Dynamics of social balance on networks

T. Antal,[1,*] P. L. Krapivsky,[1] and S. Redner[2,†]
[1]*Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
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We study the evolution of social networks that contain both friendly and unfriendly pairwise links between individual nodes. The network is endowed with dynamics in which the sense of a link in an imbalanced triad—a triangular loop with one or three unfriendly links—is reversed to make the triad balanced. With this dynamics, an infinite network undergoes a dynamic phase transition from a steady state to "paradise"—all links are friendly—as the propensity p for friendly links in an update event passes through $1/2$. A finite network always falls into a socially balanced absorbing state where no imbalanced triads remain. If the additional constraint that the number of imbalanced triads in the network not increase in an update is imposed, then the network quickly reaches a balanced final state.

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I. INTRODUCTION

In this work, we investigate the role of friends and enemies on the evolution of social networks. We represent individuals as nodes in a graph and a relationship between individuals as a link that joins the corresponding nodes. To quantify a relationship, we assign the binary variable $s_{ij}=\pm 1$ to link ij, with $s_{ij}=1$ if nodes i and j are friends and $s_{ij}=-1$ if i and j are enemies (Fig. 1). A basic characterization of relationships between mutual acquaintances is the notion of *social balance* [1,2]. The triad ijk is defined as balanced if the sign of the product of the links in the triad, $s_{ij}s_{jk}s_{ki}$, equals 1, while the triad is imbalanced otherwise. We define a triad to be of type Δ_k if it contains k negative links. Thus Δ_0 and Δ_2 are balanced, while Δ_1 and Δ_3 are imbalanced. A balanced triad therefore fulfills the adage that: (i) a friend of my friend is my friend, (ii) an enemy of my friend is my enemy, (iii) a friend of my enemy is my enemy, and (iv) an enemy of my enemy is my friend. On the other hand, an imbalanced triad is analogous to a frustrated plaquette in a random magnet [3].

A network is balanced if each constituent triad is balanced [1,2]. An ostensibly more general definition of a balanced network is that *every* cycle in the network is balanced. Cartwright and Harary showed [4] that the cycle-based and triad-based definitions of balance are equivalent on complete graphs. Their result implies that if an imbalanced cycle of any length exists in a complete graph, an imbalanced triad also exists.

Balance theory has been initiated by Heider [1] and other social psychologists [5,6], and the subject remains an active research area [2,7–12]. Much of this work was devoted to classifying balanced stable states of networks when relationships are static. A fundamental result from these studies is that balanced societies are remarkably simple: either all individuals are mutual friends (we call such a state "paradise") or the network segregates into two antagonistic cliques where individuals within the same clique are mutual friends and individuals from distinct cliques are enemies (we call such a state "bipolar") [4]. Balance theory also has natural applications to international relations [13]. As a particularly compelling example, the Triple Alliance (1882) pitted Germany, Austria-Hungary, and Italy against the Triple Entente (1907) countries of Britain, France, and Russia [14]. This bipolar state of competing alliances clearly contributed to the onset of World War I.

A large network is almost surely imbalanced if the relationships are randomly chosen to be friendly or unfriendly. Clearly such a network is socially unstable and the web of relations must evolve to a more stable state if the individual nodes behave rationally. In this work, we go beyond a static description of social relations and investigate how an initially imbalanced society achieves balance by endowing a network with a prototypical social dynamics that reflects the natural human tendency to reduce imbalanced triads. A related line of investigation, based on evolving social networks with continuous interaction strengths, has also recently appeared [15].

II. MODELS

We first consider what we term *local triad dynamics* (LTD). In an update step of LTD, we first choose a triad at

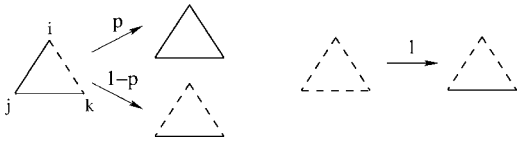

FIG. 1. Imbalanced triads Δ_1 (left) and Δ_3 (right) and the possible outcomes after an update step by local triad dynamics. Solid and dashed lines represent friendly (e.g., $s_{ij}=1$) and unfriendly (e.g., $s_{ik}=-1$) relations, respectively.

*On leave from Institute for Theoretical Physics, HAS, Eötvös University, Budapest, Hungary.
†Permanent address: Department of Physics, Boston University, Boston, MA 02215, USA. Electronic address: redner@bu.edu

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random. If this triad is balanced (Δ_0 or Δ_2), no evolution occurs. If the triad is imbalanced (Δ_1 or Δ_3), we change $s \to -s$ on a randomly-chosen link as follows: $\Delta_1 \to \Delta_0$ occurs with probability p, and $\Delta_1 \to \Delta_2$ occurs with probability $1-p$, while $\Delta_3 \to \Delta_2$ occurs with probability 1 (Fig. 1). One unit of time is defined as L update events, where L is the total number of links. Notice that for the special case of $p=1/3$, each link of an imbalanced triad is flipped equiprobably.

After an update step in LTD, the imbalanced target triad becomes balanced, but other balanced triads that share a link with this target may become imbalanced. These triads can subsequently evolve and return to balance, leading to new imbalanced triads. Such an interaction cascade is familiar in social settings. For example, if a married couple breaks up, the acquaintances of the couple may then be obliged to redefine their relations with each partner in the couple so as to maintain balanced triads. These redefinitions, may lead to additional relationship shifts, etc.

For $p<1/2$, we shall show that LTD quickly drives an infinite network to a quasistationary dynamic state where global characteristics, such as the densities of friendly relations or imbalanced triads, fluctuate around stationary values. As p passes through a critical value of $1/2$, the network undergoes a phase transition to a paradise state where no unfriendly relations remain. On the other hand, a finite network always reaches a balanced state. For $p<1/2$, this balanced state is bipolar and the time to reach this state scales faster than exponentially with network size. For $p \geqslant 1/2$, the final state is paradise. The time to reach this state scales algebraically with N when $p=1/2$ and logarithmically in N for $p>1/2$.

We also investigate *constrained triad dynamics* (CTD). Here, we select a random link and change $s \to -s$ for this link if the total number of imbalanced triads decreases. If the total number of imbalanced triads is conserved in an update, then the update occurs with probability $1/2$. Updates that would increase the total number of imbalanced triads are not allowed. We again define the unit of time as L update events, so that on average each link is changed once in unit of time. The global constraint accounts for the socially plausible feature that an agent considers all of its mutual acquaintances before deciding to change the character of a relationship. CTD also corresponds to an Ising model with a three-spin interaction between the links of a triad, $\mathcal{H}=-\Sigma_{ijk} s_{ij} s_{jk} s_{ki}$, where the sum is over all triads ijk, with zero-temperature Glauber dynamics [16]. As we shall see, a crucial outcome of CTD is that a network is quickly driven to a balanced state in a time that scales as $\ln N$.

In the following two sections we analyze the dynamics of networks that evolve by LTD or CTD. For simplicity, we consider networks with a complete graph topology—everyone knows everyone else. This limit is appropriate for small networks, such as the diplomatic relations of countries. We then summarize and discuss some practical implications of our results in Sec. IV.

III. LOCAL TRIAD DYNAMICS

A. Evolution equations

We begin with essential preliminaries for writing the governing equations for the various triad densities. Let N, $L=\binom{N}{2}$, and $N_\Delta=\binom{N}{3}$ be the numbers of nodes, links, and triads in the network. Define N_k as the number of triads that contain k negative links, with $n_k=N_k/N_\Delta$ the respective triad densities and L^+ (L^-) the number of positive (negative) links. The number of triads and links are related by

$$L^+ = \frac{3N_0 + 2N_1 + N_2}{N-2}, \quad L^- = \frac{N_1 + 2N_2 + 3N_3}{N-2}. \tag{1}$$

The numerator counts the number of positive links in all triads while the denominator appears because each link is counted $N-2$ times. The density of positive links is therefore $\rho=L^+/L=(3n_0+2n_1+n_2)/3$, while the density of negative links is $1-\rho=L^-/L$.

A fundamental network characteristic is N_k^+, which is defined as follows: for each positive link, count the number of triads of type Δ_k that are attached to this link. Then N_k^+ is the average number of such triads over all positive links. This number is

$$N_k^+ = \frac{(3-k)N_k}{L^+}. \tag{2}$$

The factor $(3-k)N_k$ accounts for the fact that each of the N_k triads of type Δ_k is attached to $3-k$ positive links; dividing by L^+ then gives the average number of such triads. Analogously, we introduce $N_k^-=kN_k/L^-$. Since the total number of triads attached to any given link equals $N-2$, the corresponding triad densities are

$$n_k^+ = \frac{N_k^+}{N-2} = \frac{(3-k)n_k}{3n_0 + 2n_1 + n_2}, \tag{3a}$$

$$n_k^- = \frac{N_k^-}{N-2} = \frac{kn_k}{n_1 + 2n_2 + 3n_3}. \tag{3b}$$

We now write the rate equations that account for changes in the various triad densities in a single update event. We choose a triad at random; if it is imbalanced (Δ_1 or Δ_3), we change one of its links as shown in Fig. 1. Let π^+ be the probability that a link changes from $+$ to $-$ in an update event and vice versa for π^-. Since a link changes from $1 \to -1$ with probability $1-p$ when $\Delta_1 \to \Delta_2$, while a link changes from $-1 \to 1$ with probability p if $\Delta_1 \to \Delta_0$ and with probability 1 if $\Delta_3 \to \Delta_2$, we have (see Fig. 1)

$$\pi^+ = (1-p)n_1, \quad \pi^- = pn_1 + n_3. \tag{4}$$

Since each update changes $N-2$ triads and we also defined one time step as L update events, the rate equations for the triad densities have the size-independent form

$$\frac{dn_0}{dt} = \pi^- n_1^- - \pi^+ n_0^+,$$

$$\frac{dn_1}{dt} = \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+,$$

$$\frac{dn_2}{dt} = \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+,$$

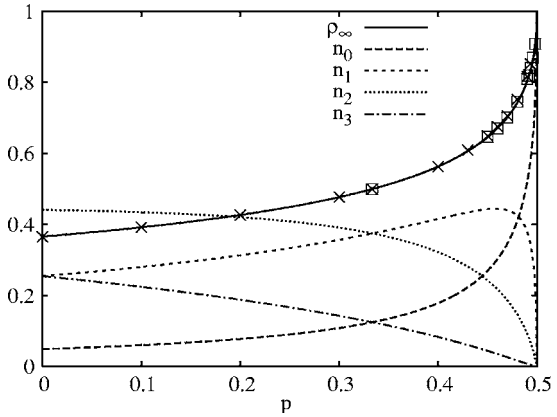

FIG. 2. Exact stationary densities $n_k(p)$ and the density of friendly relations ρ_∞ as a function of p. Simulation results for ρ_∞ for $N=64$ (crosses) and 256 (boxes) are also shown.

$$\frac{dn_3}{dt} = \pi^+ n_2^+ - \pi^- n_3^-. \tag{5}$$

B. Stationary states

We first study stationary states. Setting the left-hand sides of Eqs. (5) to zero and also imposing $\pi^+ = \pi^-$ to ensure a fixed friendship density, we obtain

$$n_0^+ = n_1^-, \quad n_1^+ = n_2^-, \quad n_2^+ = n_3^-.$$

By forming products such as $n_0^+ n_2^- = n_1^+ n_1^-$, these relations are equivalent to

$$3n_0 n_2 = n_1^2, \quad 3n_1 n_3 = n_2^2. \tag{6}$$

Substituting π^+ and π^- from Eq. (4) into the stationarity condition $\pi^+ = \pi^-$ gives $n_3 = (1-2p)n_1$. Using this result, as well as the normalization $\Sigma n_k = 1$, in Eqs. (6), we find, after some straightforward algebra,

$$n_j = \binom{3}{j} \rho_\infty^{3-j} (1-\rho_\infty)^j, \tag{7}$$

where

$$\rho_\infty = \begin{cases} 1/[\sqrt{3(1-2p)}+1], & p \leqslant 1/2, \\ 1, & p \geqslant 1/2, \end{cases} \tag{8}$$

is the stationary density of friendly links. Equation (7) shows that relationships are *uncorrelated* in the stationary state. As shown in Fig. 2, the density of friendly links ρ_∞ monotonically increases with p for $0 \leqslant p \leqslant 1/2$, while for $p \geqslant 1/2$, paradise is reached where all people are friends. Near the phase transition, the density of unfriendly links $1-\rho_\infty$ vanishes as $\sqrt{3\epsilon} + O(\epsilon)$, as $\epsilon \equiv 1-2p \to 0$.

C. Temporal evolution

A remarkable feature of Eqs. (5) is that if the initial triad densities are given by Eq. (7)—uncorrelated densities—the densities will remain uncorrelated forever. For such initial conditions it therefore suffices to study the time evolution of the density of friendly links $\rho(t)$. This time evolution can be extracted from Eqs. (5), or it can be established directly by noting that $\rho(t)$ increases if $\Delta_3 \to \Delta_2$ or $\Delta_1 \to \Delta_0$ and decreases if $\Delta_1 \to \Delta_2$. Taking into account that the respective probabilities for these processes are 1, p, and $1-p$, we find

$$\frac{d\rho}{dt} = 3(2p-1)\rho^2(1-\rho) + (1-\rho)^3. \tag{9}$$

Thus the time dependence of the density of friendly links is given by the implicit relation

$$\int_{\rho_0}^{\rho} \frac{dx}{3(2p-1)x^2(1-x) + (1-x)^3} = t. \tag{10}$$

When $p < 1/2$, the stationary density of Eq. (8) is approached exponentially in time:

$$\rho(t) - \rho_\infty \sim e^{-Ct}, \quad C = \frac{6\epsilon}{1+\sqrt{3\epsilon}},$$

where again $\epsilon = 1-2p$. At the threshold value $p = 1/2$, the friendship density is given by

$$\rho = 1 - \frac{1-\rho_0}{\sqrt{1+2(1-\rho_0)^2 t}}. \tag{11}$$

Here the approach to paradise is algebraic in time—viz., $1-\rho \to 1/\sqrt{2t}$ as $t \to \infty$. As a consequence, the leading behavior is

$$(n_0, n_1, n_2, n_3) \to \left(1 - \frac{3}{\sqrt{2t}}, \frac{3}{\sqrt{2t}}, \frac{3}{2t}, \frac{1}{(2t)^{3/2}}\right). \tag{12}$$

Finally, when $p > 1/2$,

$$1 - \rho \sim \exp[-3(2p-1)t], \tag{13}$$

so that paradise is reached exponentially quickly.

D. Fate of a finite society

Although an infinite network reaches a dynamic steady state for $p < 1/2$, a finite network ultimately falls into an absorbing state for all p. Such absorbing states are necessarily balanced, because any network that contains an imbalanced triad continues to change. To see why such an absorbing state must eventually be reached, consider the evolution in which *at each step* an unfriendly link changes to friendly in an imbalanced triad. Since the number of unfriendly links always decreases, a balanced state is reached in a finite number steps. Finally, because this particular route to a balanced state has a nonzero probability to occur, any initial network ultimately reaches an absorbing balanced state.

Our simulations show that a finite network evolves to a bipolar state for $p < 1/2$, independent of the initial state. The size difference of the two final cliques is virtually independent of the initial configuration because the network spends an enormous time in a quasistationary state before reaching the absorbing state. For $p < 1/2$, we estimate the time to reach a bipolar state by the following crude argument [17]. Consider a nearly balanced network. When a link is flipped

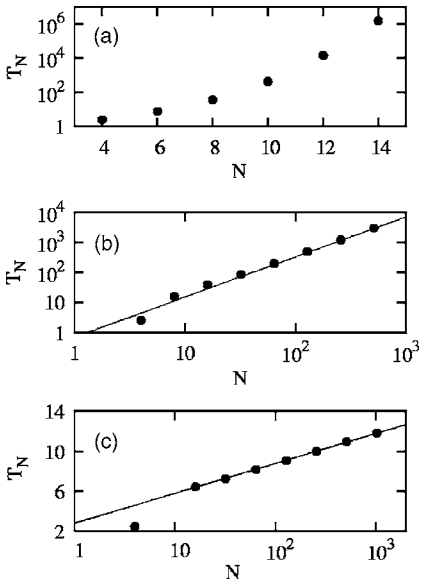

FIG. 3. Average absorption time as a function of N for an initially antagonistic society ($\rho_0=0$) for (a) $p=1/3$, (b) $p=1/2$, and (c) $p=3/4$. The line in (b) has slope 4/3.

on an imbalanced triad, then of the order of N new imbalanced triads will be created in the adjacent triads that contain the flipped link. Thus, starting near a balanced state, local triad dynamics is equivalent to a biased random walk in the state space of all network configurations, in which the bias is directed away from the balanced state, with the bias velocity v proportional to N. Conversely, when the network is far from balance, local triad dynamics is diffusive in character because the number of imbalanced triads will change by the order of $\pm N$ equiprobably in a single update. The corresponding diffusion coefficient D is then proportional to N^2. Since the total number of triads in a network of N nodes is $N_\Delta \sim N^3$, we therefore expect that the time T_N to reach balance will scale as

$$T_N \sim e^{vN_\Delta/D} \sim e^{N^2}. \tag{14}$$

When $p \geqslant 1/2$, paradise is reached with a probability that quickly approaches 1 as $N \to \infty$. At the threshold $p=1/2$, a naive estimate for the time T_N to reach paradise is given by the time at which the density of unfriendly links $a(t) \equiv 1-\rho(t)$ is of the order of N^{-2}, corresponding to one unfriendly link in the network. From Eq. (11), the criterion $a(T_N) \sim N^{-2}$ gives $T_N \sim N^4$. While simulations show that T_N does scales algebraically with N, the exponent value is much smaller [Fig. 3(b)]. The source of this smaller exponent is the existence of anomalously large fluctuations in the number of unfriendly links.

To determine these fluctuations in the thermodynamic limit, we write the number of unfriendly links $A(t) \equiv L^-(t)$ in the canonical form [18]

$$A(t) = La(t) + \sqrt{L}\eta(t), \tag{15}$$

where $a(t)$ is deterministic and $\eta(t)$ is a stochastic variable. Both a and η are size independent in the thermodynamic limit ($L \gg 1$), and the form of Eq. (15) assures that the average $\langle A \rangle$ and variance $\langle A^2 \rangle - \langle A \rangle^2$ grow linearly with the total number of links L. In the Appendix, we show that $\sigma \equiv \langle \eta^2 \rangle$ grows as

$$\sigma \sim \sqrt{t} \quad \text{as } t \to \infty. \tag{16}$$

Thus the time to reach paradise T_N is determined by the criterion that fluctuations in A become of the same order as the average—viz.,

$$\sqrt{L\sigma(T_N)} \sim La(T_N). \tag{17}$$

Using $a(t) \sim 1/\sqrt{t}$ from Eq. (11) together with Eq. (16) and $L \sim N^2$, we rewrite Eq. (17) as $N^2 T_N^{-1/2} \sim N T_N^{1/4}$. This leads to the estimate

$$T_N \sim N^{4/3}. \tag{18}$$

Above the threshold $p > 1/2$, paradise is approached exponentially quickly [see Eq. (13)] and the time to paradise scales logarithmically with network size:

$$T_N \sim (2p-1)^{-1} \ln N. \tag{19}$$

Interestingly, the estimates (18) and (19) coincide when $2p-1 \sim N^{-4/3} \ln N$. That is, there is a finite-width critical region near the phase transition due to finite-size effects. In the Appendix, we estimate this width by analyzing the fluctuations below the threshold $p < 1/2$ and obtain essentially this same result.

Summarizing, the asymptotics for the absorption time are

$$T_N \propto \begin{cases} \exp(N^2), & p < 1/2, \\ N^{4/3}, & p = 1/2, \\ (2p-1)^{-1} \ln N, & p > 1/2, \end{cases} \tag{20}$$

in agreement with the simulation results in Fig. 3.

IV. CONSTRAINED TRIAD DYNAMICS

A. Jamming and absorption time

In constrained triad dynamics, the number of imbalanced triads cannot increase in an update event, and the final state can either be balanced or jammed. A jammed state is one in which imbalanced triads exist and for which the flip of any link increases the number of imbalanced triads. Since this type of update is forbidden in CTD, there is no escape from a jammed state. Moreover, jammed states turn out to be much more numerous than balanced states (see Sec. IV D). In spite of this fact, we find that the probability for the network to reach a jammed state, $P_{\rm jam}(N)$, quickly goes to zero as N increases, except for the case of an initially antagonistic society ($\rho_0=0$), where $P_{\rm jam}(N)$ decays slowly with N (Fig. 4). Thus the final network state is almost always balanced for large N and consists either of one clique (paradise) or two antagonistic cliques. It is worth mentioning that we never observed "blinkers" [19] in simulations, although we cannot prove that such states do not exist. These are trajectories in the state space that evolve forever and correspond to a network in which all possible updates involve no change in the number of imbalanced triads.

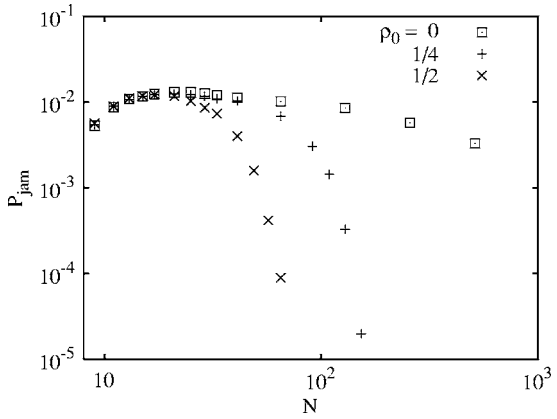

FIG. 4. Probability of reaching a jammed state P_{jam} as a function of N for several values of ρ_0.

Another fundamental feature of CTD is that the time T_N for a network of N nodes to reach its final state generically scales as $\ln N$. While T_N now depends on the initial condition, in contrast to LTD, this dependence occurs either in the amplitude or in lower-order corrections of the absorption time. Thus the logarithmic growth of T_N with N is a robust feature of CTD.

B. Final clique sizes

An unexpected feature of CTD is the phase transition for the difference in sizes C_1 and C_2 of the two final cliques as a function of ρ_0 (Fig. 5). We quantify this asymmetry by the scaled size difference $\delta=(C_1-C_2)/N$. For $\rho_0 \lesssim 0.4$ the cliques sizes in the final bipolar state are nearly the same size and $\langle \delta^2 \rangle \approx 0$. As ρ_0 increases toward $\rho_0^* \approx 0.65$, the size difference of the two cliques continuously increases. A sudden change occurs at ρ_0^*, beyond which the final network state is paradise. The probability distribution for δ is sharply peaked about its average value as $N \to \infty$ (Fig. 6). Since $\langle \delta^2 \rangle$ and the density of friendly links ρ_∞ are related by $\langle \delta^2 \rangle = 2\rho_\infty - 1$ in a large balanced society, uncorrelated initial relations generically lead to $\rho_\infty > \rho_0$. Thus CTD tends to drive a network into a friendlier final state.

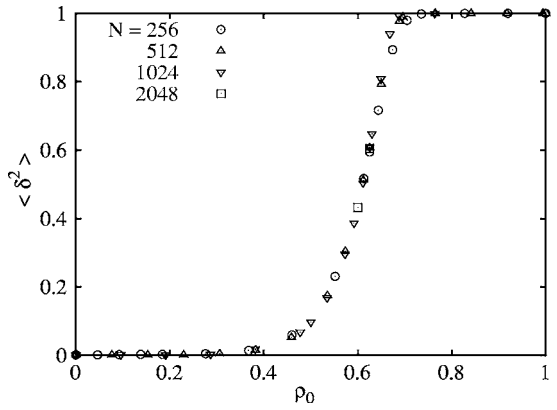

FIG. 5. Asymmetry of the final state as a function of the initial friendship density ρ_0 for several network sizes.

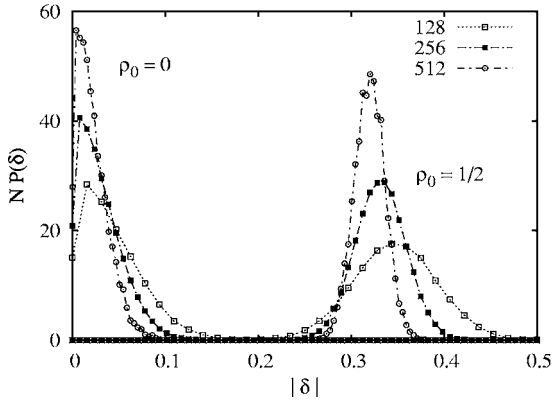

FIG. 6. Scaled probability distribution of the relative difference between the final clique sizes for $\rho_0=0$ and 1/2.

While we do not have a detailed understanding of this phase transition, we give a qualitative argument that suggests that a large network undergoes a sudden change from $\rho_\infty=0$ (two equal-size cliques) when $\rho_0<1/2$ to $\rho_\infty=1$ (paradise) when $\rho_0>1/2$. The fact that the transition appears to be located near $\rho_0^* \approx 0.65$ (Fig. 5) rather than at $\rho_0=1/2$ indicates that our approach is not a complete description for the transition.

We first assume, as observed in simulations of large networks, that jammed states do not arise. We also assume that a network remains uncorrelated during its early stages of evolution. Consequently the densities $\mathbf{n}^+ \equiv (n_0^+, n_1^+, n_2^+, n_3^+)$ of triads that are attached to a positive link are

$$\mathbf{n}^+ = (\rho^2, 2\rho(1-\rho), (1-\rho)^2, 0). \tag{21}$$

For a link to change from + to −, it is necessary that $n_1^+ + n_3^+ > n_0^+ + n_2^+$. From Eq. (21), this condition is equivalent to $4\rho(1-\rho) > 1$, which never holds. Similarly, the densities $\mathbf{n}^- \equiv (n_0^-, n_1^-, n_2^-, n_3^-)$ of triads attached to a negative link are

$$\mathbf{n}^- = (0, \rho^2, 2\rho(1-\rho), (1-\rho)^2). \tag{22}$$

The requirement $n_1^- + n_3^- > n_0^- + n_2^-$ now reduces to $1 > 4\rho(1-\rho)$, which is valid when $\rho \neq 1/2$.

Thus, for a large uncorrelated network, only negative links flip in CTD. Since the density of negative links is $1-\rho$, the governing rate equation is

$$\frac{d\rho}{dt} = 1 - \rho, \tag{23}$$

from which

$$\rho = 1 - (1-\rho_0)e^{-t}. \tag{24}$$

From the criterion $1-\rho(T_N) \sim N^{-2}$, corresponding to one unfriendly link remaining in the network, the time to reach paradise is given by $T_N \sim \ln N$, in agreement with simulations. According to Eq. (24) a network should evolve to paradise for any initial condition.

However, our simulations indicate that this homogeneous solution is unstable for $\rho_0<1/2$. In this case, the density of friendly links ρ initially still increases according to Eq. (24) until $\rho \approx 1/2$. At this point, correlations in the relationship

structure begin to arise and these ultimately lead to a bipolar society with $\rho_\infty \approx 1/2$. We now give a qualitative argument to support these observations.

When $\rho(t) \approx 1/2$, there are many partitions of the network into two subnetworks $\mathcal{S}_1$ and $\mathcal{S}_2$ of nearly equal sizes $C_1 = |\mathcal{S}_1|$ and $C_2 = |\mathcal{S}_2|$, for which the densities of friendly links within each subnetwork, ρ_1 and ρ_2, slightly exceed 1/2, while the density β of friendly links between subnetworks is slightly less than 1/2. Our basic point is that this small fluctuation is amplified by CTD so that the final state is two nearly equal-size cliques.

To appreciate how such an evolution can occur, we assume that relationships within each subnetwork and between subnetworks are homogeneous. Consider a negative link in $\mathcal{S}_1$. The densities of triads attached to this link are given by Eq. (22), with ρ replaced by β when the third vertex in the triad belongs to $\mathcal{S}_2$, and by Eq. (22), with ρ replaced by ρ_1 when the third vertex belongs to $\mathcal{S}_1$. The requirement that a link can change from $-$ to $+$ according to CTD now becomes

$$C_1[1 - 4\rho_1(1-\rho_1)] + C_2[1 - 4\beta(1-\beta)] > 0, \qquad (25)$$

which is always satisfied. Additionally, negative links within each subnetwork can change to positive with rate 1, while positive links within each subnetwork can never change.

Consider now a positive link between the subnetworks. The triad densities attached to this link are given by

$$\mathbf{n}_j^+ = (\beta\rho_j, \beta(1-\rho_j) + \rho_j(1-\beta), (1-\beta)(1-\rho_j), 0)$$

when the third vertex belongs to $\mathcal{S}_j$. Since

$$\beta(1-\rho_j) + \rho_j(1-\beta) - \beta\rho_j - (1-\beta)(1-\rho_j) = (2\rho_j - 1)(1-2\beta),$$

the change $+ \to -$ is possible if

$$[C_1(2\rho_1 - 1) + C_2(2\rho_2 - 1)](1 - 2\beta) > 0. \qquad (26)$$

Thus, if the situation arises where $\rho_1 > 1/2$, $\rho_2 > 1/2$, and $\beta < 1/2$, the network subsequently evolves to increase the density of intrasubnetwork friendly links and decrease the density of intersubnetwork friendly links. These link densities thus evolve according to the rate equations

$$\frac{d\rho_1}{dt} = 1 - \rho_1, \quad \frac{d\rho_2}{dt} = 1 - \rho_2, \quad \frac{d\beta}{dt} = -\beta, \qquad (27)$$

and give the instability needed to drive the network to a final bipolar state.

The last step in our argument is to note that when $C_1 \approx C_2 \approx N/2$, the number of ways, $\binom{N}{C_1}$, to partition the original network into the two nascent subnetworks $\mathcal{S}_1$ and $\mathcal{S}_2$ is maximal. Consequently, the partition $C_1 = C_2$ has the highest likelihood of providing the initial link density fluctuation, after which the homogeneous evolution (23) is replaced by the clique evolution (27) so that a homogeneous network organizes into two nearly equal-size cliques. Although our argument fails to account for the quantitative details of the transition shown in Fig. 5, the primary behaviors of $\langle \delta^2 \rangle$ in the two limiting cases of $\rho_0 \to 0$ and $\rho_0 \to 1$ are described correctly.

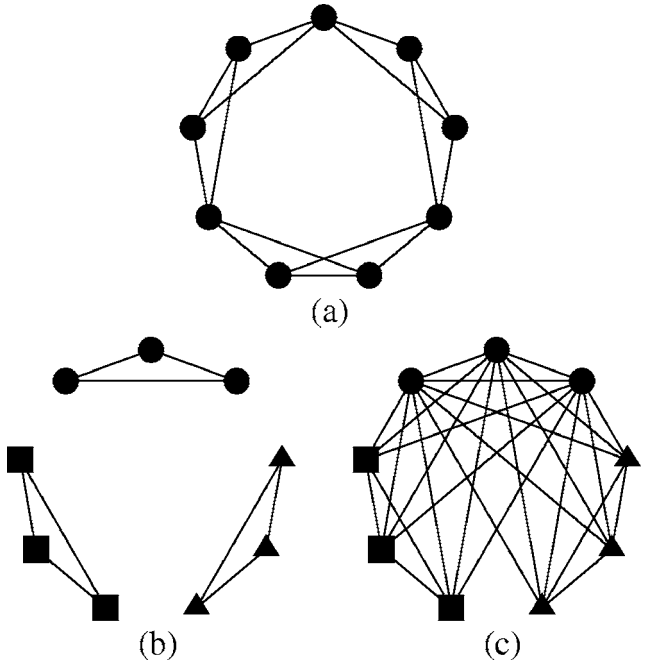

FIG. 7. Examples of jammed configurations for $N=9$ (only friendly links are displayed). (a) A jammed configuration that appeared in simulations. (b) A jammed state consisting of three mutually antagonistic cliques. (c) A jammed state derived from (b) in which the top clique from (b) is friendly toward the remaining two cliques.

C. Structure of jammed configurations

While jammed configurations can arise in CTD, we will now show that *jammed states are possible if and only if the network size is* $N=9$ *or* $N \geqslant 11$. To prove this statement, we first explicitly construct jammed configurations for $N=9$ and $N \geqslant 11$. Figure 7 shows three jammed configurations for $N=9$, the smallest possible N where jammed configurations can occur. The example in Fig. 7(a) was observed in simulations, while the jammed configuration in Fig. 7(b) consists of three antagonistic cliques of three nodes each. We now generalize this latter construction of jammed states to arbitrary $N \geqslant 11$.

Consider three mutually antagonistic cliques of sizes (m_1, m_2, m_3), with $m_1 + m_2 + m_3 = N$. A link within a clique is necessarily stable, as all attached triads are of type Δ_0 or Δ_2. Conversely, a negative link between clique 1 [circles in Fig. 7(b)] and clique 2 (squares) is attached to both stable and imbalanced triads. There are $m_1 - 1 + m_2 - 1$ attached stable triads of type Δ_2, where the third node of the triad is either within clique 1 or clique 2, and m_3 attached imbalanced triads of type Δ_3, where the third node is in clique 3 (triangles). The requirement for link stability among these cliques is then

$$m_1 + m_2 > m_3 + 2,$$

$$m_2 + m_3 > m_1 + 2,$$

$$m_3 + m_1 > m_2 + 2, \tag{28}$$

where the last two equations arise by cyclic permutations of the first.

We term a partition (m_1, m_2, m_3) "jammed" if it satisfies the inequalities (28). By summing pairs of Eqs. (28), we find $m_j \geq 3$, $j=1,2,3$. Thus jammed partitions are possible only for networks of size $N=m_1+m_2+m_3 \geq 9$. Following the rules in Eq. (28), the following partitions are jammed: for $N=3k$ with $k \geq 3$, partitions of the form (k,k,k); for $N=3k+2$ with $k \geq 3$, $(k,k+1,k+1)$; and for $N=3k+1$ with $k \geq 4$, $(k,k,k+1)$. Thus jammed partitions indeed exist for $N=9$ and $N \geq 11$.

Finally, we show that jammed states are impossible for $N \leq 8$ and $N=10$. As a preliminary, we need the following.

Lemma. Let ABC be imbalanced, ADC balanced, and $s_{AC}=-1$. Then one of the two triads ABD and BDC is balanced, and the other is imbalanced.

Proof. Let $s_{XYZ}=s_{XY}s_{YZ}s_{ZX}$ be the sign of the triad XYZ. For the imbalanced ABC triad we have $s_{ABC}=-1$ while for the balanced triad $s_{ACD}=1$. Using additionally the identities $s_{XY}=s_{YX}$, $s_{XY}^2=1$, we obtain the product of the signs of triads ABD and BDC:

$$\begin{aligned} s_{ABD}s_{BDC} &= s_{AB}s_{BD}s_{DA}s_{BD}s_{DC}s_{CB} = s_{AB}s_{DA}s_{DC}s_{CB} \\ &= s_{AB}s_{BC}s_{CA}s_{AC}s_{CD}s_{DA} = s_{ABC}s_{ACD} = -1, \end{aligned}$$

from which the lemma immediately follows.

Now suppose that there is a jammed state in a network with an even number of nodes $N=2k$. By definition, there is at least one imbalanced triad in the jammed state; let ABC be such an imbalanced triad with $s_{AC}=-1$. Since the state is stable, out of the $N-2=2k-2$ triads attached to the link AC, at least k are balanced. [Note that this construction requires $k \geq 3$; however, it is trivial to show that there is no jammed state for the case $k=2(N=4)$]. Take k such balanced triads and denote them AD_jC, $j=1,\ldots,k$. To each pair of triads ABC and AD_jC we now apply the lemma. Then there is a certain number x of imbalanced triads among ABD_j and a certain number y of imbalanced triads among CBD_j, with $x+y=k$. Stability ensures that there are at most $k-2$ imbalanced triads attached to the link AB. Recalling that ABC is imbalanced and that there are x imbalanced triads ABD_j, we obtain $x+1 \leq k-2$. A similar argument applied to link CB leads to $y+1 \leq k-2$. Summing these inequalities and using $x+y=k$ gives $k \geq 6$ or $N \geq 12$ for even N.

The case of odd N is similar. We set $N=2k-1$, with $k \geq 3$. Now each link is attached to at least k balanced triads and at most $k-1$ imbalanced triads. Repeating the same argument as for the even case we obtain the conditions $x+1 \leq k-1$ and $y+1 \leq k-1$ with $x+y=k$, which leads to $N \geq 9$ for odd N.

D. Number of jammed configurations

Last, we show that the number of jammed configurations greatly exceeds the number of balanced configurations. The total number of distinct network configurations is 2^L. Each balanced state has the form $(m_1, m_2 | m_1+m_2=N)$, and we enumerate all classes of balanced states by counting the integer solutions of $m_1+m_2=N$, with $0 \leq m_1 \leq m_2$. Therefore the number of classes of balanced states is $B=k+1$ for $N=2k$ and $N=2k+1$. The total number $\mathcal{B}$ of balanced states is determined from

$$\mathcal{B} = \sum_{m_1+m_2=N} \binom{m_1+m_2}{m_1} = 2^N \tag{29}$$

and is thus much larger than the number of classes of balanced states.

For the number of classes of jammed states J and the number of jammed states $\mathcal{J}$, we can only establish lower bounds. For large N, instead of exact counting we employ a continuum description. From Eq. (28), the number of jammed partitions is equal to N^2 times the area $\mathcal{A}$ of the region inside the triangle $x_1+x_2+x_3 \leq 1$ defined by inequalities $x_1+x_2 \geq x_3$, $x_2+x_3 \geq x_1$, and $x_3+x_1 \geq x_2$; this area is $\mathcal{A}=1/8$. We also divide by $3!=6$ to account for overcounting different permutations of m_1, m_2, m_3. Thus $J > N^2/48$. We could improve this bound by counting additional jammed states built from the construction in Fig. 7(c), but this contribution would not affect the N dependence of the bound. However, we do not know whether $J \propto N^2$ or J grows faster than N^2 due to the existence of jammed states, such as those in Fig. 7(a), that are not in the classes described in Sec. IV C.

We obtain a lower bound for $\mathcal{J}$ in a similar manner to that in Eq. (29) for counting the number of balanced states,

$$\mathcal{J} > \sum_{\text{jammed}} \binom{m_1+m_2+m_3}{m_1,m_2,m_3} \approx 3^N, \tag{30}$$

where the sum is over jammed partitions and the trinomial coefficient is

$$\binom{m_1+m_2+m_3}{m_1,m_2,m_3} = \frac{(m_1+m_2+m_3)!}{m_1!m_2!m_3!}.$$

The summand in Eq. (30) is sharply peaked around $m_1=m_2=m_3=N/3$, and therefore the sum is very close to 3^N which is the sum over *all* partitions. Again, the lower bound may be weak because of the neglect of nontripartite jammed configurations.

In summary, we find that $\mathcal{J} > 3^N \gg 2^N = \mathcal{B}$. Thus the total number of jammed states greatly exceeds the total number of balanced states. Nevertheless, for a random initial condition, the probability to end in a balanced state is very close to 1 while the probability to end in a jammed state is negligible; that is, the basin of attraction of balanced states greatly exceeds the basin of attraction of jammed states.

V. SUMMARY AND DISCUSSION

In social relations, we may encounter the uncomfortable situation of an imbalanced triad. If you have two friends that develop a mutual animosity, then an imbalanced triad of relations exists. You will then likely have to choose between these two friends, thereby resolving the social conflict and restoring the relationship triads to balance. In this work, we implemented simple and prototypical dynamical rules for healing imbalanced triads and we investigated the resulting evolution of these social networks.

In the case of local triad dynamics, a finite network falls into a socially balanced state, where no frustrated triads remain. The time to reach this final state depends very sensitively on the propensity p for forming friendly links in the update events that heal social imbalance. For an infinite network, the balanced state is never reached when $p<1/2$ and the system remains in a stationary state. The density of unfriendly links gradually decreases and the network undergoes a dynamical phase transition to an absorbing, paradise state for $p \geqslant 1/2$.

We also examined the dynamics in which an additional global constraint is imposed that the number of imbalanced triads in the entire network cannot increase in an update event. The virtue of this dynamics is that the final outcome is always reached quickly. A downside, however, is that the final configuration of the network may be jammed—these are states that are not balanced, but where flipping any link increases the number of imbalanced triads. Fortunately, the probability of reaching a jammed state is vanishingly small and the final state is either a two-clique bipolar state or paradise.

As alluded to in the Introduction, a natural application for social balance ideas is to international relations, with the prelude to World War I being a particularly appropriate example. For example, the Three Emperors' League (1872, revived in 1881) aligned Germany, Austria-Hungary, and Russia, leaving France isolated. However, the Turkish-Russian war (1877) and tension between Austria-Hungary and the Balkan states unraveled Russia's participation in the League, and a bipartite agreement between Germany and Russia lapsed in 1890. In the meantime, the Triple Alliance was formed in 1882 that joined Germany, Austria-Hungary, and Italy into a bloc that continued until World War I.

On the other hand, a French-Russian alliance was formed over the period 1891–1894 that ended France's diplomatic isolation with respect to the Triple Alliance. Subsequently an Entente Cordiale between France and Great Britain was consummated in 1904 and then a British-Russian agreement in 1907, after long-standing tensions between these two countries, that then bound France, Great Britain, and Russia into the Triple Entente. While our historical account of these Byzantine maneuvers is very incomplete (see Ref. [14] for more information), the basic point is that among the six countries that comprised the two major alliances, bipartite relationships changed as triads became unbalanced and there was a reorganization into a balanced state of the Triple Alliance and the Triple Entente that became the two main protagonists at the start of World War I.

On the theoretical side, there are several avenues for additional research. One possibility is to relax the definition of imbalanced somewhat. This is the direction followed by Davis [9] who proposed the "clusters model" in which triads with three unfriendly relations are deemed acceptable. The clusters model thus allows for the possibility that "an enemy of my enemy may still be my enemy." This more relaxed definition for imbalanced triads may lead to interesting dynamical behavior that will be worthwhile to explore.

Another natural generalization of the balance model would be to ternary relationships of positive $+$, negative $-$, or indifferent 0. These relations may lead to the emergence of cliques (groups of mutual friends who dislike other people) and communities (groups of mutual friends with no relations with other people). It would be interesting to study the number of cliques and number of communities as a function of network size and the density of indifferent relationships. Communities on the Web can be effectively identified [20], and these results may allow for useful comparisons between data and model predictions. Finally, relations need not be symmetric; that is, s_{ij} may be different from s_{ji}, and it may be interesting to generalize the basic notions of balance to networks with such asymmetric interactions.

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APPENDIX: FLUCTUATIONS IN LOCAL TRIAD DYNAMICS

In this appendix, we compute the normalized variance $\sigma=\langle \eta^2 \rangle$. We focus on the most interesting case of the critical regime $p=1/2$, where fluctuations exhibit the asymptotic behavior of Eq. (16). Then we briefly discuss the two regimes $p<1/2$ and $p>1/2$.

We first note that A changes according to

$$A \to \begin{cases} A-1 \text{ rate } N_3, \\ A-1 \text{ rate } pN_1, \\ A+1 \text{ rate } (1-p)N_1, \end{cases} \tag{A1}$$

which describe the processes $N_3 \to N_2$, $N_1 \to N_0$, and $N_1 \to N_2$, respectively. From (A1) we obtain

$$\frac{d}{dt}\langle A \rangle = -\langle N_3 \rangle - p\langle N_1 \rangle + (1-p)\langle N_1 \rangle, \tag{A2}$$

which simplifies to

$$\frac{d}{dt}\langle A \rangle = -\langle N_3 \rangle \tag{A3}$$

at the threshold $p=1/2$. Since $\langle A \rangle \propto a$ and $\langle N_3 \rangle \propto \langle A^3 \rangle \propto a^3$ to lowest order, Eq. (A3) can be written as

$$\frac{da}{dt} = -a^3, \tag{A4}$$

whose solution is given by Eq. (11). Similarly from (A1) we obtain

$$\frac{d}{dt}\langle A^2 \rangle = \langle (-2A+1)N_3 \rangle + p\langle (-2A+1)N_1 \rangle + (1-p)\langle (2A+1)N_1 \rangle, \tag{A5}$$

which, for $p=1/2$, simplifies to

$$\frac{d}{dt}\langle A^2\rangle = \langle N_3\rangle + \langle N_1\rangle - 2\langle AN_3\rangle. \tag{A6}$$

Subtracting Eq. (A3) multiplied by $2\langle A\rangle$ from Eq. (A6), we obtain the evolution equation for the variance:

$$\frac{d}{dt}(\langle A^2\rangle - \langle A\rangle^2) = \langle N_3\rangle + \langle N_1\rangle + 2(\langle A\rangle\langle N_3\rangle - \langle AN_3\rangle).$$

Now using standard methods [18] to compute moments of the stochastic variable $A = La + \sqrt{L}\eta$, Eq. (15), we obtain the leading behavior $\langle A^2\rangle - \langle A\rangle^2 \propto \sigma$, while $\langle N_3\rangle \propto a^3$ and $\langle N_1\rangle \propto 3a(1-a)^2$. Similarly, the leading terms in $\langle A\rangle\langle N_3\rangle$ and $\langle AN_3\rangle$ cancel, while the next correction is

$$\langle A\rangle\langle N_3\rangle - \langle AN_3\rangle \propto 3a^2\sigma - 6a^2\sigma. \tag{A7}$$

Using these results for the various moments, the variance satisfies

$$\frac{d\sigma}{dt} = -6a^2\sigma + a^3 + 3a(1-a)^2. \tag{A8}$$

Dividing Eq. (A8) by Eq. (A4) we obtain

$$\frac{d\sigma}{da} = \frac{6}{a}\sigma - \frac{a^3 + 3a(1-a)^2}{a^3}. \tag{A9}$$

Since $\sigma = a^6$ solves the homogeneous equation $d\sigma/da = 6\sigma/a$, we seek a solution of the inhomogeneous equation (A9) in the form $\sigma = a^6 s(a)$. Equation (A9) becomes

$$\frac{ds}{da} = -\frac{a^3 + 3a(1-a)^2}{a^9} = -\frac{3}{a^8} + \frac{6}{a^7} - \frac{4}{a^6},$$

whose solution is

$$s = \frac{3}{7}\frac{1}{a^7} - \frac{1}{a^6} + \frac{4}{5}\frac{1}{a^5} + C. \tag{A10}$$

Thus $\sigma = a^6 s(a)$, with $s(a)$ given by Eq. (A10). The integration constant C is fixed to satisfy the initial condition $\sigma(a_0) = 0$. In particular, for a totally antagonistic initial network ($\rho_0 = 0$ or $a_0 = 1$), $C = -8/35$. In this case $a = 1/\sqrt{1+2t}$, and the variance becomes

$$\sigma = \frac{3}{7}\frac{1}{a} - 1 + \frac{4}{5}a - \frac{8}{35}a^6. \tag{A11}$$

The leading asymptotic behavior $\sigma \to (3\sqrt{2}/7)t^{1/2}$ holds independent of the initial condition. Hence we establish the crucial result (16), which leads to the asymptotic behavior (18) for the absorption time for $p = 1/2$.

For $p \neq 1/2$, or $\epsilon = 1 - 2p \neq 0$, we recast Eq. (A2) into

$$\frac{da}{dt} = 3\epsilon a(1-a)^2 - a^3, \tag{A12}$$

which is of course identical to Eq. (9). Following the same steps that led to Eq. (A8), we then derive for the variance

$$\frac{d\sigma}{dt} = a^3 + 3a(1-a)^2 - 6\sigma[a^2 - \epsilon(1-a)(1-3a)]. \tag{A13}$$

When $p < 1/2$, both a and σ quickly approach stationary values

$$a_\infty = \frac{\sqrt{3\epsilon}}{\sqrt{3\epsilon}+1}, \quad \sigma_\infty = \frac{3}{4}\frac{1}{\sqrt{3\epsilon}}\frac{1+\epsilon}{(\sqrt{3\epsilon}+1)^2}. \tag{A14}$$

The density of unfriendly links in a finite system therefore exhibits fluctuations of the order of $\sqrt{\sigma_\infty/L}$ about the average density a_∞. Close to the phase transition point, the magnitude of fluctuations eventually becomes comparable with the average. From $a_\infty \sim \sqrt{\sigma_\infty/L}$ and Eq. (A14), we find that this equality occurs when $\epsilon \sim N^{-4/3}$; this gives an estimate of the width of the phase transition region due to finite-size effects.

When $p > 1/2$, both a and σ vanish as $t \to \infty$. A straightforward asymptotic analysis of Eq. (A13) yields

$$\sigma \to \frac{a}{|\epsilon|} \equiv \frac{a}{2p-1} \text{ as } a \to 0. \tag{A15}$$

Fluctuations become comparable with the deterministic part when $a \sim \sqrt{\sigma/L} \sim \sqrt{a/(2p-1)L}$—that is, when $La \sim (2p-1)^{-1}$. Using $a \sim e^{-3(2p-1)t}$ from Eq. (13) we estimate the time to reach paradise to be

$$T_N \sim (2p-1)^{-1}\ln[(2p-1)N^2]. \tag{A16}$$

The difference between this result and Eq. (19), which was established using the naive criterion $La \sim 1$, is small because the factor $2p-1$ appears inside the logarithm.

[1] F. Heider, Psychol. Rev. **51**, 358 (1944); F. Heider, J. Psychol. **21**, 107 (1946); F. Heider, *The Psychology of Interpersonal Relations* (Wiley, New York, 1958).

[2] S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications* (Cambridge University Press, New York, 1994).

[3] K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, England, 1991).

[4] D. Cartwright and F. Harary, Psychol. Rev. **63**, 277 (1956); F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs* (Wiley, New York, 1965).

[5] K. Lewin, *Field Theory in Social Science* (Harper, New York, 1951).

[6] T. M. Newcomb, *The Acquaintance Process* (Holt, Rinehart & Winston, New York, 1961).

[7] R. K. Leik and B. F. Meeker, *Mathematical Sociology* (Prentice-Hall, Englewood Cliffs, NJ, 1975).

[8] P. Bonacich, http://www.sscnet.ucla.edu/soc/faculty/bonacich

[9] J. A. Davis, Hum. Relat. **20**, 181 (1967).

[10] N. P. Hummon and T. J. Fararo, J. Math. Sociol. **20**, 145 (1995).
[11] P. Doreian and D. Krackhard, J. Math. Sociol. **25**, 43 (2001).
[12] N. P. Hummon and P. Doreian, Soc. Networks **25**, 17 (2003).
[13] See, e.g., M. Moore, Eur. J. Soc. Psychol. **9**, 323 (1979).
[14] W. L. Langer, *European Alliances and Alignments 1871-1890*, 2nd ed. (Knopf, New York, 1950); B. R. Schmitt, *Triple Alliance and Triple Entente* (Holt, Rinehart, and Winston, New York, 1934).
[15] K. Kulakowski, P. Gawronski, and P. Gronek, Int. J. Mod. Phys. C (to be published). P. Gawronski, P. Gronek, and K. Kulakowski, e-print physics/0501160.
[16] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).
[17] We use the fact that the first-passage time to an absorbing point in a finite one-dimensional interval of length L with a bias away from the absorbing point is of the order of $e^{vL/D}$. See S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, New York, 2001).
[18] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 2003).
[19] Blinker states were found to occur in the Ising model in three and higher dimensions with zero-temperature Glauber kinetics. See V. Spirin, P. L. Krapivsky, and S. Redner, Phys. Rev. E **63**, 036118 (2001); **65**, 016119 (2002).
[20] L. A. Adamic and E. Adar, Soc. Networks **25**, 211 (2003).