. The opinion dynamics on the more realistic two-clique graph mirror our analytical results for the complete graph system.

Acknowledgments. We thank Mirta Galesic for inspiring discussions and gratefully acknowledge financial support from NSF Grant No. DMR-1608211.

[1] L. A. Adamic and N. Glance, in *LinkKDD '05, Proceedings of the Third International Workshop on Link Discovery* (Association for Computing Machinery, 2005), pp. 36–43.

[2] D. Baldassarri and A. Gelman, Am. J. Soc. **114**, 408 (2008).

[3] M. P. Fiorina and S. J. Abrams, Annu. Rev. Pol. Sci. **16**, 101 (2013).

[4] N. J. Stroud, J. Commun. **60**, 556 (2010).

[5] M. Prior, Annu. Rev. Pol. Sci. **11**, 563 (2008).

[6] S. Iyengar and S. J. Westwood, Am. J. Pol. Sci. **59**, 690 (2014).

[7] See https://www.pewresearch.org/topics/political-polarization.

[8] S. Iyengar, T. Konitzer, and K. Tedin, J. Politics **80**, 1326 (2018).

[9] P. Clifford and A. Sudbury, Biometrika **60**, 581 (1973).
[10] R. A. Holley and T. M. Liggett, Ann. Probab. **3**, 643 (1975).
[11] J. T. Cox, Ann. Probab. **17**, 1333 (1989).
[12] T. M. Liggett, *Interacting Particle Systems* (Springer, Berlin, 1985).
[13] S. Redner, C. R. Phys. **20**, 275 (2019).
[14] S. Iyengar and K. S. Hahn, J. Commun. **59**, 19 (2009).
[15] M. S. Levendusky, Am. J. Pol. Sci. **57**, 611 (2013).
[16] G. J. Martin and A. Yurukoglu, Amer. Econ. Rev. **107**, 2565 (2017).
[17] M. Mobilia, Phys. Rev. Lett. **91**, 028701 (2003).
[18] M. Mobilia and I. T. Georgiev, Phys. Rev. E **71**, 046102 (2005).
[19] M. Mobilia, A. Petersen, and S. Redner, J. Stat. Mech. (2007) P08029.
[20] S. Galam and F. Jacobs, Physica A **381**, 366 (2007).
[21] C. Castellano, D. Vilone, and A. Vespignani, Europhys. Lett. **63**, 153 (2003).
[22] K. Suchecki, V. M. Eguíluz, and M. San Miguel, Phys. Rev. E **72**, 036132 (2005).
[23] V. Sood and S. Redner, Phys. Rev. Lett. **94**, 178701 (2005).
[24] V. Sood, T. Antal, and S. Redner, Phys. Rev. E **77**, 041121 (2008).
[25] C. Nardini, B. Kozma, and A. Barrat, Phys. Rev. Lett. **100**, 158701 (2008).
[26] C. Castellano, M. A. Muñoz, and R. Pastor-Satorras, Phys. Rev. E **80**, 041129 (2009).
[27] N. Masuda, N. Gibert, and S. Redner, Phys. Rev. E **82**, 010103(R) (2010).
[28] J. Xie, S. Sreenivasan, G. Korniss, W. Zhang, C. Lim, and B. K. Szymanski, Phys. Rev. E **84**, 011130 (2011).
[29] N. Masuda and S. Redner, J. Stat. Mech. (2011) L02002.
[30] D. Volovik and S. Redner, J. Stat. Mech. (2012) P04003.
[31] T. Rogers and T. Gross, Phys. Rev. E **88**, 030102(R) (2013).
[32] J. Fernández-Gracia, K. Suchecki, J. J. Ramasco, M. San Miguel, and V. M. Eguíluz, Phys. Rev. Lett. **112**, 158701 (2014).
[33] B. Fotouhi and M. Rabat, IEEE J. Sel. Top. Signal Process. **7**, 347 (2013).
[34] N. Khalil, M. San Miguel, and R. Toral, Phys. Rev. E **97**, 012310 (2018).
[35] V. V. Vasconcelos, S. A. Levin, and F. L. Pinheiro, J. R. Soc., Interface **16**, 20190196 (2019).
[36] N. Masuda, Phys. Rev. E **90**, 012802 (2014).
[37] O. Artime, N. Khalil, R. Toral, and M. San Miguel, Phys. Rev. E **98**, 042143 (2018).
[38] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, New York, 1985).
[39] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 2nd ed. (North-Holland, Amsterdam, 1997).
[40] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, UK, 2001).
[41] Details are given in D. Bhat and S. Redner, arXiv:1910.01200.
[42] H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).
[43] R. Lambiotte and S. Redner, J. Stat. Mech. (2007) L10001.
[44] R. Lambiotte, J. Saramäki, and V. D. Blondel, Phys. Rev. E **79**, 046107 (2009).
[45] O. Hirschberg, D. Mukamel, and G. M. Schütz, J. Stat. Mech. (2012) P08014.
[46] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).