Aging and its distribution in coarsening processes

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(Received 19 September 1996)

We investigate the age distribution function $P(\tau,t)$ in prototypical coarsening processes. Here $P(\tau,t)$ is the probability density that in a time interval $(0,t)$ a given site was last crossed by an interface in the coarsening process at time τ. We determine $P(\tau,t)$ exactly in one dimension for the (deterministic) two-velocity ballistic annihilation process and the (stochastic) infinite-state Potts model with zero-temperature Glauber dynamics. Surprisingly, we find that in the scaling limit, $P(\tau,t)$ is identical for these two models. We also show that the average age, i.e., the average time since a site was last visited by an interface, grows linearly with the observation time t. This latter property is also found in the one-dimensional Ising model with zero-temperature Glauber dynamics. We also discuss the age distribution in dimension $d\geqslant 2$ and find similar qualitative features to those in one dimension. [S1063-651X(97)09406-3]

PACS number(s): 02.50.Ga, 05.70.Ln, 05.40.+j

I. INTRODUCTION AND PROBLEM STATEMENT

Coarsening underlies various natural nonequilibrium processes, e.g., phase separation in binary alloys, grain growth, and growth of soap bubbles [1]. A common feature of coarsening phenomena is the scale-invariant morphology that arises in the late stage [1,2]. Such a behavior is a signature of dynamical scaling. If dynamical scaling holds, the average domain size $l(t)$ typically exhibits algebraic growth $l(t)\sim t^{1/z}$.

It has recently been appreciated that knowledge of the dynamical exponent z does *not* provide a comprehensive description of the coarsening dynamics. In particular, the exponent λ, which describes the dependence of the autocorrelation function $A(t)\equiv\langle s(x,0)s(x,t)\rangle$, where $s(x,t)$ is the order parameter at position x and time t, on the average domain size $A(t)\sim l(t)^{-\lambda}$ [2–4], and the exponent θ, which characterizes (in magnetic language) the fraction of spins that have never flipped $P_0(t)\sim t^{-\theta}$ [5–7], were found to be independent of the dynamical exponent z. The latter quantity $P_0(t)$ naturally suggests the generalization to $P_n(t)$, the fraction of spins that have flipped exactly n times up to time t [8] as a detailed and fundamental characterization of the temporal history of spin flips.

In this paper, we probe the temporal history of coarsening processes, at a deeper level than that provided by the autocorrelation function [9,10], by focusing on the time τ when the last spin flip occurs (Fig. 1). More generally, we may introduce $P_n(\tau,t)$ as the probability that a given spin flips n times up to time t *and* that the last spin flip occurs at time τ. Here we investigate $P_+(\tau,t)$, which focuses on the last spin flip and does not specify the total number of flips $P_+(\tau,t)=\Sigma_{n\geqslant 1}P_n(\tau,t)$. If we view a spin as being "reborn" each time it flips, then $P_+(\tau,t)$ gives the density of spins of "age" $t-\tau$. There is also a finite fraction of spins that have not yet flipped; these spins should be treated as spins of age t. The total age distribution density of the spins is therefore

$$P(\tau,t)=P_0(t)\delta(\tau)+P_+(\tau,t). \tag{1}$$

The density $P(\tau,t)$ should satisfy the normalization condition $\int_0^t d\tau P(\tau,t)=1$, while the average age of the system is defined via

$$T=\int_0^t d\tau(t-\tau)P(\tau,t)=tP_0(t)+\int_0^t d\tau(t-\tau)P_+(\tau,t). \tag{2}$$

The age distribution $P(\tau,t)$ will be of primarily importance in systems with history-dependent dynamics, such as glassy systems [11], and in systems with infinite memory where actual aging takes place. Generally, when a two-time correlation function $C(\tau,t)=\langle s(x,\tau)s(x,t)\rangle$ becomes a function of a single variable τ/t, instead of being a function of $t-\tau$ (as in an equilibrium system), this is interpreted as a signature of aging [12–14]. According to this definition, aging is a characteristic of coarsening processes and the scaling

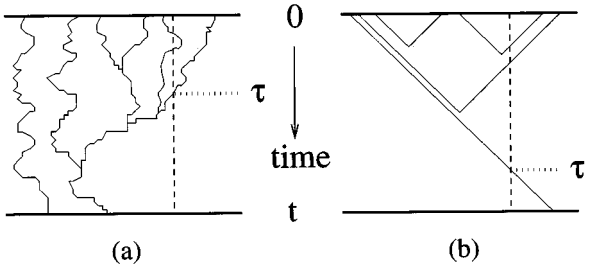

FIG. 1. Graphical definition of $P(\tau,t)$ for one-dimensional coarsening processes. At the point marked by the dashed line, the spin last flips, or, equivalently, is visited by a domain wall, at time τ. The specific examples shown are (a) the infinite-state Potts model (in which the domain walls undergo diffusive single-species coalescence) and (b) the deterministic coarsening of a three-state system with cyclic interactions (in which the domain walls undergo ballistic single-species annihilation).

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dependence $P(\tau,t)\simeq t^{-1}f(\tau/t)$ has been found in a number of pertinent examples [15,16].

The age distribution will also play a fundamental role when the dynamics of a system is explicitly time dependent. A potentially interesting situation is that of the "adaptive" voter model. The conventional voter model [17] is a two-state lattice system in which a voter (site) randomly chooses one of its nearest neighbors and assumes the state of this neighbor. In the adaptive extension of this model the probability that a given voter changes its opinion depends on the local environment (as in the usual voter model) *and* on the time interval since this particular voter last changed its opinion. This might be viewed as a model to describe the increasing conservatism of people when they are not stimulated by contact with those of differing opinions. This adaptive voter model exhibits rather unexpected coarsening dynamics that is ultimately driven by the underlying age distribution [18]. In particular, we find coarsening for all spatial dimensions, while the conventional voter model coarsens only for spatial dimension $d\leqslant 2$.

In Secs. II and III, we consider the age distribution for two specific one-dimensional coarsening processes for which exact results can be obtained. In Sec. II, we first treat a deterministic three-state model of coarsening in which the dynamics of the domain walls is simply that of two-velocity ballistic annihilation in one dimension. Because of this equivalence, it is possible to obtain the exact expression for $P(\tau,t)$ by simple means. In Sec. III, we investigate the age distribution in two stochastic coarsening models. The first is the infinite-state one-dimensional Potts model in which the domain wall dynamics is simply diffusion-limited coalescence process, which may be represented as $A+A\rightarrow A$. We find that the scaling form of the age distribution is identical to that found in the deterministic coarsening process. We also consider the age distribution for the Ising model with zero-temperature Glauber dynamics in which, for one dimension, the domain wall dynamics coincides with single-species diffusion-limited annihilation process, which may be represented as $A+A\rightarrow 0$. In this case, the age distribution has a bimodal "smiling" form as a function of τ, a result that can be understood intuitively. We also discuss the age distribution for the dynamical Ising model in dimension $d\geqslant 2$, for which the qualitative behavior is the same as in one dimension. Finally, we give an exact expression for the age distribution in the mean-field limit. Section IV gives a brief summary and outlook.

II. AGING IN A DETERMINISTIC MODEL OF COARSENING IN ONE DIMENSION

We first examine the age distribution in a deterministic coarsening model that describes phase ordering dynamics in a cyclic one-dimensional system with three equilibrium states A, B, and C. The dynamics is cyclic so that the B phase invades the A phase, C invades B, and A invades C. Corresponding to this dynamics, interfaces between dissimilar domains move toward the subordinate domain with a fixed velocity. A domain that is besieged by two dominant domains shrinks and eventually disappears, leading to the merging of the neighboring domains. The interfaces therefore undergo ballistic motion with annihilation occurring whenever two interfaces meet. These rules are precisely those of the ballistically driven single-species annihilation reaction. The simplicity and rich phenomenology of this reaction has stimulated extensive fundamental work [19–23], as well as related applications to growth processes [24–27], and the dynamics of interacting populations [28–30].

We start by describing the behavior [19,22] of the ballistic annihilation model for the domain walls. In this model, the densities of right-moving and left-moving walls are equal, with velocities that can be taken to be ± 1 without loss of generality. From the exact solution [19], the probability $S(t)$ for an arbitrary interface to survive up to time t is

$$S(t)=e^{-2t}[I_0(2t)+I_1(2t)]. \tag{3}$$

Here I_j denotes the modified Bessel function of order j and the initial spatial distribution of interfaces is assumed to be Poissonian (no correlations), with the initial densities of $\pm$ interfaces taken to be equal 1/2.

To obtain the age distribution for the coarsening process induced by this domain wall dynamics, first consider $P_0(t)$, the fraction of space that has not been crossed by any interface in the time interval $(0,t)$. One can interpret $P_0(t)$ as the probability that a stationary "target" particle, which is placed at the origin, for example, is not hit by any moving domain wall. It is convenient to consider an auxiliary one-sided problem with interfaces distributed only to the right of the origin. For this case, the survival probability of the stationary particle $S_0(t)$ is [21]

$$S_0(t)=e^{-t}[I_0(t)+I_1(t)]. \tag{4}$$

Indeed, the relative velocity between a stationary particle and its reaction partner is a factor of 2 smaller than the relative velocity between two moving reaction partners. Hence $S_0(t)=S(t/2)$ and Eq. (4) follows from Eq. (3). Clearly, the survival probability $P_0(t)$ in the original two-sided problem is

$$P_0(t)=S_0(t)^2. \tag{5}$$

The continuous part of the age distribution $P_+(\tau,t)$ can also be expressed in terms of the survival probabilities $S(t)$ and $S_0(t)$. We first note that for the origin to be crossed by an interface during the time interval $(\tau,\tau+d\tau)$, a left-moving interface should be initially located in the spatial interval $\tau<x<\tau+d\tau$ or a right-moving interface should be located in the spatial interval $-\tau-d\tau<x<-\tau$. Each of these events occurs with probability $d\tau/2$ for an initial interface density of unity.

Suppose that the origin is crossed by a left-moving interface (Fig. 2). Then this interface will ultimately be annihilated with some right-moving interface at some future time t_1, which satisfies $t_1>\tau$. If $t_1>(t+\tau)/2$, then the origin cannot be crossed by a right-moving interface during the time interval (τ,t). The contribution of these type of configurations to $P_+(\tau,t)$ is

$$S\left(\frac{t+\tau}{2}\right)S_0(t-\tau). \tag{6}$$

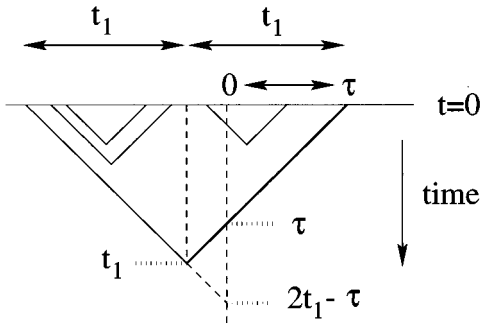

FIG. 2. Illustration of a typical configuration that contributes to $P_+(\tau,t)$ in the deterministic coarsening process generated by domain walls that undergo ballistic single-species annihilation. The left-moving domain wall trajectory that crosses the origin at time τ is shown as a heavy line. This domain wall is annihilated at a time $t_1>\tau$ such that any right-moving trajectory cannot reach the origin before time $t=2t_1-\tau$.

The first factor is just the probability that the left-moving interface survives up to time $(t+\tau)/2$. The latter factor in Eq. (6) is the probability that the initial location of the left-moving interface has not been crossed by any other left-moving interface during the time interval $(0,t-\tau)$, which, in turn, ensures that the origin remains uncrossed from the right during the time interval (τ,t).

Consider now the complementary situation when the left-moving interface that crosses the origin in the time interval $(\tau,\tau+d\tau)$ survives to time t_1 with $\tau<t_1<(t+\tau)/2$. In this case, additional right-moving interfaces can cross the origin before time t. The contribution of such configurations to $P_+(\tau,t)$ is

$$S_0(t-\tau)\int_{\tau}^{(t+\tau)/2} S_0(t-2t_1+\tau)[-\dot{S}(t_1)]dt_1. \qquad (7)$$

Here $S_0(t-\tau)$ again ensures that the origin remains uncrossed from the right during the time interval (τ,t). Similarly, $S_0(t-2t_1+\tau)$ guarantees that the origin remains uncrossed from the left. Finally, $-\dot{S}(t_1)dt_1$ is the probability that the left-moving interface is annihilated in the time interval (t_1,t_1+dt_1). Combining these contributions gives the final exact expression for the age distribution density

$$P(\tau,t)=S_0(t)^2\delta(\tau)+S\left(\frac{t+\tau}{2}\right)S_0(t-\tau) - S_0(t-\tau)\int_{\tau}^{(t+\tau)/2} S_0(t-2t_1+\tau)\dot{S}(t_1)dt_1. \qquad (8)$$

The singular part of the age distribution $S_0(t)^2\delta(\tau)$ corresponds to the fraction of space that has not been traversed by any interface; in the long-time limit, this fraction decays as t^{-1}. To determine the asymptotic behavior of the continuous part of the age distribution, we substitute into Eq. (8) the asymptotic expressions $S(t)\sim 1/\sqrt{\pi t}$ and $S_0(t)\sim\sqrt{2/\pi t}$, which are found by using the asymptotic relations for the modified Bessel functions, $I_j(z)\to e^z/\sqrt{2\pi z}$ as $z\to\infty$ and j fixed [31]. The contribution of the third term of Eq. (8) turns out to be asymptotically negligible, while the second term leads to the scaling form

$$P_+(\tau,t)\simeq t^{-1}f(\xi), \qquad (9)$$

in the scaling limit

$$t\to\infty, \quad \tau\to\infty, \quad \xi=\tau/t, \qquad (10)$$

with the scaling function given by

$$f(\xi)=\frac{2}{\pi}\frac{1}{\sqrt{1-\xi^2}}. \qquad (11)$$

A prominent feature of the age distribution is that τ scales as t. That is, the average age

$$T=\langle t-\tau\rangle\simeq t\int_0^1 d\xi(1-\xi)f(\xi)=\left(1-\frac{2}{\pi}\right)t \qquad (12)$$

grows linearly with the observation time t.

III. AGING IN STOCHASTIC MODELS OF COARSENING

A. One dimension

The ballistic annihilation model is perhaps the simplest one-dimensional coarsening process with *deterministic* dynamics. We now consider simple examples of one-dimensional coarsening processes with *stochastic* dynamics. Consider first the q-state Potts model for $q=\infty$, with zero-temperature Glauber dynamics and with the initial condition where each spin is in a different state. The dynamics proceeds as follows: During the time interval dt a given spin assumes the state of one of its nearest neighbor with overall probability $dt/2$. In one dimension, the interfaces between domains of identical spins therefore diffuse and coalesce whenever two domains meet. The domain wall dynamics is thus identical to the diffusion-limited coalescence reaction, which may be represented as $A+A\to A$.

Because of this equivalence between the Potts model and the coalescence reaction, the age distribution can be calculated exactly. Since interfaces coalesce upon colliding, only the interfaces that are the nearest neighbors of a particular site are important in determining its age distribution. In constructing the age distribution, first note that the spin will not change its color up to time t if neither of the two neighboring interfaces reaches the spin. The probability $P_0(t)$ is thus equivalent to the square of the probability $Q(t,1)$ that a random walker on a lattice starting at position $x_0=1$ will not reach the origin up to time t. The probability $Q(t,1)$ is readily computable [32] and gives the fraction of "persistent" spins

$$P_0(t)=\{e^{-t}[I_0(t)+I_1(t)]\}^2. \qquad (13)$$

To compute the contribution to the age distribution from configurations where an interface has previously reached the spin (which we may take to be at the origin), let us assume that this spin takes on a new color from its left neighbor at time τ. This spin is now the right extremity of a domain of same color spins (see Fig. 3).

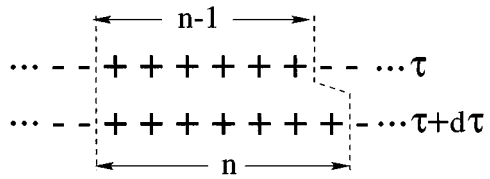

FIG. 3. Illustration of one process that enters in the computation of $P_+(\tau,t)$ for the infinite-state Potts model. Shown is the spin configuration at times τ and $\tau+d\tau$ just as one spin changes its state. For the state of this spin to remain unchanged until time t, both the domain wall a distance 1 to the right and the domain wall a distance n to the left must not reach the position of the newly flipped spin.

Let the size of this domain be n. The position of the interface that defines the left edge of this domain is distributed according to the domain size distribution $F(n-1,\tau)$. The spin at the origin will then not change its color up to time t if the two surrounding interfaces do not cross the origin. The continuous part of the age distribution can thus be written as

$$P_+(\tau,t)=\sum_{n=2}^{\infty} F(n-1,\tau)Q(t-\tau,n)Q(t-\tau,1). \quad (14)$$

The last factor is just the probability that the domain that is one lattice spacing to the right of the spin at the origin does not reach the origin between time τ and time t, while the first two factors give the corresponding probability for the left-neighboring domain that is a distance n from the origin.

Each of the factors in this equation are well known. The domain size distribution is given by $F(n-1,\tau)=E(n-1,\tau)-2E(n,\tau)+E(n+1,\tau)$, where $E(k,t)$ is the probability to find at least k successive spins of the same color at time t [33]. For a discrete lattice system, this latter distribution satisfies a lattice diffusion equation, with boundary condition $E(0,t)=1$ and initial condition $E(k,0)=\delta_{k,0}$, corresponding to the initial condition where each spin is different. The expression for $E(k,t)$ is [32]

$$E(k,t)=1-e^{-2t}\left[I_0(2t)+2\sum_{j=1}^{k-1} I_j(2t)+I_k(2t)\right] \quad (15)$$

and thus

$$F(n-1,\tau)=\frac{e^{-2\tau}}{\tau}nI_n(2\tau). \quad (16)$$

In a similar vein, the probability $Q(t,k)$ that a random walker that starts at $x=k$ does not hit the origin during the time interval $(0,t)$ is [32]

$$Q(t,k)=e^{-t}\left[I_0(t)+2\sum_{j=1}^{k-1} I_j(t)+I_k(t)\right]. \quad (17)$$

So we finally obtain

$$P_+(\tau,t)=\frac{e^{-2t}}{\tau}[I_0(t-\tau)+I_1(t-\tau)]\sum_{n=1}^{\infty} nI_n(\tau) \times\left[I_0(t-\tau)+2\sum_{k=1}^{n-1} I_k(t-\tau)+I_n(t-\tau)\right]. \quad (18)$$

In the scaling limit (10), the dominant contribution to the sum in Eq. (18) is provided by terms with $n\propto\sqrt{t}$. In this region we use the asymptotic form of the Bessel functions $I_n(t)\simeq\exp(t-n^2/2t)/\sqrt{2\pi t}$. A lengthy but elementary computation then yields

$$P_+(\tau,t)\simeq\frac{2}{\pi\sqrt{t^2-\tau^2}}, \quad (19)$$

which is exactly of the same form as Eqs. (10) and (11). At first sight, it may seem surprising to find the same scaling function, as well as the same expression for $P_0(t)$, as in the ballistic annihilation problem. Indeed, Eq. (4) can be computed from a mapping of the initial distribution of the interfaces onto a random-walk process. $S_0(t)$ can then be computed in the same way as the probability $Q(t,1)$ shown above. Whenever we can determine a property of the infinite-state Potts model via the behavior of two independent random walks, we should recover the same results as in the ballistic annihilation problem. Nevertheless, some properties of these two systems are very different. For example, the domain size distribution in ballistic annihilation exhibits a nontrivial behavior that is characterized by an infinite number of singularities [22,34].

Let us now consider the age distribution of spins in the two-state Potts model with zero-temperature spin-flip dynamics, i.e., the kinetic Ising-Glauber model [32]. Since the solution for $P_0(t)$ in the Ising-Glauber model is difficult [35], one can anticipate that calculation of $P_+(\tau,t)$ is also subtle. We therefore study this problem numerically and give heuristic arguments to explain the limiting behaviors of the age distribution $P_+(\tau,t)$.

Our numerical results, which are based on simulations of the equivalent $A+A\rightarrow 0$ reaction process, confirm that the scaling ansatz (9)and (10) still applies (Fig. 4).

The singular behavior of the scaling part of the age distribution function $f(\xi)$ in the limits $\xi\downarrow 0$ and $\xi\uparrow 1$ can be accounted for by matching to the known behaviors in these limits. When $\tau=O(1)$, $P_+(\tau,t)\sim P_0(t)\sim t^{-3/8}$ [35]. Matching this with Eq. (9) at $\xi=\tau/t=O(t^{-1})$ implies the $f(\xi)\sim\xi^{-5/8}$ as $\xi\downarrow 0$. This asymptotic behavior agrees well with our simulations. In the opposite limit of $\tau\rightarrow t$, the corresponding limiting form of the age distribution is determined by domain walls that have crossed the origin at time τ close to t: This happens with probability $t^{-1/2}$ since the number of domain walls decreases with time as $t^{-1/2}$ [32]. The diffusing domain wall should then not cross the origin again in the following time interval (τ,t): This happens with probability $(t-\tau)^{-1/2}$ [36]. Thus $P(\tau,t)\sim t^{-1/2}(t-\tau)^{-1/2}$, which implies that $f(\xi)\sim(1-\xi)^{-1/2}$ as $\xi\uparrow 1$, in agreement with our numerical results. Indeed, the product of these two asymptotic forms $f_{\rm guess}(\xi)=\mathcal{B}\xi^{-5/8}(1-\xi)^{-1/2}$ provides a

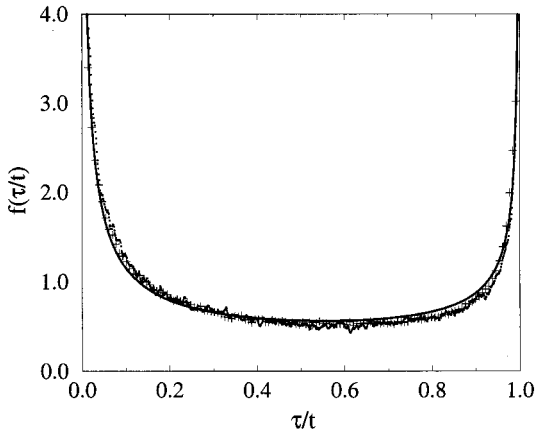

FIG. 4. Simulation data for the age distribution in the one-dimensional Ising-Glauber model. Shown is the scaling function $f(\xi)$ versus ξ for $t=(1.5)^{12}$ (+) and $t=(1.5)^{17}$ (○), with the latter data averaged (smoothed) over five consecutive points. The solid line is the guess $f_{\rm guess}(\xi)=\mathcal{B}\xi^{-5/8}(1-\xi)^{-1/2}$, with $\mathcal{B}=0.259\,349\ldots$, as explained in the text.

reasonable fit to the data over most of the range of ξ. If one uses this guess over the entire range of ξ, then the normalization condition $\int d\xi\, f_{\rm guess}(\xi)=1$, requires the numerical prefactor to be $\mathcal{B}=\Gamma(7/8)/\Gamma(3/8)\Gamma(1/2)=0.259\,349\ldots$.

For the general q-state Potts model with zero-temperature Glauber dynamics, we may also expect that the age distribution scales, with the limiting behaviors of the scaling function given by

$$f(\xi)\sim\begin{cases}\xi^{\theta(q)-1}, & \xi\downarrow 0\\ (1-\xi)^{-1/2}, & \xi\uparrow 1.\end{cases}\qquad(20)$$

The persistence exponent $\theta(q)$, found analytically in Ref. [35], increases from 3/8 to 1 as q increases from 2 to ∞. Thus the smiling form of the age distribution in the Ising case (Fig. 4) gradually transforms into the half-smiling form of the infinite-state model [see Eq. (19)].

B. Higher dimensions

In more than one dimension, aging of spins in the kinetic Ising model is expected to depend on the temperature. If an initially disordered system is quenched to a final temperature $T_f>0$, the average age is expected to be finite for all $d>1$. This follows because for nonconserved dynamics, even spins embedded within a large region of aligned spins will flip at a finite rate for all positive temperatures. On the other hand, for a quench to zero temperature, we anticipate that the average age will grow with time since spin flips can occur only at interfaces and these eventually disappear. To test this expectation, we performed numerical simulations of the two-dimensional kinetic Ising-Glauber model on the square lattice and found that the average age of the spins grows linearly in time and that scaling still applies. Moreover, the age distribution function has the same qualitative smiling form of the one-dimensional system (Fig. 4).

In the small-age limit $t-\tau\ll t$, the numerical data suggests a behavior of the age distribution that is consistent with $P(\tau,t)\sim t^{-1/2}(t-\tau)^{-1/2}$. To understand this result, which is identical to that of the one-dimensional counterpart, first note that the density of domain walls decays as $t^{-1/2}$. This arises because for nonconserved dynamics, the average domain size grows as $t^{1/2}$ [2] and domains appear to be compact. Consequently, the domain wall density is expected to be the reciprocal of the average domain size. The perimeter of a domain has typically a vicinal shape, with the kinks and antikinks that define terraces undergoing diffusive motion (this diffusion does not cost energy and is therefore allowed at zero temperature). This diffusional motion is one dimensional in character and thus a step (either kink or antikink) that has crossed a bond at time τ will not cross it again in the following time interval (τ,t) with probability $(t-\tau)^{-1/2}$. The age distribution is then given by the product of step density and the above no-return probability, which gives $P(\tau,t)\sim t^{-1/2}(t-\tau)^{-1/2}$. In fact, the evolution of interfaces is a much more involved process—kinks and antikinks annihilate upon colliding and spin flips at the corner give birth to a pair of steps (horizontal and vertical)—but in the small-age limit these additional complexities should not qualitatively affect the age distribution.

In the large-age limit $\tau\ll t$, the scaled age distribution is expected to behave as $f(\xi)\sim\xi^{\theta-1}$, similarly to one dimension. Indeed, we confirmed numerically such a power-law behavior and found that $\theta\approx 0.21$ provides the best fit to our data. This is consistent with previous simulations of the two-dimensional Ising-Glauber model for which the fraction of persistent spins $P_0(t)$ was found to decay as $t^{-0.22}$ [5,37].

To determine the form of the age distribution for the kinetic Ising-Glauber model in higher dimensions, we apply a mean-field approach. It is simple to solve for $P(\tau,t)$ in the mean-field limit (e.g., for the Ising model on a complete graph) since the dynamics in the zero-temperature case is simple: Spins from the majority phase do not change their state, while spins from the minority phase change their state with a constant rate that we may set equal to one. Suppose that the system starts from an initial condition where the fraction of + and − spins is equal to p and $q=1-p$, respectively (with $p\geq q$ without loss of generality). Clearly, the fraction of spins that never change their state until time t is equal to $p+qe^{-t}$. The probability that a minority spin changes its state in the time interval $(\tau,\tau+d\tau)$ is equal to $e^{-\tau}d\tau$. Thus

$$P(\tau,t)=(p+qe^{-t})\,\delta(\tau)+qe^{-\tau}.\qquad(21)$$

This result violates the scaling form of Eq. (9), but still implies that the average age [see Eq. (2)] increases linearly in time:

$$T=(p+qe^{-t})t+q(t-1+e^{-t})=t-q(1-e^{-t}-te^{-t}).\qquad(22)$$

IV. SUMMARY AND OUTLOOK

The age distribution in coarsening processes with a nonconserved order parameter has been investigated by analytical and numerical techniques. These approaches indicate that the average age grows linearly with the observation time of the system. Exact results for two prototypical coarsening processes in one dimension, the deterministic ballistic annihilation, and the stochastic infinite-state Potts model with

zero-temperature Glauber dynamics have been obtained. For the general q-state Potts model with zero-temperature Glauber dynamics, asymptotic behaviors have been established.

Various results for the aging of spins in the Ising-Glauber model in general dimension have been obtained. The interesting situation, for nonconserved spin-flip dynamics, is that of zero temperature where domain walls ultimately disappear so that the system undergoes aging. In particular, numerical results in two dimensions were found to be qualitatively similar to corresponding one-dimensional results. We anticipate that the bimodal smiling form of the age distribution will arise for all spatial dimension $d<4$. When $d\geq 4$, however, the age distribution is expected to exhibit features similar to the easily derived mean-field solution [see Eq. (21)]. In particular, the fraction of spins that never flip should saturate at a finite value even in the symmetric case of $p=q=1/2$. This has apparently been observed [37], although it is hard to definitively settle this issue by numerical means, especially in the marginal case of $d=4$.

It is worth noting that for the models discussed in this work, the only possibilities found are systems where the average age saturates to a finite value or where the average age increases linearly in time. The saturation of the age in the first class of systems arises because a steady state is reached. On the other hand, for systems that coarsen it is perhaps worth investigating whether there are examples where the average age grows slower than linear in time. Numerical evidence shows that the average age in the two-dimensional voter model is growing slower than linearly and perhaps logarithmically in time. This intriguing possibility merits further consideration.

For the coarsening processes examined in this work, the dynamics determines the age distribution. It may be instructive to study models with feedback, in which the aging process influences the coarsening dynamics [18]. The adaptive voter model is one such example. Another possibly intriguing extension would be to consider coarsening processes with conservative dynamics.

ACKNOWLEDGMENTS

The research reported here was supported in part by the Swiss National Foundation, by the ARO (Grant No. DAAH04-93-G-0021), and by the NSF (Grant No. DMR-9219845). We also thank J. F. Mendes for helpful discussions.

[1] A general review of coarsening is given by J. S. Langer, in *Solids Far From Equilibrium*, edited by C. Godrèche (Cambridge University Press, Cambridge, 1992).
[2] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[3] D. S. Fisher and D. A. Huse, Phys. Rev. B **38**, 373 (1988).
[4] C. Yeung, M. Rao, and R. C. Desai, Phys. Rev. E **53**, 3073 (1996).
[5] B. Derrida, A. J. Bray, and C. Godrèche, J. Phys. A **27**, L357 (1994).
[6] P. L. Krapivsky, E. Ben-Naim, and S. Redner, Phys. Rev. E **50**, 2474 (1994).
[7] A. J. Bray, B. Derrida, and C. Godrèche, Europhys. Lett. **27**, 175 (1994).
[8] E. Ben-Naim, L. Frachebourg, and P. L. Krapivsky, Phys. Rev. E **53**, 3078 (1996).
[9] L. F. Cugliandolo and D. S. Dean, J. Phys. A **28**, 4213 (1995).
[10] A. Barrat, R. Burioni, and M. Mézard, J. Phys. A **29**, 1311 (1996).
[11] C. A. Angell, Science **267**, 1924 (1995); H. Rieger, in *Annual Review of Computers in Physics II*, edited by D. Stauffer (World Scientific, Singapore, 1996).
[12] C. Bachas and B. Haberman, Phys. Rev. Lett. **57**, 1985 (1986).
[13] M. Alba, M. Ocio, and J. Hammann, Europhys. Lett. **2**, 45 (1986).
[14] J. P. Bouchaud, J. Phys. (France) I **2**, 1705 (1992); C. Godrèche and J. M. Luck, J. Phys. A **29**, 1915 (1996).
[15] N. V. Brilliantov, P. L. Krapivsky, and Yu. A. Andrienko, J. Phys. A **27**, L381 (1994).
[16] S. Ispolatov, P. L. Krapivsky, and S. Redner, Phys. Rev. E **54**, 1274 (1996).
[17] T. M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985)
[18] L. Frachebourg, P. L. Krapivsky, J. F. Mendes, and S. Redner (unpublished).
[19] Y. Elskens and H. L. Frisch, Phys. Rev. A **31**, 3812 (1985).
[20] E. Ben-Naim, S. Redner and F. Leyvraz, Phys. Rev. Lett. **70**, 1890 (1993).
[21] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. E **51**, 3977 (1995).
[22] J. Piasecki, Phys. Rev. E **51**, 5535 (1995).
[23] M. Droz, P.-A. Rey, L. Frachebourg, and J. Piasecki, Phys. Rev. E **51**, 5541 (1995); Phys. Rev. Lett. **75**, 160 (1995).
[24] J. Krug and H. Spohn, Phys. Rev. A **38**, 4271 (1988).
[25] K. Sekimoto, Int. J. Mod. Phys. B **5**, 1843 (1991).
[26] C. H. Bennett, M. Büttiker, R. Landauer, and H. Thomas, J. Stat. Phys. **24**, 419 (1981).
[27] N. Goldenfeld, J. Phys. A **17**, 2807 (1984).
[28] M. Bramson and D. Griffeath, Ann. Prob. **17**, 26 (1989).
[29] R. Fisch, Physica D **45**, 19 (1990); Ann. Prob. **20**, 1528 (1992).
[30] L. Frachebourg, P. L. Krapivsky, and E. Ben-Naim, Phys. Rev. Lett. **76**, 2125 (1996); Phys. Rev. E **54**. 6186 (1996).
[31] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, Singapore, 1984).
[32] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).
[33] D. ben-Avraham, M. A. Burshka, and C. Doering, J. Stat. Phys. **60**, 695 (1990).
[34] L. Frachebourg and P. L. Krapivsky, Phys. Rev. E **55**, 252 (1997).
[35] B. Derrida, V. Hakim, and V. Pasquier, Phys. Rev. Lett. **75**, 751 (1995); J. Stat. Phys. **85**, 763 (1996).
[36] W. Feller, *An Introduction to Probability Theory, Vols. 1 and 2* (Wiley, New York, 1971).
[37] D. Stauffer, J. Phys. A **27**, 5029 (1994).