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Wealth distributions in asset exchange models

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Abstract. A model for the evolution of the wealth distribution in an economically interacting population is introduced, in which a specified amount of assets are exchanged between two individuals when they interact. The resulting wealth distributions are determined for a variety of exchange rules. For "random" exchange, either individual is equally likely to gain in a trade, while "greedy" exchange, the richer individual gains. When the amount of asset traded is fixed, random exchange leads to a Gaussian wealth distribution, while greedy exchange gives a Fermi-like scaled wealth distribution in the long-time limit. Multiplicative processes are also investigated, where the amount of asset exchanged is a finite fraction of the wealth of one of the traders. For random multiplicative exchange, a steady state occurs, while in greedy multiplicative exchange a continuously evolving power law wealth distribution arises.

PACS. 02.50.Ga Markov processes – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes – 05.40.+j Fluctuation phenomena, random processes, and Brownian motion

1 Introduction and models

Recent applications of ideas developed in statistical physics, such as scaling, self-organization, optimization in a complex landscape, *etc.*, are helping to establish a new conceptual framework for the scientific analysis of economic activities [1–4]. In this spirit of seeking universal mechanisms, as well as simplicity and concreteness of modeling, we introduce microscopic "asset exchange" models as an attempt to account for the wealth distribution of an economically interacting population. We investigate basic consequences of our model and attempt to understand how generic features of a wealth distribution emerge. We are particularly interested in constructing specific examples which generate a power law wealth distribution, a form which has been observed in many countries [5].

The basis of our modeling is that the elemental kernel of economic activity in a dynamic economy is the interaction between two individuals which results in a redistribution of their assets. Here, we view an "individual" as a single person or a self-contained economic entity, such as a company. We also regard an "asset" as any economic attribute – cash or physical asset – which contributes to the overall wealth of an individual. As an example, a farmer pays cash to buy a tractor from a store owner. With the tractor, the farmer produces a crop which is then sold at a profit. If this sale is again with the same dealer, one may view the composite interaction as an "exchange" in which the farmer gains cash while the store owner pays out some cash (the price difference of the tractor and the crop), and has exchanged a tractor for crop. Depending on the price at which this crop is re-sold, the assets of the store owner may increase or decrease by these transactions. With this example in mind, the economic activity of a population may be viewed as many two-body exchanges of assets between randomly-chosen pairs of traders. Through this process a global wealth distribution develops, and we wish to understand how generic features of this distribution depend on the nature of the two-body interaction. While our idealized asset exchange models omit many important features of real economic activity, one has the advantages of simplicity and solvability of the basic equations of motion. Further, the insights gained in the study of our models may help guide the construction of more comprehensive descriptions of the evolution of the wealth distribution.

In the next section, we first treat "additive" processes in which a fixed amount of asset is exchanged between two traders, independent of their wealth before the interaction [6]. While the restriction to fixed assets is unrealistic, the resulting models are soluble and provide a starting point for more economically-motivated generalizations. Within a (mean-field) rate equation description, the wealth distribution typically exhibits scaling, from which both the time dependence of the average wealth and the form of the scaled wealth distribution can be obtained. We first give an exact solution for the example of *random* additive exchange, where either trader is equally likely to profit in an interaction. The scaling approach is then applied to *greedy* exchange, in which the richer

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person profits in an interaction. The asymptotic scaled wealth distribution is found to be Gaussian for random exchange, and resembles the Fermi distribution for greedy exchange.

We next consider "multiplicative" processes in Section 3, in which a fixed fraction of the current wealth of one of the traders is exchanged in an interaction. This rule is motivated by the observation that fractional exchange underlies many economic transactions – for example, loan interest or investment return is quoted in percentage terms. In parallel with our discussion of additive processes, we consider both random and greedy multiplicative exchange. The former leads to a steady state; this arises because the randomness of the exchange ensures that a rich individual experiences many wealth reducing interactions and is thus driven towards the middle class. Greedy multiplicative exchange is the most interesting of our models because, for a broad range of parameters, a power law form for the wealth distribution results in which the rich get richer and the poor get poorer. Since real wealth distributions are often a power law, our model may provide a framework for their quantitative description.

In Section 4, we summarize, mention a number of extensions, and discuss several limitations of our approach. In the Appendix, we treat additive exchange in a low-dimension "economic" space, when diffusion is the transport mechanism which brings trading partners together.

2 Additive asset exchange

Consider a population of traders, each of which possesses a certain amount of assets which are assumed to be quantized in units of a minimal asset. Taking this latter quantity as the basic unit, the fortune of an individual is restricted to the integers. The wealth of the population evolves by the repeated interaction of random pairs of traders. In each interaction, one unit of asset is transferred between the trading partners. To complete the description, we specify that if a poorest individual (with one unit of asset) loses this last unit of asset by virtue of a "loss", the bankrupt individual is considered to be economically dead and no longer participates in trading activity.

In the following subsections, we consider three specific realizations of additive asset exchange. In "random" exchange, the direction of the exchange is independent of the relative assets of the traders. While this rule has little economic basis, the resulting model is completely soluble and thus provides a helpful pedagogical starting point. We next consider "greedy" exchange in which a richer person takes one unit of asset from a poorer person in a trade. Such a rule is a reasonable starting point for describing exploitive economic activity. Finally, we consider a more heartless version – "very greedy" exchange – in which the rate of exchange is proportional to the difference in assets between the two traders. These latter two cases can be solved by a scaling approach. The primary result is that the scaled wealth distribution resembles a finite-temperature Fermi distribution, with an effective temperature that goes to zero in the long-time limit.

2.1 Random exchange

In this process, one unit of asset is exchanged between trading partners, as represented by the reaction scheme $(j,k) \to (j\pm 1, k\mp 1)$. Let $c_k(t)$ be the density of individuals with assets k. Within a mean-field description, $c_k(t)$ evolves according to

$$\frac{\mathrm{d}c_k(t)}{\mathrm{d}t} = N(t)\left[c_{k+1}(t)+c_{k-1}(t)-2c_k(t)\right], \tag{1}$$

with $N(t) \equiv M_0(t) = \sum_{k=1}^{\infty} c_k(t)$ the population density. The first two terms account for the gain in $c_k(t)$ due to the interactions $(j,k+1) \to (j+1,k)$ and $(j,k-1) \to (j-1,k)$, respectively, while the last term accounts for the loss in $c_k(t)$ due to the interactions $(j,k) \to (j\pm 1, k\mp 1)$. By defining a modified time variable,

$$T = \int_0^t \mathrm{d}t' N(t'), \tag{2}$$

equation (1) is reduced to the discrete diffusion equation

$$\frac{\mathrm{d}c_k(T)}{\mathrm{d}T} = c_{k+1}(T)+c_{k-1}(T)-2c_k(T). \tag{3}$$

The rate equation for the poorest density has the slightly different form, $\mathrm{d}c_1/\mathrm{d}T = c_2 - 2c_1$, but may be written in the same form as equation (3) if we impose the boundary condition $c_0(T) = 0$.

Equation (3) may be readily solved for arbitrary initial conditions [7]. For illustrative purposes, let us assume that initially all individuals have one unit of asset, $c_k(0) = \delta_{k1}$. The solution to equation (3) subject to these initial and boundary conditions is

$$c_k(T) = \mathrm{e}^{-2T}\left[I_{k-1}(2T) - I_{k+1}(2T)\right], \tag{4}$$

where I_n denotes the modified Bessel function of order n [8]. Consequently, the total density $N(T)$ is

$$N(T) = \mathrm{e}^{-2T}\left[I_0(2T) + I_1(2T)\right]. \tag{5}$$

To re-express this exact solution in terms of the physical time t, we first invert equation (2) to obtain $t(T) = \int_0^T \mathrm{d}T'/N(T')$, and then eliminate T in favor of t in the solution for $c_k(T)$. For simplicity and concreteness, let us consider the long-time limit. From equation (4),

$$c_k(T) \simeq \frac{k}{\sqrt{4\pi T^3}} \exp\left(-\frac{k^2}{4T}\right), \tag{6}$$

and from equation (5),

$$N(T) \simeq (\pi T)^{-1/2}. \tag{7}$$

Equation (7) also implies $t \simeq \frac{2}{3}\sqrt{\pi T^3}$ which gives

$$N(t) \simeq \left(\frac{2}{3\pi t}\right)^{1/3}, \tag{8}$$

and

$$c_k(t) \simeq \frac{k}{3t} \exp\left[-\left(\frac{\pi}{144}\right)^{1/3} \frac{k^2}{t^{2/3}}\right]. \tag{9}$$

Note that this latter expression may be written in the scaling form $c_k(t) \propto N^2 x\,\mathrm{e}^{-x^2}$, with the scaling variable $x \propto kN$. One can also confirm that the scaling solution represents the basin of attraction for almost all exact solutions. Indeed, for any initial condition with $c_k(0)$ decaying faster than k^{-2}, the system reaches the scaling limit $c_k(t) \propto N^2 x\,\mathrm{e}^{-x^2}$. On the other hand, if $c_k(0) \sim k^{-1-\alpha}$, with $0 < \alpha < 1$, such an initial state converges to an alternative scaling limit which depends on α, as discussed, *e.g.*, in reference [9]. These solutions exhibit a slower decay of the total density, $N \sim t^{-\alpha/(1+\alpha)}$, while the scaling form of the wealth distribution is

$$c_k(t) \sim N^{2/\alpha}\mathcal{C}_\alpha(x), \quad x \propto kN^{1/\alpha}, \tag{10}$$

with the scaling function

$$\mathcal{C}_\alpha(x) = \mathrm{e}^{-x^2} \int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u^2}\sinh(2ux)}{u^{1+\alpha}}\cdot \tag{11}$$

Evaluating the integral by the Laplace method gives an asymptotic distribution which exhibits the same $x^{-1-\alpha}$ as the initial distribution. This anomalous scaling in the solution to the diffusion equation is a direct consequence of the extended initial condition. This latter case is not economically relevant, however, since the extended initial distribution leads to a divergent initial wealth density.

2.2 Greedy exchange

In greedy exchange, when two individuals meet, the richer person takes one unit of asset from the poorer person, as represented by the reaction scheme $(j,k) \to (j+1,k-1)$ for $j \geq k$. In the rate equations, the densities $c_k(t)$ now evolve according to

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = c_{k-1}\sum_{j=1}^{k-1} c_j + c_{k+1}\sum_{j=k+1}^{\infty} c_j - c_k N - c_k^2. \tag{12}$$

The first two terms account for the gain in $c_k(t)$ due to the interaction between pairs of individuals with assets $(j,k-1)$, with $j<k$ and $(j,k+1)$ with $j>k$, respectively. The last two terms correspondingly account for the loss of $c_k(t)$. One can check that the wealth density $M_1 \equiv \sum_{k=1}^{\infty} k\,c_k(t)$ is conserved and that the population density obeys

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -c_1 N. \tag{13}$$

Equations (12) are conceptually similar to the Smoluchowski equations for aggregation with a constant reaction rate [10]. Mathematically, however, they appear to be more complex and we have been unable to solve them analytically. Fortunately, equation (12) is amenable to a scaling solution [11]. For this purpose, we first re-write equation (12) as

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = -\,c_k(c_k + c_{k+1}) + N(c_{k-1} - c_k) + (c_{k+1} - c_{k-1})\sum_{j=k}^{\infty} c_j. \tag{14}$$

Taking the continuum limit and substituting the scaling ansatz

$$c_k(t) \simeq N^2\mathcal{C}(x), \quad \text{with} \qquad x = kN, \tag{15}$$

transforms equations (13) and (14) to

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathcal{C}(0)N^3, \tag{16}$$

and

$$\mathcal{C}(0)[2\mathcal{C} + x\mathcal{C}'] = 2\mathcal{C}^2 + \mathcal{C}'\left[1 - 2\int_x^\infty \mathrm{d}y\mathcal{C}(y)\right], \tag{17}$$

where $\mathcal{C}' = \mathrm{d}\mathcal{C}/\mathrm{d}x$. Note also that the scaling function must obey the integral relations

$$\int_0^\infty \mathrm{d}x\,\mathcal{C}(x) = 1, \quad \text{and} \qquad \int_0^\infty \mathrm{d}x\,x\,\mathcal{C}(x) = 1. \tag{18}$$

The former follows from the definition of the density, $N = \sum c_k(t) \simeq N\int \mathrm{d}x\,\mathcal{C}(x)$, while the latter follows if we set, without loss of generality, the (conserved) wealth density equal to unity, $\sum_k kc_k(t) = 1$.

Introducing $\mathcal{B}(x) = \int_0^x \mathrm{d}y\,\mathcal{C}(y)$ recasts equation (17) into $\mathcal{C}(0)[2\mathcal{B}' + x\mathcal{B}''] = 2\mathcal{B}'^2 + \mathcal{B}''[2\mathcal{B} - 1]$. Integrating twice gives $[\mathcal{C}(0)x - \mathcal{B}][\mathcal{B} - 1] = 0$, with solution $\mathcal{B}(x) = \mathcal{C}(0)x$ for $x < x_\mathrm{f}$ and $\mathcal{B}(x) = 1$ for $x \geq x_\mathrm{f}$, from which we conclude that the scaled wealth distribution $\mathcal{C}(x) = \mathcal{B}'(x)$ coincides with the zero-temperature Fermi distribution,

$$\mathcal{C}(x) = \begin{cases} \mathcal{C}(0), & x < x_\mathrm{f}; \\ 0, & x \geq x_\mathrm{f}. \end{cases} \tag{19}$$

Hence the scaled profile has a sharp front at $x = x_\mathrm{f}$, with $x_\mathrm{f} = 1/\mathcal{C}(0)$ found by matching the two branches of the solution for $\mathcal{B}(x)$. Making use of the second integral relation (18) gives $\mathcal{C}(0) = 1/2$ and thereby closes the solution. Thus the unscaled wealth distribution $c_k(t)$ reads

$$c_k(t) = \begin{cases} 1/(2t), & k < 2\sqrt{t}; \\ 0, & k \geq 2\sqrt{t}; \end{cases} \tag{20}$$

and the total density is $N(t) = t^{-1/2}$.

We checked these predictions by numerical simulations (Fig. 1). In the simulation, two individuals are randomly chosen to undergo greedy exchange and this process is repeated. When an individual reaches zero assets, he is eliminated from the system, and the number of active traders

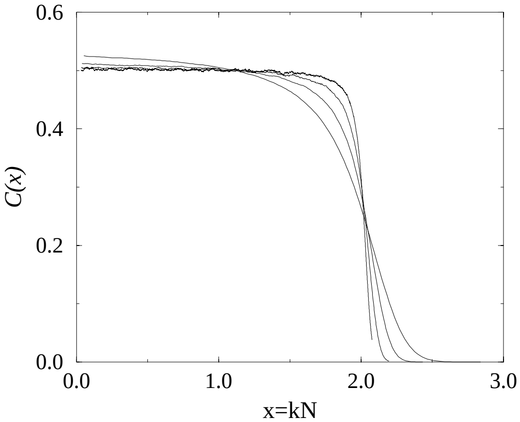

Fig. 1. Simulation results for the wealth distribution in greedy additive exchange based on 2500 configurations for 10^6 traders. Shown are the scaled distributions $\mathcal{C}(x)$ *versus* $x = kN$ for $t = 1.5^n$, with n = 18, 24, 30, and 36; these steepen with increasing time. Each data set has been averaged over a range of $\approx 3\%$ of the data points to reduce fluctuations.

is reduced by one. After each reaction, the time is incremented by the inverse of the number of active traders. While the mean-field predictions are substantially corroborated, the scaled wealth distribution for finite time actually resembles a finite-temperature Fermi distribution (Fig. 1). As time increases, the wealth distribution becomes sharper and approaches equation (20). In analogy with the Fermi distribution, the relative width of the front may be viewed as an effective temperature. Thus the wealth distribution is characterized by two scales; one of order $\sqrt{t}$ characterizes the typical wealth of active traders and a second, smaller scale which characterizes the width of the front [12].

To quantify the spreading of the front, let us include the next corrections in the continuum limit of the rate equations (14). This gives,

$$\frac{\partial c}{\partial t} = 2\frac{\partial}{\partial k}\left[c\int_k^\infty \mathrm{d}j\, c(j)\right] - c\frac{\partial c}{\partial k} - N\frac{\partial c}{\partial k} + \frac{N}{2}\frac{\partial^2 c}{\partial k^2}\cdot \tag{21}$$

Here the second and fourth terms on the right-hand side represent the next corrections. Since the "convective" (third) term determines the location of the front to be at $k_\mathrm{f} = 2\sqrt{t}$, it is natural to expect that the (diffusive) fourth term describes the spreading of the front. The term $c\frac{\partial c}{\partial k}$ turns out to be negligible in comparison to the diffusive spreading term and is henceforth neglected.

The dominant convective term can be removed by transforming to a frame of reference which moves with the front, namely, $k \to K = k - 2\sqrt{t}$. Among the remaining terms in the transformed rate equation, the width of the front region W can now be determined by demanding that the diffusion term has the same order of magnitude as the reactive terms, that is, $N\frac{\partial^2 c}{\partial k^2} \sim c^2$. This implies $W \sim \sqrt{N/c}$. Combining this with $N = t^{-1/2}$ and $c \sim t^{-1}$ gives $W \sim t^{1/4}$, or a relative width $w = W/k_\mathrm{f} \sim t^{-1/4}$. This suggests the appropriate scaling ansatz for the front region is

$$c_k(t) = \frac{1}{t}X(\xi), \quad \xi = \frac{k - 2\sqrt{t}}{t^{1/4}}\cdot \tag{22}$$

Substituting this ansatz into equation (21) gives a nonlinear single variable integro-differential equation for the scaling function $X(\xi)$. Together with the appropriate boundary conditions, this represents, in principle, a more complete solution to the wealth distribution. However, the essential scaling behavior of the finite-time spreading of the front is already described by equation (22), so that solving for $X(\xi)$ itself does not provide additional scaling information. Analysis of our data by several rudimentary approaches gives $w \sim t^{-\alpha}$ with $\alpha \approx 1/5$. We attribute this discrepancy to the fact that w is obtained by differentiating $\mathcal{C}(x)$, an operation which generally leads to an increase in numerical errors.

2.3 Very greedy exchange

We now consider the variation in which trading occurs at a rate equal to the difference in the assets of the two traders. That is, an individual is more likely to take assets from a much poorer person rather than from someone of slightly less wealth. For this "very greedy" exchange, the corresponding rate equations are

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = c_{k-1}\sum_{j=1}^{k-1}(k-1-j)c_j + c_{k+1}\sum_{j=k+1}^{\infty}(j-k-1)c_j - c_k\sum_{j=1}^{\infty}|k-j|c_j, \tag{23}$$

while the total density obeys

$$\frac{\mathrm{dN}}{\mathrm{d}t} = -c_1(1-N), \tag{24}$$

under the assumption that the (conserved) total wealth density is set equal to one, $\sum kc_k = 1$.

These rate equations can again be solved by applying scaling. For this purpose, it is first expedient to rewrite the rate equations as

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = (c_{k-1} - c_k)\sum_{j=1}^{k-1}(k-j)c_j - c_{k-1}\sum_{j=1}^{k-1}c_j + (c_{k+1} - c_k)\sum_{j=k+1}^{\infty}(j-k)c_j - c_{k+1}\sum_{j=k+1}^{\infty}c_j. \tag{25}$$

Taking the continuum limit gives

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial k} - N\frac{\partial}{\partial k}(kc). \tag{26}$$

We now substitute the scaling ansatz, equation (15), to yield

$$\mathcal{C}(0)[2\mathcal{C}+x\mathcal{C}'] = (x-1)\mathcal{C}'+\mathcal{C}, \tag{27}$$

and

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathcal{C}(0)N^2. \tag{28}$$

Solving the above equations gives $N \simeq [\mathcal{C}(0)t]^{-1}$ and

$$\mathcal{C}(x) = (1+\mu)(1+\mu x)^{-2-1/\mu}, \tag{29}$$

with $\mu = \mathcal{C}(0)-1$. It may readily be verified that this expression for $\mathcal{C}(x)$ satisfies both integral relations of equation (18). The scaling approach has thus found a family of solutions which are parameterized by μ, and additional information is needed to determine which of these solutions is appropriate for our system. For this purpose, note that equation (29) exhibits different behaviors depending on the sign of μ. When $\mu > 0$, there is an extended non-universal power-law distribution, while for $\mu = 0$ the solution is the pure exponential, $\mathcal{C}(x) = \mathrm{e}^{-x}$. These solutions may be rejected because the wealth distribution cannot extend over an unbounded domain if the initial wealth extends over a finite range.

The accessible solutions therefore correspond to $-1 < \mu < 0$, where the distribution is compact and finite, with $\mathcal{C}(x) \equiv 0$ for $x \geq x_{\mathrm{f}} = -\mu^{-1}$. To determine the true solution, let us re-examine the continuum form of the rate equation (26). From naive power counting, the first two terms are asymptotically dominant and they give a propagating front with k_{f} exactly equal to t. Consequently, the scaled location of the front is given by $x_{\mathrm{f}} = Nk_{\mathrm{f}}$. Now the result $N \simeq [\mathcal{C}(0)t]^{-1}$ gives $x_{\mathrm{f}} = 1/\mathcal{C}(0)$. Comparing this expression with the corresponding value from the scaling approach, $x_{\mathrm{f}} = [1-\mathcal{C}(0)]^{-1}$, selects the value $\mathcal{C}(0) = 1/2$. Remarkably, this scaling solution coincides with the Fermi distribution that found for the case of constant interaction rate. Finally, in terms of the unscaled variables k and t, the wealth distribution is

$$c_k(t) = \begin{cases} 4/t^2, & k < t; \\ 0, & k \geq t. \end{cases} \tag{30}$$

Following the same reasoning as the previous section, the discontinuity in the vicinity of the front is smoothed out by diffusive spreading.

Another interesting feature is that if the interaction rate is sufficiently greedy, "gelation" occurs [13], whereby a finite fraction of the total assets are possessed by a single individual. For interaction rates, or kernels $K(j,k)$ between individuals with assets j and k which do not give rise to gelation, the total density typically varies as a power law in time, while for gelling kernels $N(t)$ goes to zero at some finite time. At the border between these regimes $N(t)$ typically decays exponentially in time [11,13]. We seek a similar transition in behavior for the asset exchange model by considering the rate equation for the density

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -c_1\sum_{k=1}^{\infty} K(1,k)c_k. \tag{31}$$

For the family of kernels with $K(1,k) \sim k^{\nu}$ as $k \to \infty$, substitution of the scaling ansatz gives $\dot{N} \sim -N^{3-\nu}$. Thus $N(t)$ exhibits a power-law behavior $N \sim t^{1/(2-\nu)}$ for $\nu < 2$ and an exponential behavior for $\nu = 2$. Thus gelation should arise for $\nu > 2$.

3 Multiplicative asset exchange

We have thus far focused on additive processes in which the amount of assets exchanged in a two-body interaction is fixed. This leads to the unrealistic feature of a vanishing density of active traders in the long time limit, as an individual who possesses the minimal amount of wealth loses all assets in an unfavorable trade. In many economic transactions, however, the amount of assets exchanged is a finite fraction of the initial assets of one of the participants. This observation motivates us to consider asset exchange models with exactly this multiplicative property. A simple realization which preserves both the number of participants and the total assets is the reaction scheme $(x,y) \to (x-\alpha x, y+\alpha x)$. Here $0 < \alpha < 1$ represents the fraction of loser's assets which are gained by the winner. In this process, the assets of any individual remains non-zero, although it can become vanishingly small.

In the following, we consider the cases of random exchange, where the winner may equally likely be the richer or the poorer of the two traders, and greedy exchange, where only the richer individual profits in the trade. The former system quickly reaches a steady state, while the latter gives rise to a continuously-evolving power-law wealth distribution.

3.1 Random exchange

To determine the rate equation for random multiplicative exchange, it is expedient to first write an integral form of the equation, for which the origin of the various terms is clear. This rate equation is

$$\frac{\partial c(x)}{\partial t} = \frac{1}{2}\int\!\!\int \mathrm{d}y\,\mathrm{d}z\, c(y)c(z) \times [-\delta(x-z) - \delta(x-y) + \delta(y(1-\alpha)-x) + \delta(z+\alpha y - x)]. \tag{32}$$

The first two terms account for the loss of $c(x)$ due to the interaction of an individual of assets x. The next term accounts for the gain in $c(x)$ by the losing interaction $(x/(1-\alpha), y) \to (x, y+\alpha x/(1-\alpha))$. The last term also accounts for gain in $c(x)$ by the profitable interaction $(y, x-\alpha y) \to (y(1-\alpha), x)$. By integrating over the delta functions, this rate equation reduces to

$$\frac{\partial c(x)}{\partial t} = -c(x) + \frac{1}{2(1-\alpha)}\,c\left(\frac{x}{1-\alpha}\right) + \frac{1}{2\alpha}\int_0^x \mathrm{d}y\, c(y)\, c\left(\frac{x-y}{\alpha}\right), \tag{33}$$

where the total density is set equal to one. In this form, the rate equation describes a diffusive-like process on a logarithmic scale, except that the third term, which corresponds hopping to the right, is non-local and two-body in character.

To help understand the nature of the resulting wealth distribution, let us first consider the moments, $M_n(t) \equiv \int_0^\infty \mathrm{d}x\, x^n c(x,t)$. From equation (33) one can straightforwardly verify that the first two moments, M_0 and M_1, the population and wealth densities, respectively, are conserved. Without loss of generality, we choose $M_0 = 1$ and $M_1 = M$. The equation of motion for the second moment is

$$\frac{\mathrm{d}M_2(t)}{\mathrm{d}t} = -\alpha(1-\alpha)M_2(t) + \alpha M^2 \tag{34}$$

with solution

$$M_2(t) = \frac{M^2}{1-\alpha} + \left[M_2(0) - \frac{M^2}{1-\alpha}\right] \mathrm{e}^{-\alpha(1-\alpha)t}. \tag{35}$$

Similarly, higher moments also exhibit exponential convergence to constant values, so that the wealth distribution approaches a steady state. The mechanism for this steady state is simply that the typical size of a profitable interaction is likely to be much smaller than an unprofitable interaction for a rich individual, while the opposite holds for a poor individual. This bias prevents the unlimited spread of the wealth distribution and stabilizes a steady state.

To determine the steady state wealth distribution, we substitute simple "test" solutions into the rate equations. By this approach, we find that the exponential form $c(x) = B\mathrm{e}^{-bx}$ satisfies the steady-state version of the rate equation (33), iff $\alpha = \frac{1}{2}$ and $B = b$. Thus when the winner receives one half the assets of the loser, the exact steady wealth distribution is a simple exponential $c(x) = M^{-1}\exp(-x/M)$. For general $0 < \alpha < 1$, the large-α tail is again an exponential, $c(x) \simeq 2b(1-\alpha)\mathrm{e}^{-bx}$. However, for $x \ll 1$, we find, by substitution and applying dominant balance, that $c(x) \sim x^\lambda$ is the asymptotic solution, with exponent $\lambda = -1 - \ln 2/\ln(1-\alpha)$.

Interestingly, λ is positive when $\alpha < 1/2$, so that the density of the poor is vanishingly small. A heuristic justification for this phenomenon is that for $\alpha < 1/2$ an unfavorable interaction leads to a relatively small asset loss, and this loss is more than compensated for by favorable interactions so that a poor individual has the possibility of climbing out of poverty. In the opposite case of $\alpha > 1/2$, unprofitable interactions are sufficiently devastating that a large and persistent underclass is formed, with a power-law divergence in the number of poor in the limit of vanishing wealth.

Our simulations substantially confirm these results (Figure 2). The scope of the simulations is less than that in additive processes, since the number of traders remains fixed, so that CPU time scales linearly in the simulation time. In contrast, for additive exchange, the CPU time scale as $\int^t \mathrm{d}t'\, N(t')$, which can be much smaller than t.

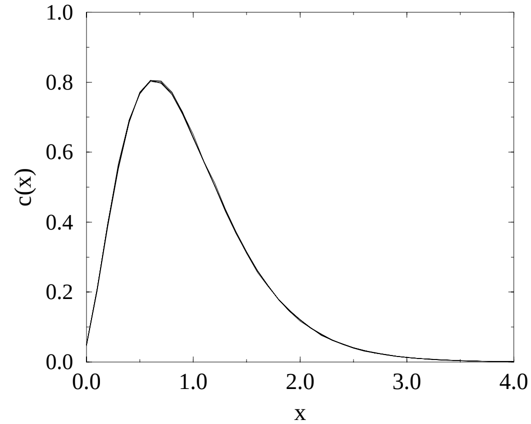

Fig. 2. Representative results for the wealth distribution in random multiplicative exchange for the case $\alpha = 0.25$ based on simulation of 10 configurations of 10^5 traders. Shown are the steady-state wealth distributions $c(x)$ as a function of wealth x for $t = 1.5^n$, with $n = 6$, 8, 10, and 12. The various curves are indistinguishable. The predicted $x^{1.409\ldots}$ small-x tail is not resolvable because of the coarseness of the data binning.

Numerically, we find that the moments $M_n(t)$ quickly converge to equilibrium values. The resulting wealth distribution is clearly a simple exponential for $\alpha = 1/2$ and exhibits either a power-law divergence or a power-law zero as $x \to 0$ for $\alpha > 1/2$ and $\alpha < 1/2$, respectively, in agreement with our analytical results.

3.2 Greedy exchange

Parallel to our discussion of additive processes, we now investigate greedy multiplicative exchange, where only the richer trader profits, as represented by the reaction $(x, y) \to (x - \alpha x, y + \alpha x)$ for $x < y$. Following the same reasoning as that used in the previous subsection, the rate equation for greedy multiplicative exchange is

$$\frac{\partial c(x)}{\partial t} = -c(x) + \frac{1}{1-\alpha}\, c\left(\frac{x}{1-\alpha}\right) N\left(\frac{x}{1-\alpha}\right) + \frac{1}{\alpha}\int_{x/(1+\alpha)}^{x} \mathrm{d}y\, c(y)\, c\left(\frac{x-y}{\alpha}\right), \tag{36}$$

where $N(x) = \int_x^\infty \mathrm{d}z\, c(z)$ is the population density whose wealth exceeds x.

Numerical simulations of this system show that the wealth distribution evolves *ad infinitum* and that the most of the population eventually becomes impoverished (Fig.3). Note that the discreteness of our linear data binning lumps the poorest into a single bin at the origin which is not visible on the double logarithmic scale. The pervasive impoverishment arises because greedy exchange causes the poor to become poorer and the rich to become richer, but wealth conservation implies that there must be many more poor than rich individuals. In the long-time limit therefore, a small fraction of the population possesses most of the wealth.

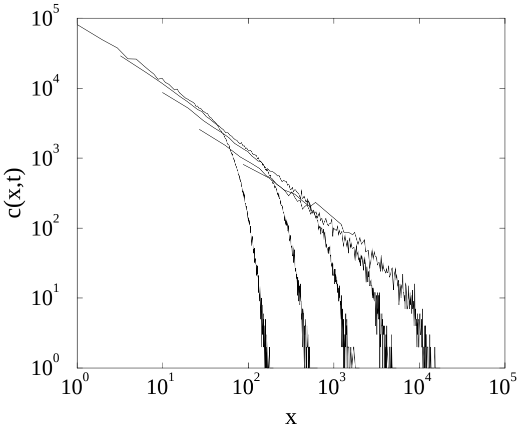

Fig. 3. The unnormalized wealth distribution in greedy multiplicative exchange for the case $\alpha = 0.5$ based on simulation of 10 configurations of 10^5 traders. Shown on a double logarithmic scale are the wealth distributions $c(x)$ as a function of wealth x for $t = 1.5^n$, with $n = 7$, 10, 13, and 16.

To understand these features analytically, first consider the extreme case of $\alpha = 1$ which reduces to classical constant kernel aggregation [10], except for the added feature that individuals of zero wealth are now included in the distribution. Consequently, the scaling form of the wealth distribution in the long time limit is

$$c(x,t) \simeq t^{-2}\mathrm{e}^{-x/t} + (1 - t^{-1})\delta(x). \qquad (37)$$

The first term is just the scaling solution to constant-kernel aggregation [10], so that the delta function term represents the population with zero wealth.

For general $0 < \alpha < 1$ a qualitatively related distribution can be anticipated which consists of a large impoverished class of negligible wealth and a much smaller and widely distributed population of wealthy. To determine this distribution, it is expedient to re-write equation (36) as

$$\frac{\partial c(x)}{\partial t} = \int_0^{x/(1+\alpha)} \mathrm{d}z\, c(z)\,[c(x-\alpha z) - c(x)] \qquad (38)$$
$$-c(x) N\left(\frac{x}{1+\alpha}\right) + \frac{1}{1-\alpha}\, c\left(\frac{x}{1-\alpha}\right) N\left(\frac{x}{1-\alpha}\right).$$

Since Figure 3 indicates that the wealth distribution is a power law, we substitute such a form in equation (38) and find that

$$c(x,t) = \frac{A}{xt}, \quad A = -\frac{1}{\ln(1-\alpha)} \qquad (39)$$

is an exact formal solution. A pathology of equation (39), however, is that all moments, $M_n(t)$, diverge. Note also that the last two terms on the right-hand side of equation (38) are divergent (although their difference is regularized to the finite value $\frac{A^2}{xt^2}\ln\frac{1+\alpha}{1-\alpha}$). These observations suggest that a solution to equation (38), which arises from any initial condition, converges to equation (39) only in the scaling region $x_1(t) < x < x_2(t)$. Outside this domain such a solution has not yet had time to become established.

To estimate these cutoffs for the scaling region, we evaluate the moments based on equation (39)

$$M_0(t) \sim \int_{x_1}^{x_2} \mathrm{d}x\, c(x,t) \sim \frac{A\ln(x_2/x_1)}{t}$$
$$M_1(t) \sim \int_{x_1}^{x_2} \mathrm{d}x\, xc(x,t) \sim \frac{Ax_2}{t}. \qquad (40)$$

Since $M_0 = 1$ and $M_1 = M$ are constant, we obtain

$$x_1(t) \sim \mathrm{e}^{-t/A} = (1-\alpha)^t, \quad x_2(t) \propto t. \qquad (41)$$

The factor $x_1(t)$ clearly gives the wealth of the poorest at time t. Since a losing interaction leads to a reduction in assets by a factor $(1-\alpha)$, the poorest individual at time t will have assets $(1-\alpha)^t$ for the monodisperse initial wealth distribution, $c_0(x) = \delta(x-1)$, and a constant reaction rate. To understand the significance of the upper cutoff, suppose that $x \gg t$. In this case, the last two terms on the right-hand side of (38) are negligible. In the remaining term, the expression in the square brackets may be replaced by $\alpha z \frac{\partial c(x,t)}{\partial x}$, so that the resulting integral is simply equal to M. With these simplifications, the rate equation reduces to $c_t + \alpha M c_x = 0$. This linear wave equation admits the general solution $c(x,t) = c_0(x - \alpha M t)$ and suggests the upper cutoff $x_2(t) \simeq \alpha M t$, consistent with equation (41).

Note, however, that there is inconsistency in our reasoning, as starting with the assumption $x \gg t$ leads to an upper cutoff of order t. In spite of this logical shortcoming, we have verified many of the resulting quantitative characteristics. For example, numerical simulation clearly yields the $1/x$ power-law tail of the wealth distribution. Furthermore, if one defines the wealthy as those whose assets exceeds some threshold ϵ, then equations (39) and (41) give the density of wealthy proportional to $\ln(t/\epsilon)/t$. It is in this sense that we can view the wealth distribution as consisting of two components: the wealthy density which is proportional to $\ln(t/\epsilon)/t$ and a complementary density of the poor. Using equation (39) one can also readily determine the behavior of the moments to be $M_n(t) \sim t^{-1}\int_0^t \mathrm{d}x\, x^{n-1} \sim t^{n-1}/n$, for $n > 1$, or equivalently $(nM_n)^{1/(n-1)}$ should grow linearly with time with the same amplitude, as is observed in our simulations (Figure 4). Least-square fits to the data with the first few data points deleted clearly indicate a growth rate which is very nearly linear.

4 Summary and discussion

We have investigated the dynamical evolution of wealth distributions in idealized asset exchange models. These models are motivated by the hypothesis that the wealth distribution of an economically active population is

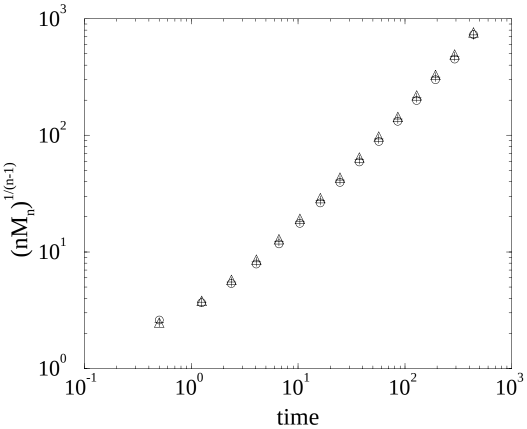

Fig. 4. The reduced moments $(nM_n)^{1/(n-1)}$ for $n = 2$ ($\circ$), 3 (+), and 4 (Δ) *versus* time for greedy multiplicative exchange for the case $\alpha = 0.5$ based on simulation of 10 configurations of 10^5 traders. These reduced moments are predicted to increase linearly in time (see text).

the result of repeated exchanges of assets between randomly-selected trading partners. In spite of the obvious shortcomings associated with considering just one mechanism – asset exchange – among the myriad of factors that influence individual wealth, our models provide realistic wealth distributions for specific exchange rules.

For additive processes, a fixed amount of asset is exchanged in any transaction, with the sense of the exchange being either random – "random" exchange – or favoring the rich – "greedy" exchange. The former leads to a Gaussian wealth distribution, while the latter gives a Fermi-like wealth distribution in the scaling limit. In both cases, the number of economically viable individuals decays as a power law in time and their average wealth correspondingly increases. A "very greedy" process was also introduced in which the trading rate between two individuals is an increasing function of their asset difference. This also leads to a Fermi-like wealth distribution, but with a faster decay in the density of active traders and a concomitant faster growth in their wealth.

There are several potentially fruitful generalizations of additive models. It would be interesting to investigate the analog of gelation [13], in which one individual acquires a finite fraction of the total assets in a finite time. This might be achieved by even more greedy versions of additive exchange. Mathematically, this "gelation" should manifest itself in the violation of wealth conservation, where the loss of wealth in the population of finitely wealthy individuals signals the appearance of an infinitely rich individual. Economically, this singularity may be reflected by the existence of monopolies or oligopolies.

The features of trading a fixed amount of asset and elimination of individuals with no wealth are unrealistic aspects of additive exchange. This motivated our consideration of multiplicative processes, $(x, y) \to (x-\alpha x, y+\alpha x)$, where a fraction of the total wealth of one individual is traded in a transaction. Such a multiplicative rule, particularly for small α, seems to underlie many real economic transactions. Multiplicative exchange leads to non-local rate equations which we have been unable to solve in closed form. Nevertheless, considerable insight was gained by analysis of the moments and the asymptotics of the wealth distribution. For random exchange, individuals with extreme wealth or extreme poverty tend to move towards the center of the distribution and a steady state is quickly reached. For greedy exchange, a continuously evolving power law wealth distribution arises, with $c(x, t) \propto 1/(xt)$, for wealth in the range $(1-\alpha)^t < x < t$. These cutoffs correspond to the poorest and richest individuals, respectively. We also find that a vanishing fraction of people, of order $\ln t/t$, possess an overwhelming amount, of order $t/\ln t$, of the total wealth. At least qualitatively, this corresponds to our naive view of how the wealth distribution evolves in developing countries. For sufficiently greedy exchange, there also seems to be the possibility of the analog of the "shattering" transition in fragmentation processes [21], where "dust" phase arises, which consists of a finite fraction of all individuals who possess no wealth.

A power law wealth distribution is relatively common in modern societies, but with the exponent of the power law typically around 2.5 (see, *e.g.*, [14] and references therein). Various empirical models, which are generally based on an underlying multiplicative process, have been invoked to explain this power law [15–20]. In contrast to these empirical approaches, greedy multiplicative exchange is microscopic (a feature also shared by [20]), and it would be worthwhile to understand if suitable generalizations of our model, which incorporate additional realistic economic features, can quantitatively reproduce the observed data. One such example is the incorporation of wealth redistribution by taxation or welfare. Another realistic aspect is heterogeneity, where the amount exchanged is either different for each individual or depends on some other aspect of the trading event. It would be particularly interesting if the wealth distributions were universal with respect to such heterogeneity.

A final interesting possibility is variable greediness, *e.g.*, the process $(x, y) \to (x - \alpha x, y + \alpha x)$ occurs at a specified rate $K(x, y)$. For example, $K(x, y) = 1/2$ corresponds to random exchange, $K(x, y) = \Theta(y - x)$ (with $\Theta(x)$ the Heaviside step function) to greedy exchange, and $K(x, y) = y/(x + y)$ to a less greedy "proportional" exchange. The latter proportional model can be solved using essentially the same approaches as in the previous section and gives the same scaled wealth distribution as greedy exchange. Perhaps other exchange rules may provide a way to control the exponent of the power law.

Our focus on conserved assets in a trade is also deficient because there is no mechanism for wealth growth. Perhaps this can be bypassed by viewing our models as being in a co-moving reference frame of the average interest rate. One could also allow interactions which increase the assets of the two traders, on average. A simple realization is random asset growth, $(j, k) \to (j + 1, k)$ or $(j, k + 1)$. The rate equations for this model are readily solved and give $c_k(t) \sim t^{-1/2} \exp[-(t - k)^2/2t]$.

This gives an (economically) fair society in which the relative fluctuation $\sqrt{\langle(\Delta k)^2\rangle}/\langle k\rangle$ decreases as $t^{-1/2}$, albeit with an unrealistic linear, rather than exponential, growth of the average wealth. However, exponential growth can be achieved by an interaction rate which equals the sum of assets of the traders, $K(i,j)=i+j$, so that the total wealth $M=\sum kc_k$ obeys $\dot{M}=2NM$. This "ideal" model is also "fair", as the relative fluctuation decreases as $\sqrt{t}\mathrm{e}^{-Nt}$. For an interaction rate which increases even more rapidly with wealth, there is a pathology analogous to gelation – infinite prosperity in a finite time. For example, for $K(i,j)=ij$, the wealth distribution for a monodisperse initial condition is $c_k(t)=(1-t)t^{k-1}$. Related exponential growth occurs in the multiplicative process $(x,y)\to(x,y+\alpha x)$, where for $\alpha=1$, we obtain $c_k(t)\sim \mathrm{e}^{-t}\exp(-x\mathrm{e}^{-t})$. Interestingly, the ideal case lies on the boundary between algebraic wealth growth and the pathological finite-time wealth divergence, a feature similar to the "life on the edge of chaos" advocated by Kauffman as a generic property of complex systems [3].

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Appendix A: Arbitrary spatial dimension

The rate equation description applies for perfectly mixed traders, a feature which should be appropriate for a highly interconnected modern economy. However, in a primitive society with limited communication, one could envision economic transactions occurring in a low dimensional space, with diffusion being the transport mechanism which brings trading partners together. For example, if economic activity were confined to a one-dimensional road, it would be appropriate to consider the spatial dimension $d=1$, while for perfect mixing the effective dimension would be $d=\infty$. From general experience about reaction kinetics, one expects deviations from the rate equation predictions when d is below an upper critical dimension d_c. In this spirit, we now consider the role of spatial dimensionality on the evolution of the wealth distribution in our additive and multiplicative exchange models.

First consider additive exchange. In arbitrary spatial dimension, an interaction occurs whenever two diffusing individuals meet. We make the simplifying (and likely unrealistic) assumption that the diffusivity is independent of an individual's assets. Accordingly, an interaction occurs when $\mathcal{N}\cdot N\approx 1$, where $\mathcal{N}(\tau)$ is the average number of distinct sites visited by a random walk in a time interval τ. This quantity scales as [7]

$$\mathcal{N}(\tau)\sim\begin{cases}\tau^{d/2}, & d<2;\\ \tau/\ln\tau, & d=2;\\ \tau, & d>2;\end{cases} \tag{A.1}$$

as $\tau\to\infty$ and thus gives the following estimates for the density dependence of the time interval between events

$$\tau\sim\begin{cases}N^{-2/d}, & d<2;\\ N^{-1}\ln(1/N), & d=2;\\ N^{-1}, & d>2.\end{cases} \tag{A.2}$$

Since the total density decreases only in events which involve the poorest individuals, we have

$$\frac{\mathrm{d}N}{\mathrm{d}t}\sim-\frac{c_1}{\tau}\,\cdot \tag{A.3}$$

Since we already know how the collision time τ depends on N, we need to express c_1 on N to solve equation (A.3) and complete the solution.

For random additive exchange in $d<2$, rate equations similar to equation (1) should apply, except for the obvious change of the collision rate N by the dimension-dependent rate τ^{-1} from equation (A.2). Thus introducing the modified time variable

$$T=\int_0^t\frac{\mathrm{d}t'}{\tau(t')} \tag{A.4}$$

reduces the governing equations to the pure diffusion equation, as in Section 2. Combining equations (7), (A.2), and (A.4), we find

$$N(t)\sim\begin{cases}t^{-d/2(d+1)}, & d<2;\\ (t/\ln t)^{-1/3}, & d=2;\\ t^{-1/3}, & d>2.\end{cases} \tag{A.5}$$

The wealth distribution is thus $c_k(t)\sim N^2x\mathrm{e}^{-x^2}$ with $N(t)$ given by equation (A.5). In particular, the density of the poorest individuals is proportional to N^3.

For greedy additive exchange in $d<2$, we assume that the scaling ansatz, equation (15), still applies, but with the slightly stronger addition condition $\mathcal{C}(0)>0$. This immediately gives $c_1\sim N^2$. Using this result, together with equation (A.2) in equation (A.3), gives

$$N\sim\begin{cases}t^{-d/(d+2)}, & d<2;\\ (t/\ln t)^{-1/2}, & d=2;\\ t^{-1/2}, & d>2.\end{cases} \tag{A.6}$$

Our results for $N(t)$ given in equations (A.5) and (A.6) also indicate that $d_\mathrm{c}=2$ is the upper critical dimension for additive asset exchange, since it demarcates dimension-independent and dimension-dependent kinetics.

Finally, we consider greedy multiplicative exchange for arbitrary spatial dimension. Because the volume explored by a diffusing individual in time t varies as $t^{d/2}$, it is straightforward to infer that the upper and lower cutoffs in equation (41) are proportional to $x_1(t) \propto \exp(-t^{d/2})$ and $x_2(t) \propto t^{d/2}$, respectively. To then satisfy the constraints that M_0 and M_1 are both constant, we find that the wealth distribution should have the form $c(x,t) \propto 1/(xt^{d/2})$. It is intriguing that the exponent of the power law for greedy multiplicative exchange is dimension independent.

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