A billiard-theoretic approach to elementary one-dimensional elastic collisions

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A simple relation is developed between the elastic collisions of freely moving particles in one dimension and a corresponding billiard system. For two particles with masses m_1 and m_2 on the half-line $x>0$ that approach an elastic barrier at $x=0$, the corresponding billiard system is an infinite wedge. The collision history of the two particles can be easily inferred from the corresponding billiard trajectory. This connection explains the classic demonstrations of the "dime on the superball" and the "baseball on the basketball" that are a staple in elementary physics courses. It also is shown that three elastic particles on an infinite line and three particles on a finite ring correspond, respectively, to the motion of a billiard ball in an infinite wedge and on a triangular billiard table. It is shown how to determine the angles of these two sets in terms of the particle masses. © *2004 American Association of Physics Teachers.*
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I. INTRODUCTION

A standard topic in introductory mechanics courses is elastic collisions. There are two very nice lecture demonstrations to accompany this topic.[1] The first is the "dime on the superball." Here one carefully places a dime on top of a superball and drops this composite system on a hard floor. Before doing so, the instructor asks the class to guess the maximum height of the dime $h_{\max}$ compared to the initial height h_0. For perfectly elastic collisions and in the limit where the mass of the dime is vanishingly small, it is easy to show that $h_{\max}=9h_0$![2] Although this theoretical limit cannot be achieved, the dime can easily hit the ceiling of a normal classroom when the superball is dropped from chest level.

Another demonstration of this genre is to carefully place a baseball (hardball) of mass m on top of a basketball of mass M and then drop the two together. The question for the class is how high does the basketball rise after a collision with the floor? It turns out that for $m=M/3$ (which is close to the actual mass ratio for a baseball and a basketball) and again for perfectly elastic collisions, the basketball hits the floor and stays there. However, be sure to warn the class to beware of the rapidly moving baseball. In the theoretically ideal situation, it rises to a height $h_{\max}=4h_0$. Working out these two examples are left as exercises for the reader.

The extension of this simple two-particle system to arbitrary mass ratios presents many interesting and unexpected challenges. In what follows, we generally ignore gravity because it plays a negligible role during the collisions. If the upper mass is much larger than the lower mass, then for two separated masses that approach an elastic barrier, there will be a large number of collisions before the two masses diverge and ultimately recede from the barrier. This dynamics can, in principle, be analyzed by applying momentum conservation to map out the particle trajectories. This approach is tedious, however, and does not provide physical insight (see the Appendix).

The goal of this article is to present a simple connection between the motion of few-particle elastically colliding systems in one dimension and a corresponding billiard system. For two particles and an elastic barrier, the corresponding billiard ball moves in a two-dimensional wedge-shaped billiard table with elastic and specular reflection each time the ball hits the boundary of the table.[3–6] Specular means that the angle of incidence equals the angle of reflection. This description can be greatly simplified by recognizing that specular reflection at a boundary is geometrically identical to passing straight through the boundary, where on the other side of the boundary there is an identical image of the wedge. By repeating this construction, the end result is that billiard motion in the wedge is equivalent to a straight trajectory in a plane that is "tiled" by a fan of wedges. By using this equivalence, it is easy to completely solve the collision history of the original two-particle and barrier system.

We then extend this approach to treat three elastically colliding particles of arbitrary masses on an infinite one-dimensional line. We ask the question: how many collisions occur when two cannonballs are approaching, with an intervening elastic ping-pong ball that is rattling between them?[7] This system can again be mapped onto the motion of a billiard ball in an infinite wedge whose opening angle depends on the three masses. Finally, we discuss the motion of three particles on a finite ring.[8,9] This system can be mapped onto the motion of a billiard ball on a triangular table. From this connection, we can gain many insights about the collisional properties of the three particle on the ring.

These exactly soluble few-body systems naturally open new issues. For example, what happens when the number of particles, N, becomes large? In one dimension, the momentum distribution of a polydisperse system of elastic particles converges to a finite-N version of the Gaussian distribution.[10] However, the transport properties appear to be anomalous. In particular, there is a lingering controversy about the nature of heat flow through such a system,[11] and the nature of the thermodynamic limit of this system is not yet fully understood.

More generally, it is natural to inquire about the roles of inelastic collisions and the spatial dimension on the dynamics. This question is a natural entry to the burgeoning field of granular media.[12] Although this area is beyond the scope of this letter, it is worth mentioning a few relevant topics. A

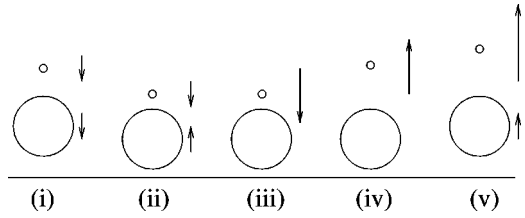

Fig. 1. Collision sequence for the dime-superball system. The arrows (drawn to scale) denote the velocities at each collision stage.

particularly intriguing feature of inelastic systems is the phenomenon of "inelastic collapse," where clumps of particles with negligible relative motion form. This collapse occurs when the number of particles is sufficiently large or when the collisions are sufficiently inelastic.[13–15] Some of the methods described here may be useful for understanding these systems. The inclusion of other parameters also leads to a wealth of new effects. For example, a variety of collisional transitions occur when inelastic particles are pushed by a massive wall,[16] while the presence of gravity in an elastic system of two particles and a wall leads to both quasiperiodic and chaotic behavior.[17] Again, a billiard-theoretic perspective may provide helpful insights into these systems.

In the next section, we discuss the dime on the superball and the baseball on the basketball problems by elementary means. Then in Sec. III we show how to map these systems onto the motion of a billiard ball in a wedge domain. The same approach is used in Sec. IV to show the equivalence of three particles on an infinite line to a billiard ball in an infinite wedge. In Sec. V the equivalence of three particles on a ring to a triangular billiard is discussed. A brief discussion is given in Sec. VI.

II. DIME ON A SUPERBALL AND A BASEBALL ON A BASKETBALL

To aid in the analysis of the dime on the superball, it is helpful to imagine that the two particles are separated. Again, gravity is neglected throughout the collisions; its only role is to give the final height of the dime in terms of its velocity immediately after the last collision. Figure 1 shows the velocities of the dime and superball given the assumption that the mass m of the dime is negligible compared to that of the superball M; all collisions are assumed to be perfectly elastic. The following collision sequence occurs:

(i) The dime and the superball both approach the ground with velocity $-v$.
(ii) The superball hits the ground and reverses direction so that its velocity is $+v$.
(iii) For $m/M\to 0$, the center-of-mass coincides with the center of the superball. In this reference frame, the dime approaches the superball with velocity $-2v$.
(iv) After the dime–superball collision in the center-of-mass frame, the dime moves with velocity $+2v$, while the superball remains at rest.
(v) Returning to the original lab frame, the superball moves with velocity $+v$, while the dime moves at velocity $+3v$. This velocity of $+3v$ leads to the dime rising to a final height that is nine times that of the superball in the presence of gravity.

For the baseball on top of the basketball, the same analysis now gives the following collision sequence (again assuming perfectly elastic collisions):

(i)–(ii) The basketball hits the ground with velocity $-v$ and reverses direction so that its velocity is $+v$.
(iii) For $m/M=1/3$, the center-of-mass has velocity $+v/2$. In the center-of-mass reference frame, the baseball has velocity $-3v/2$, while the basketball has velocity $+v/2$.
(iv) After the collision between the two balls in the center-of-mass frame, their velocities are reversed.
(v) In the original lab frame, the baseball has velocity $+2v$, while the basketball is at rest.

In both these cases, there are just two collisions—an initial collision of the lower ball with the floor and a second collision between the two balls. Subsequently, the upper ball moves faster than the lower ball and there would be no more collisions in the absence of gravity. However, if the upper ball is heavier than the lower ball, there will be many collisions before the two balls recede from the floor and from each other. How many total collisions occur for this system? What are the details of the collision sequence? These questions should be simple to answer, because only energy and momentum conservation are involved. However, when the upper ball is much heavier than the lower ball, the number of collisions is large and a direct solution is tedious. As we discuss in Sec. III there is an elegant mapping of this collision problem to an equivalent billiard system that provides a remarkable simplification.

III. BILLIARD MAPPING

We now map the problem of two colliding particles and an elastic barrier into an equivalent billiard system. From this approach, the entire particle collision history can be inferred in a simple geometric manner. To be general, suppose that the particles have masses m_1 and m_2 and are located, respectively, at x_1 and x_2, with $x_1<x_2$ (and $x_1,x_2>0$). The trajectories of the two particles in one dimension are equivalent to the trajectory $[x_1(t),x_2(t)]$ of an effective billiard ball in the two-dimensional domain defined by $x_1,x_2>0$ and $x_1<x_2$. The billiard ball hitting the boundary $x_1=0$ corresponds to a collision between the lower particle and the floor, while the ball hitting the boundary $x_1=x_2$ corresponds to a collision between the two particles.

Now define the following billiard rescaling:[3–6]

$$y_i=x_i\sqrt{m_i}\quad w_i=v_i\sqrt{m_i}, \tag{1}$$

for $i=1$ and 2. In these coordinates, the constraint $x_1<x_2$ becomes

$$y_2>\sqrt{\frac{m_2}{m_1}}\,y_1. \tag{2}$$

Thus the allowed region is now a wedge-shaped domain (see Fig. 2) with opening angle

$$\alpha\equiv\tan^{-1}\sqrt{\frac{m_1}{m_2}}. \tag{3}$$

The crucial feature of this rescaling is that it ensures that all collisions of the billiard ball with the boundary of the domain are specular. To demonstrate this point, we take the energy and momentum conservation statements:

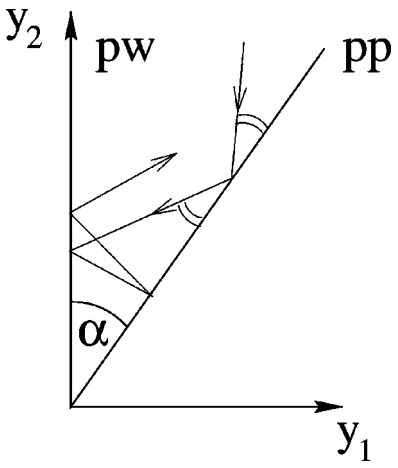

Fig. 2. Allowed wedge in y_i coordinates. A sample billiard ball trajectory is shown. Hitting the y_2 axis corresponds to a particle-wall collision (denoted by pw), while hitting the line $y_2=y_1\sqrt{m_2/m_1}$ corresponds to a particle–particle collision (denoted by pp).

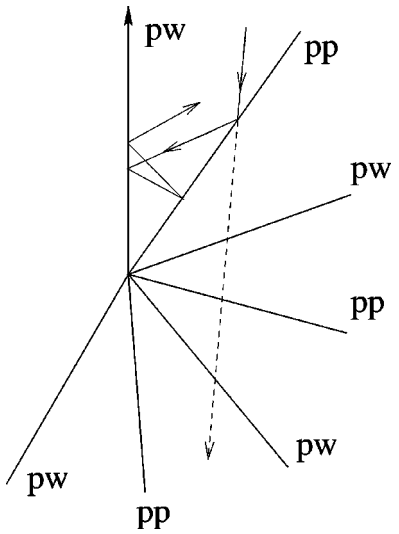

Fig. 3. Periodic extension of the allowed wedge. The trajectory in the original wedge is equivalent to the straight trajectory shown (dashed).

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_2v_2'^2, \tag{4a}$$

$$m_1v_1+m_2v_2=m_1v_1'+m_2v_2', \tag{4b}$$

where the prime denotes a particle velocity after a collision, and rewrite these conservation laws in rescaled coordinates to give

$$w_1^2+w_2^2=w_1'^2+w_2'^2, \tag{5a}$$

$$\sqrt{m_1}w_1+\sqrt{m_2}w_2=\sqrt{m_1}w_1'+\sqrt{m_2}w_2'. \tag{5b}$$

Equation (5a) states that the speed of the billiard ball is unchanged by a collision. Equation (5b) can be rewritten as $(\sqrt{m_1},\sqrt{m_2})\cdot(w_1,w_2)$ remains constant in a collision. Because the vector $(\sqrt{m_1},\sqrt{m_2})$ is tangent to the constraint line $y_2=y_1\sqrt{m_2/m_1}$, the projection of the rescaled velocity onto this line is constant in a particle–particle collision. It also is intuitively clear that in a particle–wall collision, the rescaled velocity is preserved. As a result, the collision sequence of two elastically colliding particles and an elastic barrier in one dimension is completely equivalent to the trajectory of a billiard ball in a two-dimensional wedge of opening angle α given by Eq. (3) in which each collision with the boundary is specular.

A more dramatic simplification arises in the y_i coordinates by recognizing that because each reflection is specular, the trajectory in the wedge is the same as a straight trajectory in the periodic extension of the wedge (see Fig. 3). Each collision is alternately a particle–particle or a particle-wall collision, so that the identity of each barrier alternates between pp and pw. From this description, we immediately deduce that the collision sequence of the two-particle system ends when the trajectory of the billiard ball no longer crosses any wedge boundary. As shown in Fig. 3, when the original trajectory is extended in this manner, it will ultimately pass through six wedges. Thus five collisions (particle–wall and particle–particle) occur in total.

The maximum number of wedges that can be packed in the half plane is π/α. A straight trajectory of the billiard ball typically passes through all these wedges. This is therefore the maximum number of collisions, $N_{\max}$, possible in the two-particle system. In the limit where $m_1\ll m_2$, we thereby find (using Eq. (3))

$$N_{\max}\simeq\pi\sqrt{\frac{m_2}{m_1}}. \tag{6}$$

Thus the total number of collisions in the original particle system emerges from extremely simple geometric considerations of the equivalent billiard.

From Fig. 3, the incidence angle of the billiard ball at each boundary increases by a factor α after each collision. Furthermore, using the constancy of the rescaled velocity w, we also can deduce the particle velocities at every collision stage. We now illustrate this approach by reconsidering our initial examples from this billiard-theoretic perspective.

IV. DIME ON A SUPERBALL AND A BASEBALL ON A BASKETBALL: A SECOND LOOK

For the dime on the superball, the opening angle of the wedge has the limiting behavior $\alpha=\pi/2-\delta$, with $\delta\approx\sqrt{m_2/m_1}$ as $m_2/m_1\to0$. In Fig. 4 the trajectory of the corresponding billiard ball is shown in the y_1–y_2 coordinate system. Because the dime and the superball have the same initial velocities, the incoming trajectory in the wedge is parallel to the initial pp boundary. The distance to the pp

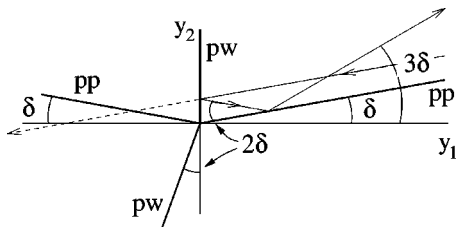

Fig. 4. Allowed wedge in y_i coordinates for the dime and superball. A trajectory corresponding to the dime and the superball approaching the wall at the same velocity is shown. The dashed line shows the trajectory in the periodic extension of the wedges.

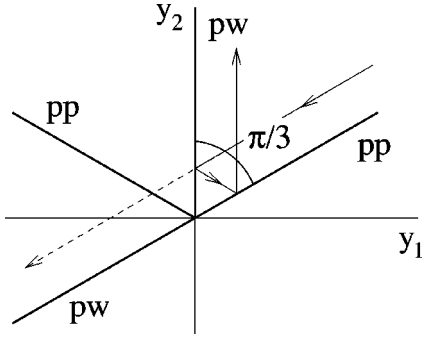

Fig. 5. Allowed wedge in y_i coordinates for the baseball–basketball system with $m_1=3m_2$. The dashed line again shows the billiard trajectory in the periodic extension of the wedges.

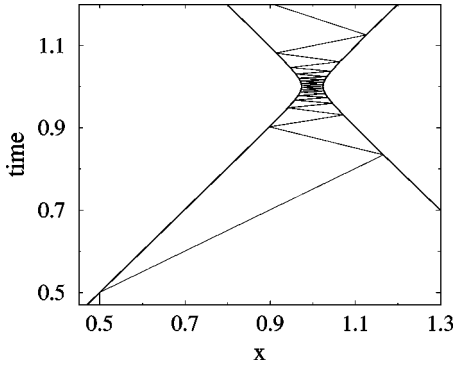

Fig. 6. Space-time diagram of the typical evolution of two cannonballs (heavy lines) approaching an initially stationary ping-pong ball (light line). The cannonballs each have mass $M=1$ and initial conditions $[x_1(0),v_1(0)]=(0,1)$ and $[x_3(0),v_3(0)]=(2,-1)$. A ping-pong ball of mass $m=0.005$ is initially at $x_2(0)=1/2$. There are 31 collisions in total before the three particles recede. The first 30 collisions are shown.

boundary is proportional to the initial separation of the dime and the superball. (If, initially, $x_2=x_1+\epsilon$, then $y_2=y_1\sqrt{m_2/m_1}+\sqrt{m_2}\epsilon$.)

After the superball collides with the wall, the billiard trajectory is incident on the pp boundary with inclination angle 2δ (see Fig. 4). After specular reflection from this boundary, the final outgoing trajectory is then inclined at an angle 3δ with respect to the horizontal. This inclination angle means that the final velocity of the dime is three times that of the superball. Thus in the presence of gravity, an ideal dime will rise to nine times its initial height.

This same result can be obtained even more simply by drawing a straight trajectory through the periodic extension of the wedges. In this case, the final trajectory is inclined at an angle of $\pi/2-3\delta$ with respect to the last periodically extended pw boundary. This construction again implies that the outgoing trajectory is inclined at an angle of 3δ with respect to the initial pp boundary.

For a basketball of mass $m_1=3m$ and a baseball of mass $m_2=m$, the opening angle of the wedge is $\alpha=60°$ (see Fig. 5). Again, there are two collisions in total and by simple geometry it easily follows that the final outgoing billiard trajectory is vertical, that is, $v_1'=0$ and $v_2'>0$. We can obtain the final speed v_2' by exploiting the constancy of the rescaled speed. Initially, $\sqrt{w_1^2+w_2^2}=\sqrt{m_1v^2+m_2v^2}=2\sqrt{m}v$, where v is the initial speed. In the final state, the rescaled speed is $w_2'=\sqrt{m}v_2'$. Therefore, $v_1'=0$ and $v_2'=2v$.

As a byproduct of the billiards approach, notice that as soon as $m_1/m_2<3$, the total angle of three wedges is less than 180°, and there necessarily must be one more pw collision. Whenever the final trajectory is tangent to either a pp or a pw line, a critical point is defined where the total number of collisions changes by one. As m_1 continues to decrease, a sequence of transitions arises. Each transition occurs when the wedge angle decreases below π/n, with n an integer. At this point the total number of collisions increases from $n-1$ to n. We therefore find that three collisions first occur when $m_1<3m_2$, four when $m_1<m_2$, five when $m_1<0.5278m_2$, six when $m_1<m_2/3$, etc.

V. THREE PARTICLES ON AN INFINITE LINE

The billiards approach gives an extremely simple way to solve the classic elastic collision problem that apparently was first posed by Sinai.[7] Consider a three-particle system on an infinite line that consists of two approaching cannonballs, each of mass M. Between them (and nonsymmetrically located) lies a ping-pong ball of mass $m\ll M$. Due to the collisions between the cannonballs and the intervening ping-pong ball, the latter rattles back and forth with rapidly increasing speed until its momentum is sufficient to drive the cannonballs apart (see Fig. 6). In the final state, the three particles are receding from each other. How many collisions occur before this final state is reached?

By using energy and momentum conservation, we can determine the state of the system after each collision and thereby find the number of collisions before the three particles mutually recede. However, this approach is complicated and provides minimal physical insight (see the appendix and Refs. 14, 15, 18, and 19.) We now present a much simpler solution by mapping the original three-particle system onto a billiard in an appropriately defined domain.

We let the coordinates of the particles be x_1, x_2, and x_3, with $x_1<x_2<x_3$. This order between the particles again translates to a geometrical constraint on the accessible region for the billiard ball in the three-dimensional x_i space. Similarly, the trajectories of the particles on the line translate to the trajectory $[x_1(t),x_2(t),x_3(t)]$ of a billiard ball in the allowed region.

As in the previous examples, we introduce the rescaled coordinates $y_i=x_i\sqrt{m_i}$. These coordinates satisfy the constraints

$$\frac{y_1}{\sqrt{M}}<\frac{y_2}{\sqrt{m}},\quad \frac{y_2}{\sqrt{m}}<\frac{y_3}{\sqrt{M}}. \tag{7}$$

(The generalization to arbitrary masses is straightforward and is made in the next section.) In y_i space, the constraints correspond, respectively, to the effective billiard ball being confined to the half-space to the right of the plane $y_1/\sqrt{M}=y_2/\sqrt{m}$ and to the half-space to the left of the plane $y_2/\sqrt{m}=y_3/\sqrt{M}$ (see Fig. 7). These constraints define the allowed region as an infinite wedge of opening angle α.

The use of rescaled coordinates ensures that all collisions between the effective billiard particle and these constraint planes are specular. Furthermore, momentum conservation gives

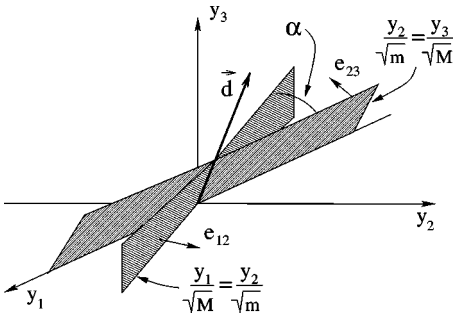

Fig. 7. Allowed wedge in the y_i coordinate system for a system of two cannonballs and an intervening ping-pong ball on an infinite line. The wedge is the region where the vectors e_{12} and e_{23} point toward.

$$Mv_1+mv_2+Mv_3=\sqrt{M}w_1+\sqrt{m}w_2+\sqrt{M}w_3=0, \quad (8)$$

where, without loss of generality, we take the total momentum to be zero. In this zero momentrum reference frame, the trajectory of the billiard ball is always perpendicular to the diagonal, $\mathbf{d}=(\sqrt{M},\sqrt{m},\sqrt{M})$. Thus we may reduce the three-dimensional billiard to a two-dimensional system in the plane perpendicular to **d**.

To complete this picture, we need to find the wedge angle α. The normals to the two constraint planes are $\mathbf{e}_{12}=(-1/\sqrt{M},1/\sqrt{m},0)$ and $\mathbf{e}_{23}=(0,-1/\sqrt{m},1/\sqrt{M})$. Consequently, the angle between these planes is given by

$$\alpha=\cos^{-1}\left(-\frac{\mathbf{e}_{12}\cdot\mathbf{e}_{23}}{|\mathbf{e}_{12}||\mathbf{e}_{23}|}\right)=\cos^{-1}\left(\frac{1}{1+\frac{m}{M}}\right). \quad (9)$$

In the limit $m/M\to 0$, Eq. (9) gives $\alpha\approx\sqrt{2m/M}$. Finally, the maximum number of possible collisions is determined by the number of wedges that fit into the half plane. This condition gives

$$N_{\max}=\frac{\pi}{\alpha}\approx\pi\sqrt{\frac{M}{2m}}. \quad (10)$$

For $m/M\to 0$, the opening angle of the wedge goes to zero and correspondingly, the number of collisions diverges.

VI. THREE PARTICLES ON A RING AND THE TRIANGULAR BILLIARD

Finally, let us consider three elastically colliding particles of arbitrary masses m_1, m_2, and m_3 on a finite ring of length L.[8,9] If we make an imaginary cut in the ring between particles 1 and 3, we can write the order constraints of the three particles as

$$x_1<x_2, \quad x_2<x_3, \quad x_3<x_1+L. \quad (11)$$

As usual, we employ the rescaled coordinates $y_i=x_i\sqrt{m_i}$ to ensure that all collisions of the billiard ball with the domain boundaries in the y_i coordinates are specular. In these coordinates, the first two constraints again confine the particle to be between the planes defined by the normal vectors $\mathbf{e}_{12}=(-1/\sqrt{m_1},1/\sqrt{m_2},0)$ and $\mathbf{e}_{23}=(0,-1/\sqrt{m_2},1/\sqrt{m_3})$. Without the offset of L, the constraint $x_3<x_1+L$ corresponds to a plane that slices the y_1-y_3 plane and passes through the origin. The offset of L means that we must translate this plane by a distance $L\sqrt{m_3}$ along y_3. The fact that x_3 is the lesser coordinate also means that the billiard ball is confined to the near side of this constraint plane. Thus the billiard ball must remain within a triangular bar whose outlines are shown in Fig. 8.

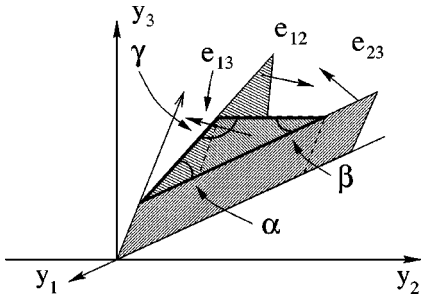

Fig. 8. Allowed region in the y_i coordinates for three particles of arbitrary masses on a ring of circumference L. The triangular billiard with angles α, β, and γ is defined by the thick solid lines.

If the total momentum of the system is zero, then $(\sqrt{m_1},\sqrt{m_2},\sqrt{m_3})\cdot(w_1,w_2,w_3)=0$, and the trajectory of the billiard ball remains within a triangle perpendicular to the long axis of the bar, with angles α, β, and γ. We compute these angles by the same approach given in Eq. (9). Thus, for example,

$$\alpha=\cos^{-1}\left(-\frac{\mathbf{e}_{12}\cdot\mathbf{e}_{23}}{|\mathbf{e}_{12}||\mathbf{e}_{23}|}\right)=\cos^{-1}\left(\sqrt{\frac{m_2m_3}{(m_1+m_2)(m_2+m_3)}}\right). \quad (12)$$

The angles β and γ can be obtained by cyclic permutations of this formula.

Therefore the elastic collisions of three particles on a finite ring can be mapped onto the motion of a billiard ball within a triangular billiard table. One can then exploit the wealth of knowledge about triangular billiards[5,6,8] to infer basic collisional properties of the three-particle system. For example, periodic or ergodic behavior of the billiard translates to periodic or nonperiodic behavior in the three-particle collision sequence.

VII. DISCUSSION

We have shown how to recast the elastic collisions of point particles in one dimension into the motion of a billiard

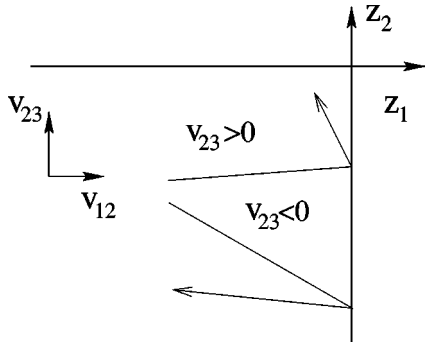

Fig. 9. Allowed region in z_1-z_2 coordinates for three particles of arbitrary masses on the line. Typical nonspecular collisions of the billiard ball with the boundary are shown for 12 collisions when $v_{23}>0$ and $v_{23}<0$ [see Eq. (A1)]. A similar picture arises for 23 collisions.

ball that moves at constant speed in a confined region of a higher-dimensional space. A crucial step in this reformulation is the introduction of the rescaled coordinates $y_i = x_i\sqrt{m_i}$. This rescaling ensures that all collisions of the billiard ball with the boundaries of its accessible region are specular. For the examples of two particles on a semi-infinite line and a reflecting wall and for three particles on an infinite line, the allowed region for the billiard ball is an infinite wedge. For three particles on a ring, the allowed region is a triangular billiard. The shape of the associated wedge or triangle is readily calculable in terms of the particle masses in the original system.

If these masses are widely disparate, the opening angle of the wedge or one angle in the triangle becomes small. If a billiard ball enters such an acute corner, a large number of bounces occurs before the ball recedes from this corner. These frequent bounces are completely equivalent to a straight line trajectory passing close to the tips of a large number of periodic extensions of the wedge over a short distance. In the original system, either picture corresponds to a large number of collisions between neighboring particles.

The mapping onto a billiard system can, in principle, be generalized to an arbitrary number of particles N. The spatial dimension of the accessible region in the corresponding billiard is now $(N-1)$ dimensional. Although less is known about such high dimensional billiards, this mapping provides a useful perspective to deal with the elastic collisions of many particles in one dimension.

Finally, it is worth mentioning that a similar wedge mapping has been applied to determine the probability that three diffusing particles on the line satisfy various constraints on their relative positions.[20] In both the collisional and diffusive systems, the order constraint leads to nearly identical wedge constructions, and these provide elegant solutions to the original respective problems.

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APPENDIX A: PING–PONG BALL BETWEEN TWO CANNONBALLS: DIRECT SOLUTION

We consider three particles with masses m_i, positions x_i, and velocities v_i for $i=1$, 2, and 3. Define the relative coordinates $z_i = x_i - x_{i+1} < 0$ and the relative velocities $v_{i,i+1} = x_i - x_{i+1}$. From elementary mechanics, these relative velocities transform as follows (see Fig. 9):

$$\text{12 collision:} \quad \begin{aligned} v'_{12} &= -v_{12} \\ v'_{23} &= v_{23} + \lambda_{12} v_{12}, \end{aligned} \tag{A1}$$

$$\text{23 collision:} \quad \begin{aligned} v'_{12} &= v_{12} + \lambda_{23} v_{23} \\ v'_{23} &= -v_{23}, \end{aligned} \tag{A2}$$

with $\lambda_{12} = 2m_1/(m_1+m_2)$ and $\lambda_{23} = 2m_3/(m_2+m_3)$. Once again, we can view the particle collisions as equivalent to the motion of a billiard ball in the third quadrant of the z plane, but with nonspecular reflections at each boundary.

It is convenient to characterize a trajectory by its polar angle $\tan\theta = v_{23}/v_{12}$. Then the above collision rules can be written as

$$\tan\theta_n = -\lambda_{12} - \tan\theta_{n-1}, \quad (12 \text{ collision}), \tag{A3a}$$

$$\cot\theta_{n-1} = -\lambda_{23}^{-1} - \cot\theta_{n-2}, \quad (23 \text{ collision}), \tag{A3b}$$

where θ_n is the angle after the nth collision with the boundary. In writing these recursion relations, we use the fact that the 12 and 23 collisions alternate. Initially, the billiard ball is heading toward the corner, but eventually it "escapes" by having a trajectory with its polar angle in the range $(\pi, 3\pi/2)$. This condition means that the three particles are all receding from each other. For a 12 collision, the incidence angle is in the range $-\pi/2 < \theta < \pi/2$, while the outgoing angle is in the range $\pi/2 < \theta < 3\pi/2$. In this case, escape means that $\tan\theta > 0$. Similarly, for a 23 collision, the incidence angle is in the range $0 < \theta < \pi$, while the outgoing angle is in the range $\pi < \theta < 2\pi$. For this case, escape means that $\cot\theta > 0$:

To solve Eq. (A3), we first write θ_n in terms of θ_{n-2}:

$$\tan\theta_n = -\lambda_{12} + \frac{1}{\dfrac{1}{\lambda_{23}} + \dfrac{1}{\tan\theta_{n-2}}}. \tag{A4}$$

Next, let $t_n = \tan\theta_n/\sqrt{\lambda_{12}\lambda_{23}}$, which simplifies Eq. (A4) to

$$t_n = -\mu + \frac{1}{\mu + \dfrac{1}{t_{n-2}}}, \tag{A5}$$

where $\mu = \sqrt{\lambda_{12}/\lambda_{23}}$. Equation (A5) can be written even more simply as

$$t_n = -\mu - \frac{1}{t_{n-1}}. \tag{A6}$$

To solve this recursion formula, we define $t_n \equiv g_n/h_n$ and find that Eq. (A6) is equivalent to the two first-order recursion relations $h_n = g_{n-1}$ and $g_n = -\mu g_{n-1} - h_{n-1}$. These relations are equivalent to the second-order recursion formula:

$$g_n = -\mu g_{n-1} - g_{n-2}. \tag{A7}$$

The general solution is $g_n = A_+\alpha_+^n + A_-\alpha_-^n$, where $A_\pm$ are constants and $\alpha_\pm = (-\mu \pm \sqrt{\mu^2-4})/2$. To complete the solution, we need an initial condition. For simplicity, we start with a billiard with incidence angle $\theta = 0$ that has undergone a single 12 collision. This situation corresponds to the initial condition $t_1 = -\mu$. Imposing this condition, and after some simple algebra, we find

$$t_n = \frac{\alpha_+^{n+1} - \alpha_-^{n+1}}{\alpha_+^n - \alpha_-^n}. \tag{A8}$$

It is more convenient to write Eq. (A8) in complex form by defining $\alpha_\pm = A\, e^{\pm i\phi}$, which leads to

$$t_n = \frac{\sin(n+1)\phi}{\sin n\phi}, \tag{A9}$$

where $\phi = \tan^{-1}\sqrt{(4-\mu^2)/\mu^2}$.

The particles are all receding when t_n first becomes negative. The maximum number of collisions until this occurs is

thus given by the condition $t_n=0$, which gives $(n+1)\phi=\pi$ or $n\approx\pi/\phi$. In the limit $m_2/m_1\equiv\epsilon_1\to 0$ and $m_2/m_3\equiv\epsilon_3\to 0$, we have

$$\mu^2=\frac{4}{(1+\epsilon_1)(1+\epsilon_3)}=4(1-\epsilon_1-\epsilon_3). \tag{A10}$$

Finally $\phi=\tan^{-1}\sqrt{(4-\mu^2)/\mu^2}\approx\sqrt{\epsilon_1+\epsilon_3}$, which gives

$$N_{\max}\approx\frac{\pi}{\sqrt{\epsilon_1+\epsilon_3}}=\pi\sqrt{\frac{m_1m_3}{m_2(m_1+m_3)}}. \tag{A11}$$

In the special case of $m_1=m_3=M$ and $m_2=m$, Eq. (A11) reduces to Eq. (10).

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[1]D. Halliday, R. Resnick, and J. Walker, *Fundamental of Physics*, 5th ed., (Wiley, New York, 1997), Vol. 1.

[2]J. Walker, *The Flying Circus of Physics* (Wiley, New York, 1977), Example 2.18.

[3]G. Galperin and A. Zemlyakov, *Mathematical Billiards* (in Russian) (Nauka, Moscow, 1990).

[4]V. V. Kozlov and D. V. Treshshëv, *Billiards: A Genetic Introduction to the Dynamics of Systems with Impacts* (American Mathematical Society, Providence, RI, 1991).

[5]S. Tabachnikov, *Billiards* (Société Mathématique de France; American Mathematical Society, Providence, RI, 1995).

[6]E. Gutkin, "Billiard in polygons: Survey of recent results," J. Stat. Phys. **81**, 7–26 (1996).

[7]This problem was apparently first posed by Sinai. See, for example, Ya. Sinai, *Introduction to Ergodic Theory* (Princeton University Press, Princeton, NJ, 1978).

[8]S. L. Glashow and L. Mittag, "Three rods on a ring and the triangular billiard," J. Stat. Phys. **87**, 937–941 (1996).

[9]S. G. Cox and G. J. Ackland, "How efficiently do three pointlike particles sample phase space?" Phys. Rev. Lett. **84**, 2362–2365 (2000).

[10]J. Rouet, F. Blasco, and M. R. Feix, "The one-dimensional Boltzmann gas: The ergodic hypothesis and the phase portrait of small systems," J. Stat. Phys. **71**, 209–224 (1993).

[11]See, for example, A. Dhar, "Heat conduction in a one-dimensional gas of elastically colliding particles of unequal masses," Phys. Rev. Lett. **86**, 3554–3557 (2001); P. L. Garrido, P. I. Hurtado, and B. Nadrowski, "Simple one-dimensional model of heat conduction which obeys Fourier's law," *ibid.* **86**, 5486–5489 (2001); O. Narayan and S. Ramaswamy, "Anomalous heat conduction in one-dimensional momentum-conserving systems," *ibid.* **89**, 200601–1-4 (2002). For a recent review, see S. Lepri, R. Livi, and A. Politi, "Thermal conduction in classical low-dimensional lattices," Phys. Rep. **377**, 1–80 (2003).

[12]For a general review, see H. M. Jaeger, S. R. Nagel, and R. B. Behringer, "Granular solids, liquids, and gases," Rev. Mod. Phys. **68**, 1259–1273 (1996).

[13]S. McNamara and W. R. Young, "Inelastic collapse and clumping in a one-dimensional granular medium," Phys. Fluids A **4**, 496–504 (1992); I. Goldhirsch and G. Zanetti, "Clustering instability in dissipative gases," Phys. Rev. Lett. **70**, 1619–1622 (1993); S. McNamara and W. R. Young, "Dynamics of a freely evolving, two-dimensional granular medium," Phys. Rev. E **53**, 5089–5100 (1996).

[14]A similar approach to that in the Appendix is given in P. Constantin, E. Grossman, and M. Mungan, "Inelastic collision of three particles on the line as a two-dimensional billiard," Physica D **83**, 409–420 (1995); see also, T. Zhou and L. P. Kadanoff, "Inelastic collapse of three particles," Phys. Rev. E **54**, 623–628 (1996).

[15]M. Hasegawa, "Broken ergodic motion of two hard particles in a one-dimensional box," Phys. Lett. A **242**, 19–24 (1998); B. Cipra, P. Dini, S. Kennedy, and A. Kolan, "Stability of one-dimensional inelastic collision sequences of four balls," Physica D **125**, 183–200 (1999).

[16]B. Bernu and R. Mazighi, "One-dimensional bounce of inelastically colliding marbles on a wall," J. Phys. A **23**, 5745–5754 (1990).

[17]N. D. Whelan, D. A. Goodings, and J. K. Cannizzo, "Two balls in one dimension with gravity," Phys. Rev. A **42**, 742–754 (1990).

[18]D. W. Jepsen, "Dynamics of simple many-body systems of hard rods," J. Math. Phys. **6**, 405–413 (1965).

[19]T. J. Murphy, "Dynamics of hard rods in one dimension," J. Stat. Phys. **74**, 889–901 (1994).

[20]S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, New York, 2001).