

Superdiffusion in Random Velocity Fields

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We discuss the *superdiffuse* motion of a random walk in a medium containing random velocity fields. For a two-dimensional layered medium with y -dependent random velocities in the x direction $u(y)$, $\langle x^2(t) \rangle \sim t^{2\nu}$, with $2\nu = \frac{3}{2}$, and with strong sample-to-sample fluctuations. The probability distribution of displacements, averaged over environments, takes a non-Gaussian scaling form at large time, $\langle P(x, t) \rangle \sim t^{-3/4} f(x/t^{3/4})$, where $f(u) \sim \exp(-u^\delta)$ for $u \gg 1$, with $\delta = \frac{4}{3}$. For an isotropic two-dimensional medium with $u_x(y) = f(y)$ and $u_y(x) = f(x)$, we find $\nu = \frac{2}{3}$ and $\delta = (1 - \nu)^{-1} = 3$.

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Stochastic transport in random media is often *subdiffusive*, where the mean-square displacement $\langle x^2(t) \rangle$ grows more slowly than linearly with time (see, e.g., Refs. 1-3 for recent reviews). In this Letter, we discuss a simple and general mechanism, based on the coupling between diffusion and convection by spatially random, but temporally static velocity fields, that leads to the complementary situation of *superdiffusion*, where $\langle x^2(t) \rangle$ grows *faster* than linearly in time.

Superdiffusive transport has been treated previously in a variety of contexts, such as chaotic systems,⁴ turbulence,⁵ flow in fractal geometries,⁶ and Levy flights.⁷ In the latter case, both subdiffusion and superdiffusion can arise by choosing a sufficiently singular single-step probability distribution. Here, we discuss a situation, relevant to ground water transport in geological aquifers,⁸ where superdiffusion arises naturally, rather than being built into microscopic-level transport laws. Measurements of tracer dispersion in aquifers indicate that the dispersivity systematically increases with distance between source and sink.⁸ A theoretical model involving random velocity fields was constructed by Matheron and de Marsily⁹ that yields a superdiffusive spread of a tracer pulse. We extend this treatment in several important respects to gain a deeper insight into the role that random velocity fields play in superdiffusion.

The basic phenomenon can be appreciated by considering a two-dimensional stratified porous medium consisting of distinct parallel layers with different transport properties in each layer [Fig. 1(a)]. When a pressure drop is applied along the strata, the longitudinal fluid velocity correspondingly varies from layer to layer. In a center-of-mass frame of reference, therefore, the steady velocities in the x direction are random zero mean func-

tions of the transverse coordinate y . Although the longitudinal bias averaged over an infinite number of layers is zero, the average over the finite number of layers that a random walk visits is a fluctuating quantity which is a decreasing function of the number of layers sampled. This nonvanishing bias underlies superdiffusive transport.⁹⁻¹²

Within a continuum description, the random-walk motion is accounted for by the Langevin equations,

$$dx/dt = u(y(t)), \quad dy/dt = \eta(t), \quad (1)$$

in which a random walker undergoes pure diffusion along y [$\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t')$, where D is the transverse diffusion coefficient] and is passively carried by the quenched random convection field $u(y)$ along x . Diffusion noise in the x direction is subdominant and thus neglected with respect to the random convection. For simplicity, we take the convection field to be a Gaussian white noise in space, $\langle u_x(y)u_x(y') \rangle_c = \sigma\delta(y-y')$, where $\langle \dots \rangle_c$ denotes an average over all velocity configurations of the medium.

To compute the moments of the longitudinal displacement, note that for a given walk in a fixed environment,

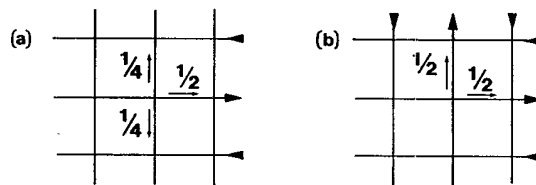


FIG. 1. (a) The random stratified medium and (b) the random isotropic medium on the square lattice. The hopping rules at a typical lattice site are indicated. In (a), the layers consist of contiguous rows of the same orientation.

the longitudinal position at time t can be written formally as

$$x(t) = \int_0^t u(y(t')) dt' = \int_{-\infty}^{+\infty} \mathcal{N}(y, t) u(y) dy, \quad (2)$$

where $\mathcal{N}(y, t) = \int_0^t dt' \delta(y - y(t'))$ is the number of times that the transverse Brownian motion $y(t)$ visits layer y after time t , having started at $y=0$. The superdiffusive behavior of this model can be understood by the following heuristic estimate of $x_{\text{rms}}(t)$ from Eq. (2). The quantity $\mathcal{N}(y, t)$ is of the order of the elapsed time t divided by the number of y layers encountered, $\sqrt{2Dt}$.

Further, the rms value of $\int dy u(y)$ is roughly $\sigma^{1/2} \times (2Dt)^{1/4}$, where the latter factor is the square root of the typical range of the integral. Thus

$$x_{\text{rms}}(t) \sim \langle u_x \rangle_t \sim \sigma^{1/2} D^{-1/4} t^{3/4}, \quad (3)$$

a remarkable result which was apparently first derived in Ref. 9 and which also can be derived in terms of a power-law decay of the longitudinal velocity correlation function.¹⁰

To justify and extend this intuitive argument to higher moments of the longitudinal displacement, we write these moments as

$$\langle \langle x^n(t) \rangle_w \rangle_c = n! \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \langle \langle u(y(t_1)) \cdots u(y(t_n)) \rangle_w \rangle_c. \quad (4)$$

The double angular brackets indicate that one should first average over all transverse Brownian trajectories for a given configuration of random velocities, and then average over all configurations. However, these two averages factorize and can be performed in either order. Thus the velocity correlation function is

$$\langle \langle u(y(t_1)) \cdots u(y(t_n)) \rangle_w \rangle_c = \int_{-\infty}^{+\infty} dy_1 dy_2 \cdots dy_n \langle u(y_1) \cdots u(y_n) \rangle_c \times p(y_n, t_n) p(y_{n-1} - y_n, t_{n-1} - t_n) \cdots p(y_1 - y_2, t_1 - t_2), \quad (5)$$

where

$$p(x, t) = (1/\sqrt{4\pi Dt}) \exp(-x^2/4Dt)$$

is the Gaussian probability distribution for the transverse motion. The product of Gaussians in Eq. (5) is the probability that a Brownian path visits the sequence of transverse positions $\{y(t_i)\}$ at times $\{t_i\}$, having started at $y=0$. For the continuous model defined by Eq. (1), $\langle u(y_1) \cdots u(y_n) \rangle_c$ is a sum of products of delta functions. Consequently, the second moment is

$$\langle \langle x(t)^2 \rangle_w \rangle_c = 2\sigma \int_0^t dt_1 \int_0^{t_1} dt_2 \int_{-\infty}^{+\infty} dy p(0, t_1 - t_2) p(y, t_2) = \frac{4\sigma}{3\sqrt{\pi D}} t^{3/2}. \quad (6)$$

Intriguingly, Eq. (6) does not fully characterize long-time transport properties, as there are anomalously large sample-to-sample fluctuations.¹³ The longitudinal displacement, averaged over all walks in a *fixed* environment, $\langle x(t) \rangle_w$, depends on the configuration, and does not necessarily converge to zero at large times. However, the average over all environments, $\langle \langle x(t) \rangle_w \rangle_c$, does equal zero in the center-of-mass reference frame. Clearly $\langle x(t) \rangle_w$ has a distribution over environments which is a Gaussian of variance $\langle \langle x(t) \rangle_w^2 \rangle_c$, which, for the continuous model of Eq. (1) is

$$\begin{aligned} \langle \langle x(t) \rangle_w^2 \rangle_c &= \sigma \int dy \mathcal{N}(y, t)^2 \\ &= (\sqrt{2} - 1) \frac{4\sigma}{3\sqrt{\pi D}} t^{3/2}. \end{aligned} \quad (7)$$

Thus both the configuration average of the mean-square displacement, $\langle \langle x(t)^2 \rangle_w - \langle x(t) \rangle_w^2 \rangle_c$, and the second moment $\langle \langle x(t)^2 \rangle_w \rangle_c$ vary as $t^{3/2}$, but with different prefactors. This effect has been pointed out previously,¹³ but actual values of these prefactors are obtained here for the first time.

An important consequence of these sample-specific fluctuations is that the probability distribution for the

longitudinal displacement, $P(x, t)$, cannot reach a configuration-independent limiting form as a function of $x/t^{3/4}$, in a fixed environment. For a single "typical" environment, the typical width of $P(x, t)$ is presumably described by $\langle \langle x(t)^2 \rangle_w - \langle x(t) \rangle_w^2 \rangle_c$, while the configuration-averaged distribution can be viewed as the "envelope" of the individual diffusion fronts for each environment. The width of the latter is equal to

$$\langle \langle x(t)^2 \rangle_w \rangle_c - \langle \langle x(t) \rangle_w \rangle_c^2 = \langle \langle x(t) \rangle_w^2 \rangle_c,$$

and is expected to be larger than the typical width, as found above.

In contrast to the individual $P(x, t)$'s for each environment, the configuration average is expected to take a well-defined scaling form in the large-time limit,

$$\langle P(x, t) \rangle_c \rightarrow t^{-3/4} f(x/t^{3/4}), \quad (8)$$

where it is understood that x and t are simultaneously large with $u \equiv x/t^{3/4}$ finite (as is the case of any central-limit theorem). For $u \gg 1$, the scaling function is expected to vary as

$$f(u) \sim \exp(-cu^5) \quad (9)$$

(with possible power-law prefactors). According to our numerical and analytical arguments, the shape exponent δ appears to take on the anomalous value $\delta = \frac{4}{3}$.

Our numerical approach to test the validity of Eq. (9), and to find δ , is based on computing dimensionless moment ratios such as $m_{2k}(t) \equiv \langle x^{2k}(t) \rangle / \langle x^2(t) \rangle^k$ and

$$n_{2k}(t) \equiv \langle x^{2k}(t) \rangle / \langle x^{2(k-1)}(t) \rangle \langle x^2(t) \rangle.$$

If $\langle x^{2k}(t) \rangle \sim \langle x^2(t) \rangle^k$, the m_{2k} and n_{2k} will approach constants at $t \rightarrow \infty$ whose values depend on δ . By attempting to match our numerical estimates for m_{2k} and n_{2k} to the moments that arise directly from Eq. (9), we infer a value of δ .

We have developed several independent calculational approaches for these moments, all of which yield consistent results. One method is to evaluate Eqs. (4) and (5), thus yielding results which are *exact* for all times. This calculation becomes progressively unwieldy at higher order, and appears to be amenable only to numerical integration. Up to sixth order, we find $m_4 \approx 3.3 \pm 0.03$ and $m_6 \approx 19.1 \pm 0.4$, where the errors denote statistical uncertainties in numerical integration. These results indicate that $f(u)$ is non-Gaussian (a Gaussian gives $m_4 = 3$, $m_6 = 15$, etc.), but are insufficient to give a reliable estimate of δ .

A complementary numerical approach is based on using exact enumeration² to find the probability distribution of longitudinal displacements exactly for a given environment, and then averaging over *all* environments in a system of finite width w . For this computation, we employed the lattice model illustrated in Fig. 1(a). Each horizontal line is randomly assigned a velocity \pm , and at each lattice site a random walk moves either in the $+y$ or $-y$ direction with probability $\frac{1}{4}$ (thus fixing D), or moves with the bias with probability $\frac{1}{2}$. We performed this calculation for the 92205 distinct velocity configurations (cyclic permutations and reflection symmetry), out of the 2^{23} states on a system of width $w = 23$ with periodic transverse boundary conditions. This procedure provides the *exact* configurational-average probability distribution for an *infinite* system up to 22 time steps.

By extrapolation of the resulting moments to $t \rightarrow \infty$, a consistent trend in the behavior of n_{2k} as a function of k can now be discerned (Fig. 2). To extrapolate the moment values $n_{2k}(t)$, we form the sequence

$$i(t) = [\sqrt{t} n_{2k}(t) - \sqrt{t-1} n_{2k}(t-1)] / [\sqrt{t} - \sqrt{t-1}],$$

which are the intercepts at $1/\sqrt{t} = 0$ of the straight line that passes through $n_{2k}(t)$ and $n_{2k}(t-1)$. This extrapolation is then repeated on the successive levels of extrapolant sequences. This approach yields a consistent picture in which the effective value of δ is a decreasing function k . Our estimate of n_4 suggests $\delta \approx 1.7$, while our estimate for n_8 is consistent with $\delta \lesssim 1.4$.

The apparent change as a function of the order of the

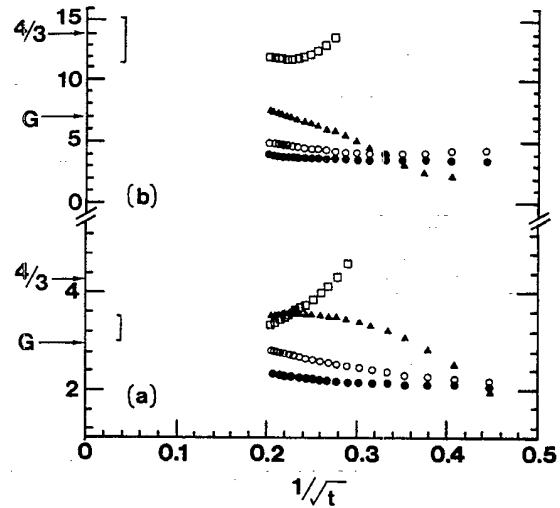


FIG. 2. Graphical estimates for the asymptotic behavior of (a) n_4 and (b) n_8 . Plotted are the values for $n_{2k}(t)$ vs $1/\sqrt{t}$ (\bullet), together with first (\circ), second (\blacktriangle), and third (\square) extrapolations based on plotting the intercepts at $1/\sqrt{t} = 0$ of successive lower-order extrapolants. The arrows indicate the asymptotic values for $\delta = 2$ (Gaussian) and $\delta = \frac{4}{3}$, while the square brackets indicate our subjective uncertainty estimates for n_{2k} .

moment suggests anomalous behavior of the tail of the distribution function.¹⁴ Consider, therefore, the averaged probability of finding a “stretched out” walk, namely, $\langle P(x \sim t, t) \rangle_c$. According to Eq. (9), this probability should vary as $\exp(-t^{\delta/4})$. On the other hand, the probability of a walk being stretched out can be bounded from below by the probability of remaining transversely confined to a region of unidirectional velocity bias. This confining probability, averaged over all environments, is isomorphic to the survival probability of a one-dimensional random walk in the presence of randomly distributed traps,¹⁵ and hence varies as $\exp(-at^{1/3})$. By comparing these two distributions, one concludes that $\delta \leq \frac{4}{3}$.

To further argue that $\delta = \frac{4}{3}$, we first exploit the fact that the average in Eq. (4) can be first taken over environments for a fixed trajectory, and then over all walks. For a fixed trajectory, Eq. (2) expresses the longitudinal displacement as a sum of independent random variables, and it is thus clear that its distribution is a Gaussian, $\exp[-x^2/Q(t)]$, whose variance is proportional to $Q(t) \equiv \sum_y \mathcal{N}(y, t)^2$. While the complete distribution for $Q(t)$ is not readily calculable, it may be argued that the large- Q tail of this distribution is also a Gaussian. Consider first typical one-dimensional random walks which visit each site \sqrt{t} times within a range \sqrt{t} . For such walks, $\langle Q(t) \rangle \sim (t/t^{1/2})^2 t^{1/2} \sim t^{3/2}$. Now consider “confined” walks which fill a region of extent $w \sim t^\alpha$ with $\alpha < \frac{1}{2}$. We assume that these walks spread uniformly over the region t^α so that $Q(t) \sim (t/t^\alpha)^2 t^\alpha \sim t^{2-\alpha}$. Consequently, for confined walks, $Q(t)/\langle Q(t) \rangle \sim t^{1/2-\alpha}$,

while the probability of such walks varies as $\exp(-t/w^2) \sim \exp(t^{1-2\alpha})$. Therefore the distribution of $Q(t)$ has the Gaussian form, $P(Q) \sim \exp\{-Q(t)/\langle Q(t) \rangle\}^2$ for $Q \gg \langle Q(t) \rangle$. Now averaging the Gaussian displacement distribution $\exp[-x^2/Q(t)]$ over all random-walk trajectories, i.e., over $P(Q)$, leads to an averaged probability distribution of the form of Eq. (9), but with $\delta = \frac{4}{3}$.

An isotropic-random-velocity-field model [Fig. 1(b)] also exhibits superdiffusion. In two spatial dimensions, a walk moves on a random "Manhattan" grid, in which the directionality along any avenue or street is fixed along its entire length, but whose orientation is random. [This model can also be shown to be equivalent to a random walk in a divergenceless random velocity field with long-range correlations $\langle \mathbf{u}(0)\mathbf{u}(\mathbf{x}) \rangle_c = |\mathbf{x}|^{-\alpha}$.¹⁶] For the random Manhattan system, we generalize the arguments of Eqs. (2) and (3) by formally decomposing the isotropic motion into transverse and longitudinal components. Assuming $x_{\text{rms}} \sim t^\nu$, and then by following the steps that lead to Eq. (3), one finds that $x_{\text{rms}} \sim t^{1-\nu/2}$. By isotropy, however, one must have $\nu = 1 - \nu/2$, or $\nu = \frac{2}{3}$. Generalizing to arbitrary spatial dimension d yields $\nu = 2/(d+1)$ for $d < d_c = 3$, $\nu = \frac{1}{2}$ for $d > d_c$, and with logarithmic corrections for $d = d_c$. For the probability distribution of displacements, even modest simulations in two dimensions indicate that Eq. (9) holds over a substantial range, with $\nu = \frac{2}{3}$ and $\delta = 3$, in accord with the usual relation¹⁷ between the shape and size exponent, $\delta = (1 - \nu)^{-1}$.

In summary, superdiffusive transport arises from the interplay between pure diffusion and convection by spatially inhomogeneous, but correlated, velocity fields. For the layered system, we find a size exponent $\nu = \frac{3}{4}$ in two dimensions, with an anomalous large-distance tail in the averaged probability distribution of displacements. For an isotropic two-dimensional Manhattan system, the probability distribution appears to exhibit conventional scaling in which $\delta = (1 - \nu)^{-1}$, and $\nu = 2/(d+1)$ for

$d < 3$.

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