A Tale of Two (and More) Altruists

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Abstract.
We introduce a minimalist dynamical model of wealth evolution and wealth sharing among N agents as a platform to compare the relative merits of altruism and individualism. In our model, the wealth of each agent independently evolves by diffusion. For a population of altruists, whenever any agent reaches zero wealth (that is, the agent goes bankrupt), the remaining wealth of the other $N-1$ agents is equally shared among all. The population is collectively defined to be bankrupt when its total wealth falls below a specified small threshold value. For individualists, each time an agent goes bankrupt (s)he is considered to be "dead" and no wealth redistribution occurs. We determine the evolution of wealth in these two societies. Altruism leads to more global median wealth at early times; eventually, however, the longest-lived individualists accumulate most of the wealth and are richer and more long lived than the altruists.

1. Introduction

Through the best of times and the worst of times, wealth inequality has persisted in modern societies [1–3]. There is a long-standing debate about how to deal with this phenomenon. An individualistic viewpoint is that each person should try to maximize her or his individual wealth without external constraints, and this opportunistic perspective is best for society as a whole because those who thrive economically can serve as wealth creators for others. An extreme altruistic viewpoint is that wealth should be shared equally by all. Naively, altruism would seem to forestall individual penury. On the other hand, it might be argued that wealth sharing stifles individual entrepreneurship, which ultimately leads to less wealth for society as a whole.

We have no pretense that we are seriously addressing these vital issues; instead, our goal is to investigate an idealized but solvable model of wealth evolution and wealth sharing, for which we can make quantitative predictions about the relative merits of altruistic and individualistic strategies. Relevant investigations of this genre includes an unpublished note on the so-called "up the river" problem by Aldous [4], in which N independent Brownian particles all start at $x = 1$ and are absorbed when they hit $x = 0$. A unit drift can be allocated arbitrarily among the surviving particles to maximize the number of particles that survive forever. The goal is to find the optimal allocation of the drift velocities for each particle. Related work by McKean and Shepp [5] investigate this same setting for two agents in which they determine the optimal allocation of drift to maximize overall societal welfare, which is related to the number of surviving agents. A number of intriguing variations of this basic problem have also been pursued [6–8]. The recent work of Abebe et al. [9] is concerned with the optimal allocation of subsidies to agents that experience negative economic shocks. A broader perspective on the same type of question is that of optimal tax-and-redistribution schemes [10, 11].

There is also extensive literature on ruin problems in which the goal is to understand the financial ruin of a single agent or firm that is subject to positive and negative financial influences of various kinds (see, e.g., [12–15]). This perspective is often applied to the financial ruin of an insurance agency: the positive financial influences are the premiums collected, the negative ones are the claims paid to policyholders. Because this class of problems involves a *single* agent that responds to external influences, there is a considerably better-developed analytical understanding of basic phenomenology, and so-called ruin probabilities may be computed in models beyond simple diffusion, such as Lévy processes [16].

While all these articles contain valuable insights, most of the problems mentioned above typically involve considerable technical challenges. Moreover, the nature of the optimal strategy depends on optimization criterion itself. In the work of McKean and Shepp, for example, the optimal allocation of drift depends on whether the goal is to maximize the probability that both agent remain solvent forever or maximize the expected number of agents that remain solvent forever.

In this work, we construct an analytically tractable model of wealth redistribution for which definitive conclusions can be reached with fairly simple calculations. In section 2 we introduce our altruism model. In the following two sections, we solve for the wealth dynamics of two individualists and two altruists, respectively. Then in section 5, we determine whether altruism or individualism is superior by studying both the time dependence of the survival probability—the probability that an agent is still economically viable—and the typical, or median, wealth of each agent. We give some concluding comments in the final section.

2. Altruism Model

Our model of wealth evolution and redistribution is defined as follows:

(i) Two diffusing particles are initially located at x_0 and at y_0 on the positive real axis. Each particle represents a person and its coordinate represents the person's wealth.

(ii) The wealth of each person subsequently evolves by free diffusion.

(iii) If a person's wealth reaches 0, (s)he has gone bankrupt. The other person—an altruist—immediately shares half of her/his current wealth with the bankrupt person. The wealth of the two people again move by free diffusion until the next individual bankruptcy. This cycle of events end when the total wealth shrinks to a specified small level that defines the bankruptcy of both agents.

We may equivalently represent the wealth of the two agents as the diffusion of a single effective particle in two dimensions that starts at (x_0, y_0) and moves in the positive quadrant (Fig. 1). When both agents are solvent, the effective particle satisfies the constraint that both coordinates x and y are positive. Whenever one coordinate hits 0, the particle is reset to the main diagonal along a ray that is perpendicular to this diagonal; this update rule corresponds to equal wealth sharing. There are several basic questions that we will address about this process:

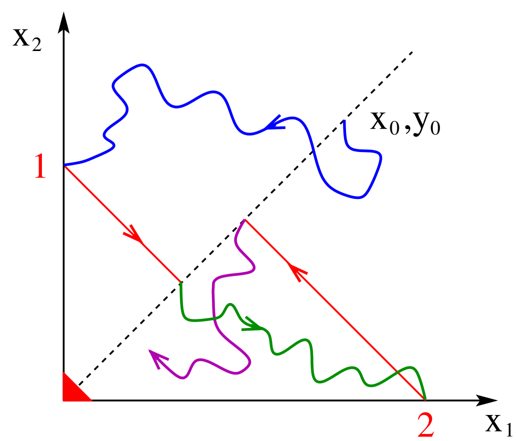


Figure 1. Trajectory of the effective particle in two dimensions that represents the wealth of two altruistic agents. Whenever a bankruptcy occurs (red numbers), where one of the coordinates reaches 0, the trajectory is reset to the main diagonal (dashed) along a ray perpendicular to the diagonal (red); this corresponds to equal-wealth sharing. Joint bankruptcy occurs when the trajectory enters the small triangle near the origin.

(i) When does the each individual bankruptcy occur? At each such bankruptcy, how much wealth does the other individual have?

(ii) What is the wealth dynamics in successive bankruptcies?

(iii) What is the wealth dynamics for more than two people?

In addition to answering these detailed questions about the dynamics, we are also interested in comparing the evolution of wealth in this altruistic society with that in an individualistic society. In the latter, the wealth of each individual also evolves by free diffusion, and whenever one person does go bankrupt, (s)he is considered to be economically "dead", while the wealth of the remaining $N-1$ solvent individuals continue to evolve by free diffusion. The existential question is: which society leads to a better outcome? As we shall see, the answer to this question depends on what is defined as "better". This same issue arises in other wealth sharing models, where the notion of optimality depends on what is actually being optimized [5]. We will first treat the simplest non-trivial case of $N = 2$ agents and then generalize to $N > 2$.

3. Dynamics of Two Diffusing Individualists

As a preliminary, we solve for the wealth dynamics of two individualists. Because diffusion in one dimension is recurrent [17–19], one agent necessarily goes bankrupt and subsequently the other necessarily goes bankrupt. In spite of bankruptcy being certain for both agents, the average time for each individual bankruptcy event is infinite. To find the time of the first bankruptcy, we again represent the wealth evolution as the motion of an effective freely diffusing particle in two dimensions, with the constraint that both x and y must be positive (Fig. 1). By the image method [17, 18], the probability distribution of this effective particle is

$$P(x,y,t) = \frac{1}{4\pi Dt}\left\{e^{-[(x-x_0)^2+(y-y_0)^2]/4Dt} - e^{-[(x+x_0)^2+(y-y_0)^2]/4Dt}\right.$$
$$\left. -e^{-[(x-x_0)^2+(y+y_0)^2]/4Dt} + e^{-[(x+x_0)^2+(y+y_0)^2]/4Dt}\right\}. \qquad (1)$$

This distribution is the sum of the initial Gaussian that starts at (x_0, y_0) plus the contribution of three image Gaussians—two negative images at $(-x_0, y_0)$ and $(x_0, -y_0)$ and one positive image at $(-x_0, -y_0)$—to enforce the condition that the probability vanishes on the quadrant boundaries.

To find the time of the first bankruptcy, we compute the first-passage probability to each boundary. The first-passage probability to the horizontal boundary, where the person with initial wealth y_0 goes bankrupt first, is

$$F_1(x,t|x_0,y_0) = D\,\frac{\partial P(x,y,t)}{\partial y}\bigg|_{y=0}. \qquad (2a)$$

That is, $F(x,t|x_0,y_0)$ is the probability that the first person goes bankrupt at time t and that the second person has wealth $x > 0$ at the time of this bankruptcy, when the initial wealth of the two individuals is (x_0, y_0). For simplicity, we do not write this initial condition dependence in what follows. Performing the derivative and doing some simple rearrangement gives

$$F_1(x,t) = \frac{y_0}{\sqrt{4\pi Dt^3}}\,e^{-y_0^2/4Dt} \times \frac{1}{\sqrt{4\pi Dt}}\left[e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt}\right]. \qquad (2b)$$

This expression is just the product of the first-passage probability for the y-coordinate to reach 0 times the probability that the x-coordinate remains positive until this first passage. Since these two events are independent, the product of these probabilities gives the desired first-passage probability.

Integrating over x, the distribution of times for the first of the two agents to go bankrupt is

$$\phi_1(t) = \int_0^\infty F_1(x,t)\,dx = \frac{y_0}{\sqrt{4\pi D t^3}}\,e^{-y_0^2/4Dt} \times \mathrm{erf}(x_0/\sqrt{4Dt})\,. \tag{3a}$$

Alternatively, this quantity is just the probability that the y coordinate first becomes zero at time t multiplied by the probability that the x coordinate always remains positive up to time t. Using $\mathrm{erf}(z) \simeq 2z/\sqrt{\pi}$ for $z \to 0$, the long-time behavior of $\phi_1(t)$ is

$$\phi_1(t) \simeq \frac{x_0 y_0}{2\pi D t^2} \qquad t \to \infty\,. \tag{3b}$$

As we might anticipate, the distribution of times for the first agent to go bankrupt asymptotically decays faster than the same distribution for a single agent (t^{-2} versus $t^{-3/2}$ [18]). Nevertheless, the average time for this first bankruptcy event is still infinite.

By similar reasoning, the distribution of times for the second person to go bankrupt, $\phi_2(t)$, is

$$\begin{aligned}\phi_2(t) = \int_0^\infty dx \int_0^t dt'\,\frac{y_0}{\sqrt{4\pi D t'^3}}\,e^{-y_0^2/4Dt'} &\times \frac{1}{\sqrt{4\pi D t'}}\left[e^{-(x-x_0)^2/4Dt'} - e^{-(x+x_0)^2 4Dt'}\right]\\ &\times \frac{x}{\sqrt{4\pi D (t-t')^3}}\,e^{-x^2/4D(t-t')}\,.\end{aligned} \tag{4}$$

The leading factor is the probability for the first person, with initial wealth y_0, to go bankrupt at a time $t' < t$. The second factor is the wealth distribution of the second person at the moment of the first bankruptcy. This distribution defines the effective initial condition of the second person. The last factor is the probability that this second person, with wealth x, goes bankrupt in the remaining time $t - t'$. Performing the integrals in (4) gives

$$\phi_2(t) = \frac{x_0}{\sqrt{4\pi D t^3}}\,e^{-x_0^2/4Dt}\,\mathrm{erfc}(y_0/\sqrt{4Dt})\,. \tag{5}$$

The complementary error function gives the probability that one person has already gone bankrupt by time t, while the remaining factor gives the probability of the other agent goes bankrupt at time t. This joint probability $\phi_2(t)$ has the same $t^{-3/2}$ asymptotic behavior as the classic first-passage probability in one dimension [17–19]. We also note that ϕ_1 and ϕ_2 may be derived using order statistics: if one defines T_1 and T_2 as the two independent random variables corresponding to the first passage times to the origin of two independent Brownian motions, then ϕ_1 is the probability distribution of the minimum of T_1 and T_2 and ϕ_2 is the distribution of the maximum.

4. Dynamics of Two Diffusing Altruists

We now incorporate altruism, in which after each individual bankruptcy, the solvent person shares half of her/his wealth with the bankrupt person. This same wealth sharing rule occurs at each subsequent individual bankruptcy. We want to understand how such a wealth-sharing rule influences the time dependence of the collective wealth.

To determine the wealth distribution of the solvent agent, $\mathcal{F}(x)$, at the moment of the first bankruptcy, we integrate $F_1(x,t)$, the time-dependent wealth distribution of the solvent agent, in Eq. (2b) over time:

$$\begin{aligned}\mathcal{F}(x) &= \int_0^\infty F_1(x,t)\,dt = \int_0^\infty \frac{y_0\, e^{-y_0^2/4Dt}}{4\pi D t^2}\left[e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt}\right] dt \\ &= \frac{y_0}{\pi}\left[\frac{1}{y_0^2+(x+x_0)^2} - \frac{1}{y_0^2+(x-x_0)^2}\right], \end{aligned} \tag{6a}$$

where the integral is performed by making the substitution $z = 1/4Dt$. The probability P_B that the person with initial wealth y_0 goes bankrupt first is

$$\begin{aligned} P_B &= \int_0^\infty \mathcal{F}(x)\,dx = \int_0^\infty dx\,\left[\frac{1}{y_0^2+(x+x_0)^2} - \frac{1}{y_0^2+(x-x_0)^2}\right] \\ &= \frac{2y_0}{\pi}\int_0^{x_0}\frac{dw}{w^2+y_0^2} = \frac{2}{\pi}\,\tan^{-1}(x_0/y_0)\,, \end{aligned}$$

where we make the substitution $w = x + x_0$ in the first integral and $w = x - x_0$ in the second. When both people possess the same initial wealth, $x_0 = y_0$, the above formula gives the obvious result $P_B = \frac{1}{2}$, while for $y_0 = 0$, $P_B = 1$.

Henceforth we focus on the symmetric initial condition, $x_0 = y_0$. From Eq. (6a) and multiplying by 2 to account for either person going bankrupt first, the wealth distribution of the solvent person at the moment of the first bankruptcy is

$$\mathcal{F}(x) = \frac{1}{\pi}\,\frac{8xx_0^2}{4x_0^4+x^4}\,. \tag{6b}$$

Notice that $\int_0^\infty \mathcal{F}(x)\,dx$, which is the probability that either person eventually goes bankrupt, equals 1, as it must. The average wealth of the solvent person at the first bankruptcy is

$$\langle x\rangle = \frac{\displaystyle\int_0^\infty x\,\mathcal{F}(x)\,dx}{\displaystyle\int_0^\infty \mathcal{F}(x)\,dx} = \int_0^\infty \frac{8x^2x_0^2}{4x_0^4+x^4}\,dx = 2x_0\,.$$

If this person now shares half of her/his wealth with the bankrupt person, then both people restart with average wealth x_0. This conservation of the average wealth is a consequence of diffusion being a martingale [20].

However, this average outcome is not representative of a typical realization of the dynamics. When an individual bankruptcy occurs, the probability that the wealth of the solvent person is less than $2x_0$ is

$$\int_0^{2x_0} \mathcal{F}(x)\,dx = \frac{2}{\pi}\tan^{-1}2 \approx 0.7048\,.$$

After sharing half of her/his wealth with the bankrupt person, the typical, or median, wealth of each person will therefore be less than x_0. Consequently, the typical wealth of the two people systematically decreases to zero upon repeated bankruptcies. This dichotomy between the typical and average outcome arises because the wealth distribution of the solvent person at the moment of bankruptcy asymptotically has an algebraic x^{-3} tail (Eq. (6b)). The average incorporates rare events where x is anomalously large, and these are absent in typical events.

To determine how the typical wealth decays with time, we rewrite $\mathcal{F}(x)$ in (6b) as

$$\mathcal{F}(x)\,dx = \frac{8}{\pi}\,\frac{x/x_0}{4+(x/x_0)^4}\,\frac{dx}{x_0}\,.$$

That is, the typical wealth of the solvent person is rescaled by $x \to \eta x$ at each bankruptcy, where the probability distribution for the random variable η is

$$P(\eta)\,d\eta = \frac{8}{\pi}\frac{\eta}{4+\eta^4}\,d\eta\,.$$

After the joint wealth is shared, the typical wealth of both people now is $x = \frac{1}{2}\eta\,x_0$. After the n^{th} bankruptcy, each person therefore has typical wealth $x_n = \frac{1}{2}\eta\,x_{n-1}$; namely, x_n follows a geometric random walk. In terms of $S_n = \ln x_n$ and $\xi = \ln(\eta/2)$, S_n follows a simple random walk in which $S_n = S_{n-1} + \xi$. The distribution of the random variable ξ is determined by the transformation $f(\xi)\,d\xi = P(\eta)\,d\eta$, which leads to

$$f(\xi) = \frac{8}{\pi}\frac{e^{2\xi}}{1+4e^{4\xi}}\,, \tag{7}$$

which is properly normalized on $[-\infty,\infty]$. The salient point is that the average value of ξ is given by $\langle\xi\rangle = \int_{-\infty}^{\infty}\xi\,f(\xi)\,d\xi = -\ln\sqrt{2}$, which means that the typical wealth of both people is reduced by a factor $1/\sqrt{2}$ after each bankruptcy.

Because the typical wealth decreases multiplicatively after each bankruptcy, the wealth of each person ultimately becomes vanishingly small. Since this wealth is always non-zero, we need to define the notion of joint bankruptcy through a cutoff. We postulate that joint bankruptcy occurs when the total wealth of both people has been reduced by a factor of 10^{-4} (see Fig. 1). At this level, we regard the wealth to be too small for an agent to be economically viable; our main results are independent of this cutoff, as long as it is sufficiently small. The number of individual bankruptcies needed to reach this state of joint bankruptcy is determined by $(1/\sqrt{2})^n = 10^{-4}$ or $n \approx 27$. This prediction agrees with simulations of the time evolution of the joint wealth in Fig. 2(a).

We now determine how long it takes for both altruists to be bankrupt when they both start with the same wealth. It is useful to focus on the longitudinal coordinate along the main diagonal, $X_a(t) = \frac{1}{2}[x_1(t)+x_2(t)]$, which is just the average wealth of the two altruists. The diffusion coefficient associated with this coordinate, $D_\parallel$, is related to D by $D_\parallel = \frac{1}{2}D$. The factor of $\frac{1}{2}$ occurs because the component of a wealth increment or decrement of either agent is reduced by a factor $1/\sqrt{2}$ when this displacement is projected onto the main diagonal.

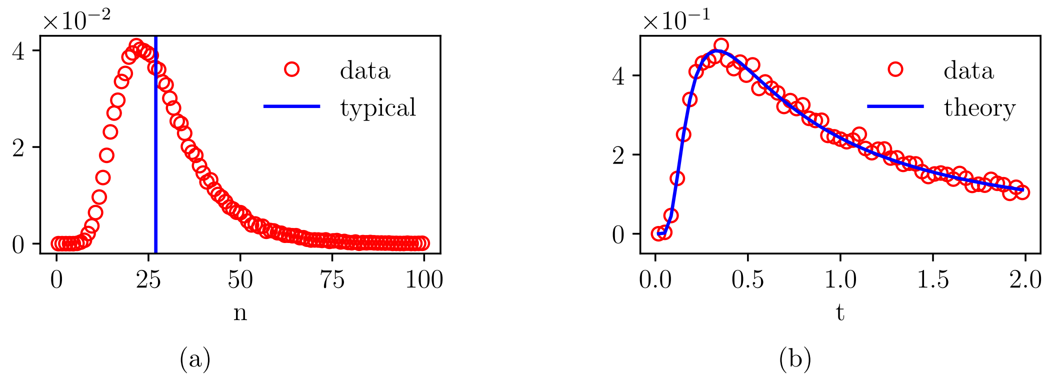


Figure 2. The probability distribution that joint bankruptcy for two altruists occurs after n individual bankruptcy events (a) or after a time t (b). The altruists start with initial wealth $x_0 = 1$. In (a), the peak is at $n = 22$, the median is at $n = 27$, and $\langle n\rangle = 29.14$. In (b), the peak is at $t = 0.33$ and the median is at $t = 2.21$.

By construction, wealth redistribution after a bankruptcy does not affect $X_a(t)$. Thus $X_a(t)$ is simply a one-dimensional Brownian motion that starts at x_0 in the presence of an absorbing boundary at $10^{-4}\,x_0$; we approximate the location of this boundary by 0. Thus the first-passage probability for the altruists to reach the cutoff is

$$F_a(x_0,t) = \frac{1}{\sqrt{4\pi D_\parallel\, t}}\; e^{-x_0^2/4D_\parallel t}\,. \tag{8}$$

The numerical data shown in Fig. 2(b) for the first-passage probability to joint bankruptcy accurately fits (8). Note that if the altruists employed an *unequal* sharing rule, the first-passage probability (8) will remain valid because the coordinate $X_a(t)$ is unaffected by an unequal redistribution rule.

The results for two altruists can be straightforwardly generalized to more altruists, and we quote some basic results for three altruists with the symmetric initial condition $x_0 = y_0 = z_0 = 1$. The analogue of (6b) for the wealth distribution of the two solvent agents when the third agent goes bankrupt is

$$\mathcal{F}(x) = \frac{1}{\pi}\left\{\frac{1}{[(x-2)x+2]\sqrt{(x-2)x+3}} - \frac{1}{[x(x+2)+2]\sqrt{x(x+2)+3}}\right\}. \tag{9}$$

This distribution has the same x^{-3} tail as in the case of two altruists. From this distribution, the average wealth of the two solvent agents when the other agent goes bankrupt is 3/2. After wealth sharing, each agent again has average wealth equal to 1. However, the typical wealth decreases after each bankruptcy by a factor of approximately 0.8022. With this reduction factor, the number of individual bankruptcies for the total wealth to be reduced by 10^{-4} is now 42. As the number of agents increases, the number of individual bankruptcies before collective bankruptcy occurs increases commensurately.

5. Which is Better: Altruism or Individualism?

We now address the fundamental question of whether altruism or individualism is better. Specifically, are altruists more likely to "live" longer than individualists? Here the term "live" means that each altruist possesses sufficient wealth (greater than the cutoff) to be economically viable. A related question is: who has more wealth—the altruists or the individualists? We first address these questions for two agents, and then extend our considerations to any number of agents.

5.1. Two Agents

A basic ingredient in the following is $S(x_0, t, D)$, the survival probability of a one-dimensional Brownian motion in the presence of an absorbing boundary at the origin [18]:

$$S(x_0, t, D) = \mathrm{erf}\left(\frac{x_0}{\sqrt{4\,D\,t}}\right). \tag{10}$$

Since the wealth of the two altruists is also described by one-dimensional Brownian motion with diffusion coefficient $D_\parallel = D/2$, the ultimate survival probability of the altruists is

$$S_a(t) = S(x_0, t, D_\parallel)\,. \tag{11}$$

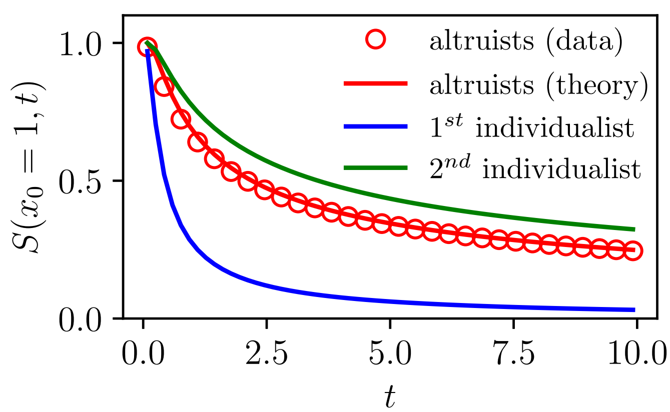


Figure 3. The survival probabilities in a society of $N = 2$ agents. The altruists (red), and the first and second individualists to go bankrupt (blue and green, respectively). Both agents start with wealth $x_0 = 1$.

The survival probability of the first individualist, which is the probability that both individualists are still alive, is

$$S_1(t) = \left[S(x_0, t, D)\right]^2. \tag{12a}$$

The survival probability of the second individualist equals the probability that both individuals are alive plus the probability that one is alive:

$$S_2(t) = S(x_0,t,D)^2 + 2\,[1 - S(x_0,t,D)]S(x_0,t,D)\,. \tag{12b}$$

As shown in Fig. 3, the individualist that goes bankrupt first has the worst possible outcome, while the individualist that goes bankrupt second has the best outcome. The survival probability of the altruists is intermediate to those of each individualist.

To determine the time dependence of the altruists' and individualists' wealth, we need $G(x,t)$, the propagator of the fraction of surviving one-dimensional Brownian motions that start at x_0 in the presence of an absorbing boundary at the origin:

$$G(x,t,D) = \frac{1}{\sqrt{4\pi Dt}}\left(e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt}\right) \Big/ \operatorname{erf}\left(\frac{x_0}{\sqrt{4Dt}}\right)\,. \tag{13}$$

The error function normalizes this propagator so that its spatial integral equals 1.

For two altruists, their wealth distribution is

$$G_a(x,t) = S_a(t)\,G(x,t,D_\|)\,. \tag{14}$$

A natural way to quantify the wealth of the agents is by its *typical* or *median* value. The median wealth for the altruists, $w_a(t)$, is obtained from the criterion

$$\int_{w_a(t)}^{\infty} dx\, G_a(x,t) = \tfrac{1}{2}\,. \tag{15}$$

Namely, at the median wealth, one-half of the wealth distribution exceeds w_a and one-half is less than w_a. This criterion leads, after a straightforward integration over x, to the following implicit equation for $w_a(t)$:

$$\operatorname{erfc}\left(\frac{w_a(t)-x_0}{\sqrt{4D_\|t}}\right) - \operatorname{erfc}\left(\frac{w_a(t)+x_0}{\sqrt{4D_\|t}}\right) = 1\,. \tag{16}$$

From this expression, we can solve for the typical altruist wealth numerically and the result is shown in Fig. 4.

For individualists, the wealth distribution of the first individualist to go bankrupt is given by

$$G_1(x,t) = S_1(t)\,G(x,t)\,, \tag{17}$$

We can once again obtain the evolution of the typical wealth of this first individualist, $w_1(t)$, by applying criterion (15) to $G_1(x,t)$. This gives the implicit equation for $w_1(t)$:

$$\operatorname{erfc}\left(\frac{w_1(t)-x_0}{\sqrt{4Dt}}\right) - \operatorname{erfc}\left(\frac{w_1(t)+x_0}{\sqrt{4Dt}}\right) = \frac{\operatorname{erf}\left(\frac{x_0}{\sqrt{4\,D\,t}}\right)}{S_1(t)} = \frac{1}{\operatorname{erf}\left(\frac{x_0}{\sqrt{4\,D\,t}}\right)}\,. \tag{18}$$

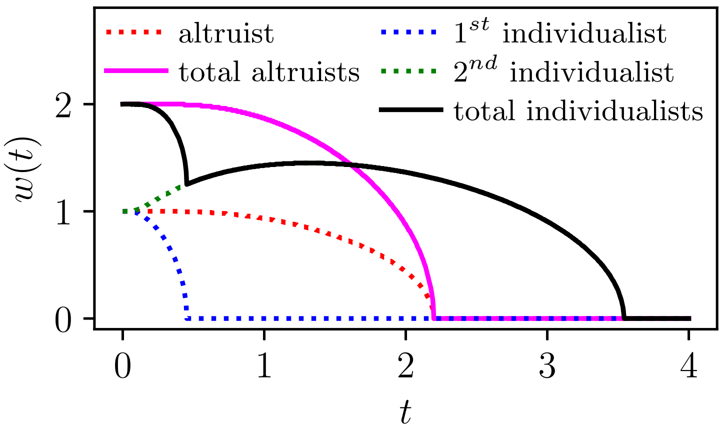


Figure 4. Evolution of the typical wealth in a society of $N = 2$ agents. The altruists (red), the first individualist to go bankrupt (blue), the second individualist (green), the altruistic society (magenta), and the individualistic society (black). Both agents have initial wealth $x_0 = 1$.

Similarly, the distribution of the wealth of the second individualist to go bankrupt is

$$G_2(x,t) = S_2(t)\, G(x,t)\,. \tag{19}$$

Applying criterion (15) to $G_2(x,t)$ yields the implicit equation for $w_2(t)$, the typical wealth of this second individualist:

$$\mathrm{erfc}\left(\frac{w_2(t)-x_0}{\sqrt{4\,D\,t}}\right) - \mathrm{erfc}\left(\frac{w_2(t)+x_0}{\sqrt{4\,D\,t}}\right) = \frac{\mathrm{erf}\left(\dfrac{x_0}{\sqrt{4\,D\,t}}\right)}{S_2(t)}\,. \tag{20}$$

The numerical solutions for the typical wealth of the individualists are also shown in Fig. 4. As in the case of the survival probability, the typical altruist wealth is intermediate to that of the two individualists. Notice also that the wealth of the longer-lived individualist initially increases before being inexorably drawn toward bankruptcy. This non-monotonicity arises because the typical time for the bankruptcy of the longer-lived individualist is larger than the diffusion time x_0^2/D. For the wealth trajectory to not reach the origin within this time period, the trajectory must initially move *away* from the origin. Related types of effective repulsion phenomena have been found to arise from a variety physically motivated constraints on first-passage trajectories.[21–24]

5.2. N agents

The calculations in the previous section can be straightforwardly extended to N agents. The survival probability of N altruists is given by

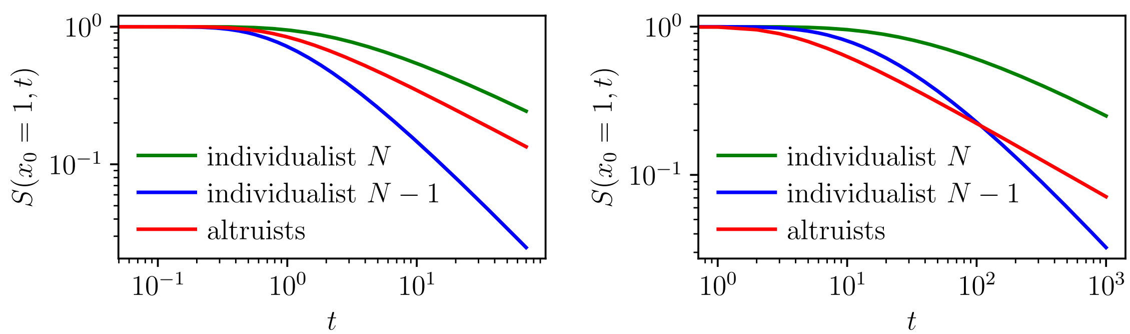
$$S_a(t) = S(x_0, t, D/N)\,. \tag{21}$$

with $S(x_0,t,D)$ from (10). The diffusion coefficient of the effective particle is now D/N because the projection of a displacement in a coordinate direction onto the diagonal

$(1, 1, \ldots, 1)$ is $1/\sqrt{N}$. The survival probability of the n^{th} individualist to go bankrupt is given by

$$S_n(t) = \sum_{m=0}^{n-1} \binom{N}{m} \operatorname{erfc}\left(\frac{x_0}{\sqrt{4\,D\,t}}\right)^m \operatorname{erf}\left(\frac{x_0}{\sqrt{4\,D\,t}}\right)^{N-m} . \tag{22}$$

where the combinatorial factor accounts for the fact that there are $\binom{N}{m}$ possible groups of m bankrupt individualists among N, with $m < n$, when the n^{th} individualist is alive. We compare the survival probabilities of last and one before last individualists to go bankrupt with the altruists for $N = 4$ and 16 in Fig. 5. We see that for large times, the survival probability of the altruists is larger than all but the most long-lived individualist.



(a) $N = 4$

(b) $N = 16$

Figure 5. The survival probability of N altruists (red), the survival probability of the last of the N individualist to go bankrupt (green), and the one before last individualist to go bankrupt (blue). All agents have initial wealth $x_0 = 1$.

For $t \to \infty$, the asymptotic decay of the altruist survival probability (21) and the survival probabilities of each of the individualists (22) is

$$\begin{aligned} S_a(t) &\sim \frac{x_0\sqrt{N}}{\sqrt{\pi D t}} \\ S_n(t) &\sim \binom{N}{n-1}\left(\frac{x_0}{\sqrt{\pi D t}}\right)^{N-n+1} . \end{aligned} \tag{23}$$

To obtain the asymptotics of $S_n(t)$, we use the fact that the sum in (22) is dominated by the term with $m = n - 1$, and that the asymptotic form of the error function is $\operatorname{erf}(x) \sim 2x/\sqrt{\pi}$ for $x \to 0$. Thus the individualist survival probabilities in (23) all decay asymptotically faster than $t^{-1/2}$, *except* for the $n = N$ individualist. Here, the asymptotic form of $S_N(t)$ is

$$S_N(t) \sim \frac{x_0 N}{\sqrt{\pi D t}} , \quad t \to \infty . \tag{24}$$

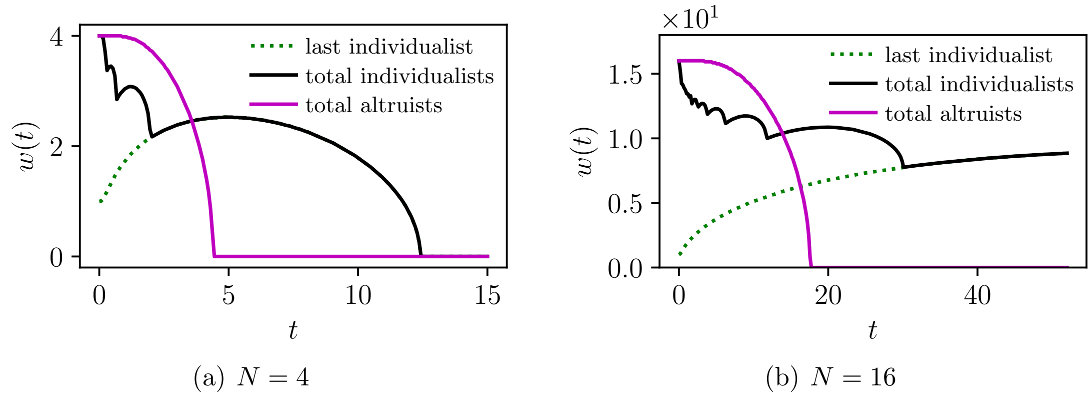


Figure 6. Evolution of the typical total wealth of N altruists (violet) and N individualists (black), and the wealth of the last individualist to go bankrupt for $N = 4$ and 16. All agents have initial wealth $x_0 = 1$.

As in the case of $N = 2$, the last individualist to go bankrupt has the largest survival probability, while the survival probability of all the other individualists decays faster than that of the altruists.

We may also compute the typical wealth of the agents using the same approach as in the case of two altruists. Now the distribution of wealth of N altruists is

$$G_a(x,t) = S_a(t)\, G(x,t)\,, \tag{25}$$

with $G(x,t)$ given in (13). The evolution of the typical wealth $w_a(t)$ of the N altruists evolves according to Eq. (16), but with $D_{\|}$ now equal to D/N. The distribution of wealth of the n^{th} individualist is

$$G_n(x,t) = S_n(t)\, G(x,t)\,, \tag{26}$$

where $G(x,t)$ again given in (13). The evolution of the typical wealth $w_n(t)$ of the n^{th} individualist evolves according to

$$\operatorname{erfc}\left(\frac{w_n(t) - x_0}{\sqrt{4\,D\,t}}\right) - \operatorname{erfc}\left(\frac{w_n(t) + x_0}{\sqrt{4\,D\,t}}\right) = \frac{\operatorname{erf}\left(\dfrac{x_0}{\sqrt{4\,D\,t}}\right)}{S_n(t)}\,. \tag{27}$$

The numerical comparison of the typical wealth for $N = 4$ and 16 altruists and the first and last of the N individualists, along with the total wealth in both societies, is presented in Fig. 6. We see that the first individualist rapidly goes bankrupt, while the last individualist accumulates the total wealth of society and survives for a long time. In contrast, the altruistic society sees its wealth decrease monotonically with a bankruptcy time that is intermediate to that of the first and the last individualists.

6. Discussion

We explored the role of redistribution in a toy model of wealth evolution, in which the wealth of each person in a population of N agents evolves by free diffusion. As a preliminary, we first studied the outcome for individualists—these are agents that do not engage in any wealth sharing when someone goes bankrupt. Even though the *average* wealth is conserved during the evolution, including bankruptcy events, the *typical* wealth decreases systematically with time. Thus individualists go bankrupt one by one until nobody remains solvent. For large N, the first few individualists that go bankrupt suffer a harsh fate, as their typical wealth decays rapidly with time. Conversely, the last few individualists to go bankrupt initially have a favorable economic outcome as their wealth grows appreciably at early times. This initial wealth growth stems from an effective repulsion of their Brownian paths from the origin (bankruptcy) because the time until their individual bankruptcies are much larger than the diffusion time x_0^2/D. To avoid bankruptcy over such a long time period, the wealth trajectory must initially be repelled by the origin. In fact, the most long-lived individualists typically accumulate the total wealth of his society. Nevertheless, every individualist ultimately suffers the same fate of bankruptcy.

In contrast, a population of altruists equally share their wealth each time an individual goes bankrupt. Again, the average wealth is conserved throughout the dynamics, but the typical wealth also systematically decreases with time. Thus eventually a population of altruists collectively goes bankrupt, in which the total wealth of the population falls below a small threshold value. We showed that at early times altruists have a better economic fate than individualists. At long times, however, an individualistic society becomes extremely inequitable, with most individualists quickly reaching a fate of having no wealth and a few having most of the societal wealth. These longest-lived individualists eventually have a better outcome than the altruists. Thus if one is faced with a choice of which society to join, being an average altruist is preferable to an average individualist.

Our model is naive in many respects and there are variety of possible extensions to consider. The notion that an individual's wealth evolves by free diffusion can clearly be made more realistic. Many people draw a regular salary, continuously spend for routine expenses, and sometimes experience negative shocks of large unexpected expenses; this latter feature was the focus of the study by Abebe et al. [9]. Thus the evolution of individual wealth is more realistically described by a process that incorporates these features of salary, spending, and shocks. More realistically, the per capita wealth of societies generally increase over the long term and it would be interesting to superimpose a slight a positive drift on the wealth dynamics, perhaps different for each agent, that more than compensates for the decrease in the typical wealth.

The equal-wealth sharing mechanism we studied is also unrealistically idealized, and we focused on this simple rule because it lead to an analytically tractable model. It may be worthwhile to study more selfish wealth sharing rules. Perhaps such modifications of the wealth sharing rules lead to better outcomes for both the survival probability and the average wealth of an altruist population compared to an individualistic population. It should also be useful to explore possible connections between wealth sharing rules and policies on optimal taxation and redistribution [10].

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References

[1] Piketty T and Saez E 2014 *Science* **344** 838–843

[2] Piketty T 2015 *The Economics of Inequality* (Harvard University Press)

[3] Gabaix X, Lasry J M, Lions P L and Moll B 2016 *Econometrica* **84** 2071–2111

[4] Aldous D 2002 Unpublished `http://www.stat.berkeley.edu/~aldous/Research/OP/river.pdf`.

[5] McKean H and Shepp L A 2006 *Journal of Mathematical Sciences* **139** 6589–6594

[6] Pal S and Pitman J 2008 *The Annals of Applied Probability* **18** 2179–2207

[7] Tang W and Tsai L C 2018 *The Annals of Probability* **46** 1597–1650

[8] Grandits P 2020 *Theory of Probability & Its Applications* **64** 646–655

[9] Abebe R, Kleinberg J and Weinberg S M 2020 Subsidy allocations in the presence of income shocks *Proceedings of the AAAI Conference on Artificial Intelligence* vol 34 pp 7032–7039

[10] Piketty T, Saez E and Stantcheva S 2014 *American Economic Journal: Economic Policy* **6** 230–71

[11] Chorro C 2016 *Statistics & Probability Letters* **112** 35–40

[12] Cramér H 1955 *Collective risk theory: A survey of the theory from the point of view of the theory of stochastic processes* (Esselte Stockholm)

[13] Andersen E S 1957 *Bulletin of the Institute of Mathematics and its Applications* **12** 275–279

[14] Thorin O 1982 *Scandinavian Actuarial Journal* **1982** 65–103

[15] Dickson D C 1998 *North American Actuarial Journal* **2** 60–68

[16] Asmussen S and Albrecher H 2010 *Ruin probabilities* vol 14 (World scientific)

[17] Feller W 2008 *An Introduction to Probability Theory and its Applications, vol 1* (John Wiley & Sons)

[18] Redner S 2001 *A Guide to First-Passage Processes* (Cambridge University Press)

[19] Bray A J, Majumdar S N and Schehr G 2013 *Advances in Physics* **62** 225–361

[20] Mörters P and Peres Y 2010 *Brownian Motion* vol 30 (Cambridge University Press)

[21] Bhat U and Redner S 2015 *Journal of Statistical Mechanics: Theory and Experiment* **2015** P06035

[22] Meerson B 2019 *Journal of Statistical Mechanics: Theory and Experiment* **2019** 013210

[23] Smith N R and Meerson B 2019 *Journal of Statistical Mechanics: Theory and Experiment* **2019** 023205

[24] Meerson B and Smith N R 2019 *Journal of Physics A: Mathematical and Theoretical* **52** 415001