

Preface to the Second Edition

Since the publication of “A Guide to First-Passage Processes” in 2001, the topic of first-passage processes has become more mainstream. What appeared to me to be an intriguing, yet somewhat boutique area in the late 1990’s has burgeoned over the past 25 years, with new theoretical developments and a host of new applications. The time therefore seems ripe for an updated exposition.

Here is an outline of changes in the second edition. Chapter 1 of the 1st edition has been substantially updated, especially Secs. 1.2, 1.4 and 1.5, and the discussion of the Kolmogorov formalism has been reorganized. Some material from “A Kinetic View of Statistical Physics” (jointly authored with Krapivsky & Ben-Naim) has been borrowed. The section on random walks and the connection to resistor networks has been moved to the (new) Chapter 10 on networks. The presentation in Chapter 2 has been updated, with cleaner and simpler subsection partitioning. A new section on the martingale method was added and the discussion of the Kolmogorov approach was expanded considerably. Chapter 3 on first passage in a heterogeneous interval is mostly new, with material about spatially random hopping rates and models of consensus formation that correspond to first passage in the interval with hopping probabilities that have a systematic dependence on position in the interval. Material in the 1st edition about the freely accelerated particle and the expanding cage have been moved to this chapter because these topics more naturally fit here.

Chapter 4 on first-passage on the semi-infinite line (Chapter 3 in the 1st edition) has been extended to incorporate the phenomenon of duality, in which certain first-passage properties of biased diffusion are independent of the sign of the bias. There is also new material about heterogeneous hopping in the semi-infinite geometry. Chapter 5 is mostly new and discusses the shapes of one-dimensional first-passage (and related) paths when viewed as two-dimensional spacetime trajectories. Many of the results here are folded in with relevant material from Sec. 3.4 in the 1st edition. Chapter 6 on applications is a much expanded version of Chapter 4 in the 1st edition, with new sections on first passage in disparate systems, including the run-and-tumble particle, the hot hand paradox, the Pólya urn, telomere dynamics and the immortality transition, first-passage resetting, and population growth of random walkers due to a localized source.

Chapter 7 on spherically symmetric systems has been more logically reorganized and now includes a new section about the physics of chemoreception that is based on the beautiful article by Berg and Purcell. Material from chapters 3 & 6 in the 1st edition on the quasi-static approximation have been combined into a single presentation here. Chapter 8 contains the material from Chapter 7 from the 1st edition on first passage in the wedge geometry, as well as new material on cones in higher dimensions and paraboloids. Chapter 9 treats aspects of first passage in simple bounded two-dimensional domains such as the circle, the square, and the triangle. The latter two cases correspond to first passage of two diffusing particles on a finite interval and the probability that they meet rather than reaching the ends of the interval.

Chapter 10 on networks contains the material from Chapter 5 in the 1st edition, as well as considerable new material about first passage in important classes on networks, including the complete graph and its extensions, the Erdős-Renyi random graph, the star graph, and complex (degree-heterogeneous) networks. The final Chapter 11 on reactions and searching includes the material in Chapter 8 of the 1st edition, as well as new sections on the kinetics of extinction and the dynamics of diffusive searching.

I hope that the additional material in this second edition together with the many corrections from the 1st edition will provide the reader with a fuller picture of fundamental aspects of first-passage processes. Although much new material has been added, I’m painfully aware that many essential topics are still not covered. To complete this revision in a finite time, however, I have to give up on including everything that I would have liked to include.

Two stylistic points: (a) I’ve moved all citations to a Notes section at the end of each chapter rather than having the text interspersed with citations. I think this construction makes for smoother reading and it allows for a more contextual perspective for the citations in these Notes sections. The citations are fairly extensive but not comprehensive, and I apologize to those whose work should have been cited. (b) In the 1st edition, I spent effort in explaining how to derive certain results or perform certain integrals, and reserved the use of Mathematica for really tedious calculations. In the second edition, I’ve used Mathematica more extensively. So when a result is simply quoted for a complicated integral or for a non-trivial calculation, most of these have been performed by Mathematica. I’ve also used ChatGPT to perform certain calculations in Chapters 9 and 11, to create some figures, and to find useful literature references.

When I wrote this book, I thought I was reasonably careful. I’ve been disabused of this presumption by

many errata discovered in the 1st edition, both by many devoted readers and also by myself. These have all been posted on my website and implemented here. I've also read through the entire original text and made numerous small changes throughout for this second edition. I also thank Harrison Hartle, James Holehouse, Max Jerdee, Aanjaneya Kumar, and Yuanzhao Zhang for collectively proofreading the entire manuscript and for their constructive suggestions. While I hope that this updated edition will not be as replete with errors as the original, I will be grateful to all who let me know of new errors (redner@santafe.edu).

I'm very grateful to collaborators and colleagues over the past 25 years who discussed or worked with me on various aspects of first-passage processes, a few of which are presented in this second edition. In addition to those acknowledged in the 1st edition, some of whom helped with this edition, I also thank Uttam Bhat, Olivier Bénichou, Benjamin de Bruyne, Aaron Clauset, Maria D'Orsogna, James Holehouse, Jon Machta, Baruch Meerson, Julien Randon-Furling, Narayan Sabhahit, and Vishal Sood. I continue to be especially grateful to Paul Krapivsky for his advice on various sections of this book and from whom I've learned so much. Thanks to Courtney McGlynn who provided the whimsical opening drawings for each chapter.

I'd like to dedicate this second edition to two very special people. The first is my late wife, Anita Zetlan Redner, who died unexpectedly only two weeks after we moved from Boston to Santa Fe in 2014 to start a new life here. She was my constant love and support during our 37-year marriage and I'm grateful and honored to have been her husband. The second is my namesake that I never knew, my Father's younger brother, Shulim Redner. Born to an orthodox Jewish family in Chrzanow Poland in 1924, the odds were stacked against him for a full life. I learned much more about him only 20 years ago via a mysterious letter that described his resistance activities during WW II, for which he was murdered/executed by the Nazis in 1943. May the memory of these two wonderful people be an inspiration and blessing to all.

Santa Fe, New Mexico, 2025

Preface to the 1st Edition

You arrange a 7pm date at a local bistro. Your punctual date arrives at 6:55, waits until 7:05, concludes that you will not show up, and leaves. At 7:06, you saunter in – “just a few minutes” after 7 (see Cover). You assume that you arrived first and wait for your date. The wait drags on and on. “What’s going on?” you think to yourself. By 9pm, you conclude that you were stood up, return home, and call to make amends. You explain, “I arrived around 7 and waited 2 hours! My probability of being at the bistro between 7 and 9pm, $P(\text{bistro}, t)$, was nearly one! How did we miss each other?” Your date replies, “I don’t care about your *occupation* probability. What mattered was your *first-passage* probability, $F(\text{bistro}, t)$, which was zero at 7pm. GOOD BYE!” Click!

The moral of this juvenile parable is that first passage underlies many stochastic processes, in which the event, such as a dinner date, a chemical reaction, the firing of a neuron, or the triggering of a stock option, relies on a variable reaching a specified value *for the first time*. In spite of the wide applicability of first-passage phenomena (or perhaps because of it), there does not seem to be a pedagogical source on this topic. For those with a serious interest, essential information is scattered and presented at diverse technical levels. In my attempts to learn the subject, I also encountered the proverbial conundrum that a fundamental result is “well-known to (the vanishingly small subset of) those who know it well”.

In response to this frustration, I attempt to give a unified presentation of first-passage processes and illustrate some of its beautiful and fundamental consequences. My goal is to help those with modest backgrounds learn essential results quickly. The intended audience is physicists, chemists, mathematicians, engineers, and other quantitative scientists. The technical level should be accessible to the motivated graduate student.

My literary inspirations for this book include *Random Walks and Electric Networks*, by P. G. Doyle and J. L. Snell (Carus Mathematical Monographs #22, Mathematical Association of America, Washington, D. C., 1984), which cogently describes the relation between random walks and electrical networks, and *A Primer on Diffusion Problems*, by R. Ghez (Wiley, 1988) which gives a nice exposition of solutions to physically-motivated diffusion problems. This book is meant to complement classic monographs, such as *An Introduction to Probability Theory and its Applications*, by W. Feller (Wiley, New York, 1968), *Aspects and Application of the Random Walks*, by G. H. Weiss, (North-Holland, Amsterdam, 1996), and *Stochastic Processes in Physics and Chemistry*, by N. G. van Kampen (North-Holland, Amsterdam, 1997). Each of these very worthwhile books discusses first-passage phenomena, but secondarily rather than as a comprehensive overview.

I begin with fundamental background in Chap. 1 and outline the relation between occupation and first-passage probabilities, as well as the connection between first passage and electrostatics. Many familiar results from electrostatics can be easily adapted to give first-passage properties in the same geometry. In Chap. 2, I discuss first passage in a one-dimensional interval. This provides a simple laboratory for answering basic questions, such as: What is the probability that a diffusing particle eventually exits at either end? How long does it take to exit? These problems are solved both by direct approaches and by developing the electrostatic equivalence. Chapter 3 treats first passage in a semi-infinite interval both by standard approaches and by the familiar image method. I also discuss surprising consequences of the basic dichotomy between certain return to the starting point and infinite mean return time.

Chapter 4 is devoted to illustrations of the basic theory. I discuss neuron dynamics, realizations of self-organized criticality, and the dynamics of spin systems. These all have the feature that they can be viewed as first-passage processes in one dimension. I also treat stochastic resonant escape from fluctuating and inhomogeneous media, for which the time-independent electrostatic formalism provides a relatively easy way to solve for mean first-passage times. Finally, I discuss the survival of a diffusing particle in a growing “cage” and near a moving “cliff”, where particularly rich behavior arises when diffusion and the motion of the boundary have the same time dependence.

In Chap. 5, I turn to first passage on branched, self-similar structures. I emphasize self-similar systems because this feature allows us to solve for the first-passage probability by renormalization. Another essential feature of branched systems is the competition between transport along the “backbone” from source to sink and detours along sidebranches. I give examples that illustrate this basic competition and the transition from scaling, in which a single time scale accounts for all moments of the first-passage time, to multiscaling, in which each moment is governed by a different time scale.

I then treat spherically-symmetric geometries in Chap. 6 and discuss basic applications, such as efficient simulations of diffusion-limited aggregation and the Smoluchowski chemical reaction rate. First passage in wedge and conical domains are presented in Chap. 7. I discuss how the wedge geometry can be solved elegantly by the mapping to electrostatics and conformal transformations. These systems provide the kernel for understanding the main topic of Chap. 8, namely, the kinetics of one-dimensional diffusion-controlled reactions. This includes trapping, the reactions among three diffusing particles on the line, as well as basic bimolecular reactions, including capture $p + P \rightarrow P$, annihilation $A + A \rightarrow 0$, coalescence $A + A \rightarrow A$, and aggregation $A_i + A_j \rightarrow A_{i+j}$. The chapter ends with a brief treatment of ballistic annihilation.

A large fraction of this book discusses either classical first-passage properties or results about first passage from contemporary literature, but with some snippets of new results sprinkled throughout. However, several topics are either significant extensions of published results or are original. This includes the time-integrated formalism to compute the first-passage time in fluctuating systems (Section 6.5), aspects of survival in an expanding interval (Section 3.5), return probabilities on the hierarchical tree and homogeneous comb (Sections 10.7 & Subsection 10.8.1), the first-passage probability on the hierarchical blob (Section 10.9), and reactions of three diffusing particles on the line (Section 11.3).

This book has been influenced by discussions or collaborations with Dani ben-Avraham, Eli Ben-Naim, Charlie Doering, Laurent Frachebourg, Slava Ispolatov, Joel Koplik, Paul Krapivsky, Satya Majumdar, Francois Leyvraz, Michael Stephen, George Weiss, David Wilkinson, and Bob Ziff to whom I am grateful for their friendship and insights. I thank Bruce Taggart of the U.S. National Science Foundation for providing financial support at a crucial juncture in the writing, as well as Murad Taqqu and Mal Teich for initial encouragement. Elizabeth Sheld helped me get this project started with her invaluable organizational assistance. I also thank Satya Majumdar for advice on a preliminary manuscript and Erkki Hellén for a critical reading of a nearly final version. I am especially indebted to Paul Krapivsky, my next-door neighbor for most of the past 6 years, for many pleasant collaborations and for much helpful advice. While it is a pleasure to acknowledge the contributions of my colleagues, errors in presentation are mine alone.

Even in the final stages of writing, I am acutely aware of many shortcomings in my presentation. If I were to repair them all, I might never finish. This book is still work “in progress” and I look forward to receiving your corrections, criticisms, and suggestions for improvements (redner@bu.edu).

Finally and most importantly, I thank my family for their love and constant support and for affectionately tolerating me while I was writing this book.