Densification and Structural Transitions in Networks that Grow by Node Copying

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We introduce a growing network model—the copying model—in which a new node attaches to a randomly selected target node and, in addition, independently to each of the neighbors of the target with copying probability p. When $p < \frac{1}{2}$, this algorithm generates sparse networks, in which the average node degree is finite. A power-law degree distribution also arises, with a non-universal exponent whose value is determined by a transcendental equation in p. In the sparse regime, the network is "normal", e.g., the relative fluctuations in the number of links are asymptotically negligible. For $p \geq \frac{1}{2}$, the emergent networks are dense (the average degree increases with the number of nodes N) and they exhibit intriguing structural behaviors. In particular, the N-dependence of the number of m-cliques (complete subgraphs of m nodes) undergoes $m-1$ transitions from normal to progressively more anomalous behavior at a m-dependent critical values of p. Different realizations of the network, which start from the same initial state, exhibit macroscopic fluctuations in the thermodynamic limit—absence of self averaging. When linking to second neighbors of the target node can occur, the number of links asymptotically grows as N^2 as $N \to \infty$, so that the network is effectively complete as $N \to \infty$.

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I. INTRODUCTION AND MODEL

A wide variety of complex networks grow by copying mechanisms. As examples, copying and redirection are key ingredients in the growth of the world-wide web, citation networks and other information networks [1–4]. In social networks, copying corresponds to triadic closure, that is, the formation of new social ties between two friends of a given individual. This mechanism appears to be important in driving social network dynamics [5, 6]. Copying also occurs in Nature. For example, the process of gene duplication, which is essentially the copying mechanism, plays a crucial role in evolution [7, 8]. Various models for protein interaction networks are also based on duplication and divergence [9–19].

From a modeling viewpoint, the copying mechanism has the advantage of being local [20–22], as the creation of new links only depends on the nearest neighborhood of each node, in contrast to global rules, such as preferential attachment. Despite the simplicity of the copying rule, there has not yet been a a rigorous mathematical analysis of networks that are generated by this mechanism.

Here we investigate networks that grow by an elementary implementation of the copying mechanism, which depends on only a single parameter—the copying probability p (Fig. 1). In our copying model, a network grows

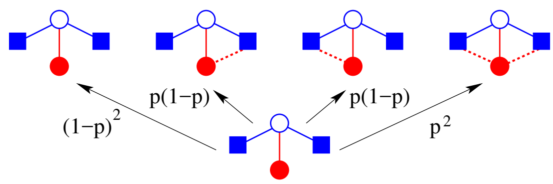


FIG. 1: The copying model. A new node (filled circle) attaches to a random target (open circle) and independently to each of the friends of the target (squares) with probability p.

by adding nodes sequentially. Each new node connects to a randomly chosen target node and, in addition, independently to each of the neighbors of the target with probability p. The simplicity of this growth mechanism allows us to develop an analytical description of many of the rich network properties that emerge.

Perhaps the most crucial structural change is the transition from sparse networks for $p < \frac{1}{2}$, where the number of links L_N in a network of N nodes grows linearly with N, to dense networks, where L_N grows super-linearly with N. In the sparse regime, the network is "normal" in the sense that a typical realization of the network is representative of the average behavior. In contrast, in the dense regime, $p > \frac{1}{2}$, network growth is not self-

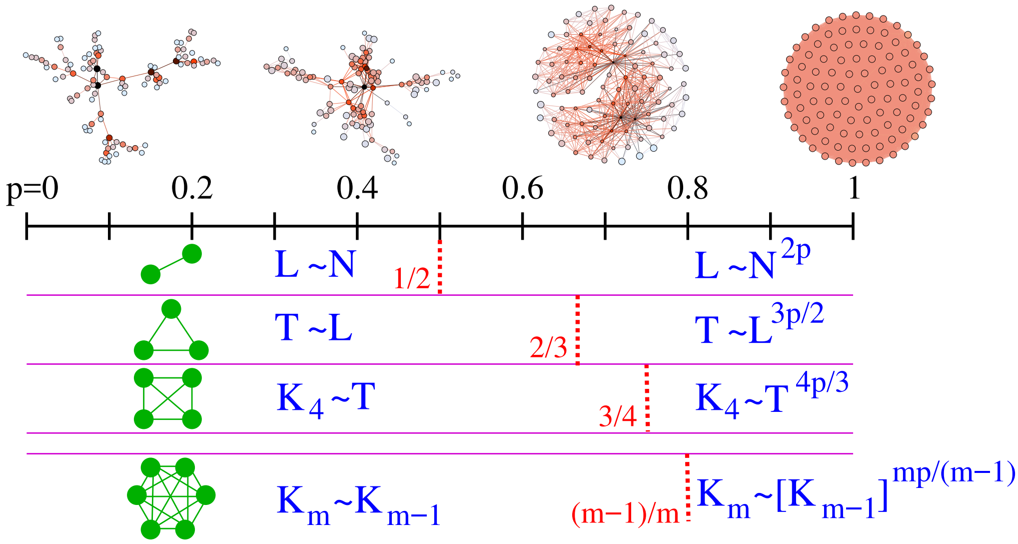


FIG. 2: Realizations of the copying model for $p = 0.1$, 0.4, 0.7, and 1 for $N = 100$, and a summary of the dense regimes. For simplicity, only the last of the structural transitions is shown (see Sec. IV.

averaging; namely sample-to-sample fluctuations do not vanish even when the number of nodes N is very large. In addition, the copying model undergoes infinitely many transitions at $p = \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ where sudden changes arise in the growth laws of the number of triangles and progressively higher-order cliques—complete subgraphs of m nodes. Moreover, for intermediate values of p, the resulting networks appear to be highly clustered (Fig. 2).

This article is organized as follows. In the next Sec. II we quantify the simplest global network characteristic, the number of links L_N. Specifically, we show that the N-dependence of the average number of links has a transition point at $p = \frac{1}{2}$, while the variance of L_N has transition points at $p = \frac{1}{4}$ and at $p = \frac{1}{2}$. We then analyze the degree distribution in Sec. III, and show that it has a power-law tail with a non-universal exponent in the sparse regime. In the dense regime, nearly all features of the degree distribution are anomalous. In Sec. IV we determine the growth laws for the average number of triangles and higher-order m-cliques. Cliques undergo a rich sequence of structural transitions as p increases. In Sec. V, we analyze the clustering properties of the network as a function of the copying probability p and argue that maximal cluster occurs at an intermediate value of p. In Sec. VI, we examine the probability distributions for the number of links L_N and triangles T_N. In Sec. VII we briefly discuss what happens if, in addition to connecting to the neighbors of the target node, connections to second neighbors are also allowed. Finally in Sec. VIII, we conclude and discuss some possible open questions.

II. NUMBER OF LINKS

A basic global characteristic of a network of N nodes is the number of links L_N. In many models, the dependence of L_N on N is trivial. For example, if each new node links to m pre-exiting nodes, then $L_N = m(N-1)$. Hereinafter we assume that the network starts with a single node, so that $L_1 = 0$. In the copying model, however, L_N is a random variable taking different values in different realizations. The exceptions are the extreme cases of $p = 0$ and $p = 1$ where the number of links is deterministic. In the former case the copying mechanism produces a tree (more precisely, a random recursive tree), so $L_N = N-1$. When $p = 1$, the copying model leads to the complete graph which has $L_N = N(N-1)/2$ links.

A. The average $L(N)$

The simplest characterization of the random quantity L_N is the *average* number of links $L(N) \equiv \langle L_N\rangle$. When a new node is added, the average number of links increases by $1 + p\langle k\rangle$, where $\langle k\rangle = 2L(N)/N$ is the average node degree. The factor 1 accounts for direct linking and the factor $p\langle k\rangle$ accounts for copying events. Indeed, for a target node of degree k, pk additional links are created on average by copying (Fig. 1). Thus the average number of links grows as

$$L(N+1) = \left(1 + \frac{2p}{N}\right) L(N) + 1\,. \tag{1}$$

Since we assume that the network starts with a single node, the initial condition is $L(1) = 0$.

The solution to the homogeneous version of recursion (1) is elementary. Using this solution as a integrating factor, we solve the inhomogeneous equation (see Appendix A), from which the asymptotic behavior is

$$L(N) = \begin{cases} \dfrac{1}{1-2p}\,N & p<\frac{1}{2}, \\ N\,\ln N & p=\frac{1}{2}, \\ A(p)\,N^{2p} & \frac{1}{2}<p\leq 1, \end{cases} \tag{2a}$$

with

$$A(p) = \frac{1}{(2p-1)\,\Gamma(1+2p)}\,, \tag{2b}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Equation (2a) shows that as the copying probability p is varied, there is a transition from sparse regimes arising when $p < \frac{1}{2}$ to dense regimes when $p \geq \frac{1}{2}$. The average degree remains finite as $N \to \infty$ in sparse regimes and diverges in dense regimes—logarithmically with N at the transition point $p = \frac{1}{2}$ and algebraically for $p > \frac{1}{2}$. The change in the dependence of $L(N)$ and many other network properties as a function of p is a major feature of the copying model.

Parenthetically, we can obtain the asymptotics of (2a), with the exception of the amplitude $A(p)$, by considering the continuum limit of (1). In this limit, we treat N as a continuous variable and recast the exact difference equation (1) into the differential equation

$$\frac{dL(N)}{dN} = 1 + 2p\,\frac{L(N)}{N}\,, \tag{3}$$

whose solution recovers the exact asymptotics given by (1) for $p \leq \frac{1}{2}$. In this range, the leading asymptotics are independent of $L(0)$; the initial condition plays no role. For $p > \frac{1}{2}$, the continuum solution has the correct N dependence, $L \sim N^{2p}$, but the amplitude depends on $L(1)$. The replacement of (1) by (3) is accurate only when $N \gg 1$. The dependence on the initial condition indicates that the behavior at small N affects the outcome and hence the continuum approach cannot be trusted whenever there is the dependence on $L(1)$.

Logarithmic and power-law densifications given in (2a) have been observed in citation graphs, the autonomous systems graph, software networks, and other social and information networks [1–3, 26]. Network densification also occurs in models that are based on accelerated network growth [27–30]. In these models, densification arises by introducing a time-dependent attractiveness to the nodes. Our approach is fundamentally distinct, as densification is an emergent property of the dynamics.

B. The variance $V(N)$

We now study the variance $V(N) \equiv \langle L_N^2\rangle - \langle L_N\rangle^2$, which characterizes the fluctuations in the random variable L_N. This variance exhibits a richer dependence on N than the average number of links $L(N)$, with a new transition at $p = \frac{1}{4}$, in addition to the transition at $p = \frac{1}{2}$. To determine the variance, we need to consider the copying process in more detail. When a new node attaches to a randomly selected target node of degree k, it also attaches to a of its neighbors by copying, where a is a random variable that can range from 0 to k. Thus the number of links changes according to

$$L_{N+1} = L_N + 1 + a. \tag{4}$$

Since connections to each of the neighbors of the target occurs independently with probability p, the probability $Q(a|k)$ that a additional links are made to the neighbors of a target of degree k is

$$Q(a|k) = \binom{k}{a}\,p^a(1-p)^{k-a}. \tag{5}$$

Averaging (4) we obtain

$$L(N+1) = L(N) + 1 + \langle \overline{a}_k\rangle. \tag{6}$$

Here $\overline{a}$ denotes the average over all possible values of a for a target node of degree k, and $\langle\ldots\rangle$ denotes the average over all target nodes and hence over all possible degrees. Using (5) we compute

$$\overline{a} = \sum_{a=0}^{k} aQ(a|k) = pk\,, \tag{7}$$

and thus (6) reduces to (1), as it must.

We now extend this approach to compute the variance. Squaring Eq. (4) gives

$$L_{N+1}^2 = L_N^2 + 1 + a^2 + 2L_N + 2a + 2L_N a\,,$$

which, after averaging, becomes

$$\langle L_{N+1}^2\rangle = \langle L_N^2\rangle + 1 + \langle\overline{a^2}\rangle + 2L(N) + 2\langle\overline{a}\rangle + 2\langle L_N\overline{a}\rangle.$$

To compute $\langle\overline{a^2}\rangle$ and $2\langle L_N\overline{a}\rangle$, we use (5) to obtain

$$\overline{a^2} = \sum_{a=0}^{k} a^2 Q(a|k) = p^2k^2 + p(1-p)k\,. \tag{8}$$

Therefore

$$\left\langle\overline{a^2}\right\rangle = p^2\langle k^2\rangle + p(1-p)\langle k\rangle. \tag{9}$$

Further

$$2\langle L_N\overline{a}\rangle = 2p\langle L_N k\rangle = \frac{2p}{N}\left\langle L_N \sum k\right\rangle\,, \tag{10}$$

where the sum is over all N nodes of the network. Since $\sum k = 2L_N$ we conclude that

$$2\langle L_N\overline{a}\rangle = \frac{4p}{N}\,\langle L_N^2\rangle \tag{11}$$

Using (9)–(11) and $\langle k\rangle = 2L(N)/N$ we find

$$\begin{aligned}\langle L_{N+1}^2\rangle &= \Big(1+\frac{4p}{N}\Big)\langle L_N^2\rangle + 1 + p^2\langle k^2\rangle \\ &\quad +2\Big(1+\frac{3p-p^2}{N}\Big)\, L(N)\,.\end{aligned}$$

Subtracting the square of (1) from this equation, we thereby find that the variance evolves according to

$$\begin{aligned}V(N+1) &= \Big(1+\frac{4p}{N}\Big)V(N) + 2p(1-p)\,\frac{L(N)}{N} \\ &\quad -\frac{4p^2}{N^2}\,L(N)^2 + p^2\langle k^2\rangle. \end{aligned} \tag{12}$$

Equation (12) is exact but not closed as it contains $\langle k^2\rangle$. To close (12) we need to express $\langle k^2\rangle$ as a function of $L(N)$ and $V(N)$. We have not found such an expression and its existence seems doubtful. To make progress, we first estimate the asymptotic behavior of (12) using arguments that should apply asymptotically. As long as we are merely interested in the dependence of $V(N)$ on N and not on amplitudes, we can replace (12) by the differential equation

$$\begin{aligned}\frac{dV(N)}{dN} &= \frac{4p}{N}\,V(N) + 2p(1-p)\,\frac{L(N)}{N} \\ &\quad -\frac{4p^2}{N^2}\,L(N)^2 + p^2\langle k^2\rangle. \end{aligned} \tag{13}$$

The first term on the right leads to superlinear growth, $V \sim N^{4p}$, when $p > \frac{1}{4}$ and linear growth for $p < \frac{1}{4}$. At $p = \frac{1}{4}$, Eq. (13) becomes $\frac{dV}{dN} = \frac{V}{N} + \text{const}$; hence the variance acquires an additional logarithmic correction: $V \sim N\ln N$. To summarize, we anticipate that the asymptotic behavior of the variance is given by

$$V(N) \sim \begin{cases} N & p<\frac{1}{4}, \\ N\,\ln N & p=\frac{1}{4}, \\ N^{4p} & \frac{1}{4}<p<1\,. \end{cases} \tag{14}$$

To derive $V(N)$ in the regime $p > \frac{1}{4}$ in a more principled way, we need $\langle k^2\rangle$, as mentioned above. To derive $\langle k^2\rangle$ requires information about the degree distribution that will be discussed in Sec. III. Here we merely quote the pertinent results that will be used to derive of $V(N)$. In the range $p < p_2 = \sqrt{2}-1$, the second moment is given by Eq. (20) in the next section. Using this result in Eq. (13), the evolution of the variance is given by

$$\frac{dV(N)}{dN} = \frac{4p}{N}\,V(N) + B(p)\,, \tag{15}$$

with

$$B(p) = \frac{2p(1-5p+2p^2)}{(1-2p)^2} + \frac{2p^2}{1-2p}\,\frac{3+2p-p^2}{1-2p-p^2}\,. \tag{16}$$

As long as $p < p_2$, the rational function $B(p)$ is finite and positive. Solving (15) gives, for $p < p_2$,

$$V(N) = \begin{cases} (1-4p)^{-1}B(p)\,N & p<\frac{1}{4}, \\ B(1/4)\,N\ln N & p=\frac{1}{4}, \\ \sim N^{4p} & p>\frac{1}{4}\,. \end{cases} \tag{17}$$

These results improve on (14) because (17) gives the amplitude in the range $p \le \frac{1}{4}$. For $p > \frac{1}{4}$, the amplitude cannot be computed within a continuum approach.

The behavior (17) is established for $p < p_2$, but we can extend the $V(N) \sim N^{4p}$ asymptotic to the $p > p_2$ range by noticing that the second, third, and fourth terms on the right-hand side of (13) are of order $\max[1, N^{2p-1}]$, $\max[1, N^{4p-2}]$, and N^{p^2+2p-1}, respectively. [The last result follows from Eq. (42).] These terms are all subdominant with respect to the first term on the right, which is of order N^{4p-1}. Thus we conclude that $V(N) \sim N^{4p}$ for all $p > \frac{1}{4}$.

The above results for the number of links and its variance lead us to the following conclusions:

1. When $p < \frac{1}{4}$, the variance $V(N)$ grows linearly with N. Fluctuations are asymptotically negligible because $\sqrt{V(N)}/L(N) \to 0$ as $N \to \infty$. Thus we anticipate that the distribution $P(L,N)$ of the number of links may be asymptotically Gaussian when $p < \frac{1}{4}$.

2. The variance scales as N^{4p} when $p > \frac{1}{4}$, thereby suggesting that the distribution $P(L,N)$ is non-Gaussian when $p > \frac{1}{4}$.

3. In the dense phase ($p > \frac{1}{2}$) the magnitude of fluctuations is the same as the average: $\sqrt{V(N)} \sim L(N)$.

The last point implies that the number of links does *not* self-average. This feature leads to a wide diversity between individual realizations of the network. In particular, the first few steps of the network growth are crucial to shaping its asymptotic evolution.

III. DEGREE DISTRIBUTION

We now study the degree distribution, both because of its fundamental nature in characterizing the network and because the second moment of this distribution is an essential ingredient in the variance $V(N)$ from the previous section. We will argue that the copying model leads to dramatically different degree distributions in the sparse ($p < \frac{1}{2}$) and dense ($p \ge \frac{1}{2}$) regimes. In the sparse regime, the degree distribution has an algebraic tail and we can also write the number of nodes of degree k in the scaling form $N_k = Nn_k$, which simplifies the analysis. In the dense regime, the degree distribution is anomalous in nearly all respects and thus far defies a complete analytical description. Finally, we will use the second moment of the degree distribution to provide a more complete derivation of $V(N)$.

In what follows, we assume that N is sufficiently large that we can employ a continuum approach. Let $N_k(N)$ be the number of nodes of degree k in a network of N nodes. The degree distribution evolves according to

$$\frac{dN_k}{dN} = \frac{N_{k-1} - N_k}{N} + p\,\frac{(k-1)N_{k-1} - kN_k}{N} + m_k. \tag{18a}$$

The first two terms on the right account for the contributions due to attachment to a randomly selected target node, the next two terms account for attachment to the neighbors of the target node, and the last term

$$m_k = \sum_{s\geq k-1} n_s \binom{s}{k-1} p^{k-1}(1-p)^{s-k+1} \tag{18b}$$

is the probability that the new node acquires a degree k. Each term in the above sum accounts for the contribution due to a target node of degree s in which the new node attaches to the $k-1$ neighbors of this target. Here $n_s \equiv N_s/N$ denotes the fraction of nodes of degree s.

Notice that the rate equations (18a) satisfy two basic sum rules: $\sum_k n_k = 1$, i.e., the network contains N nodes, and the value of $\sum_k kn_k$ is consistent with the total number of links growing according to (3). The first sum rule is verified by summing Eq. (18a) over all $k \geq 1$. The first four terms on the right trivially give zero. For the last term, we use $\sum_{s\geq 1} n_s = 1$ and the binomial identity, $\sum_{0\leq a\leq s}\binom{s}{a}p^a(1-p)^{s-a} = 1$, to conclude that $\sum_{k\geq 1} m_k = 1$, thus giving $\sum_k n_k = 1$. In a similar spirit, multiplying (18a) by k and summing over $k \geq 1$ gives (3).

A. Sparse regime

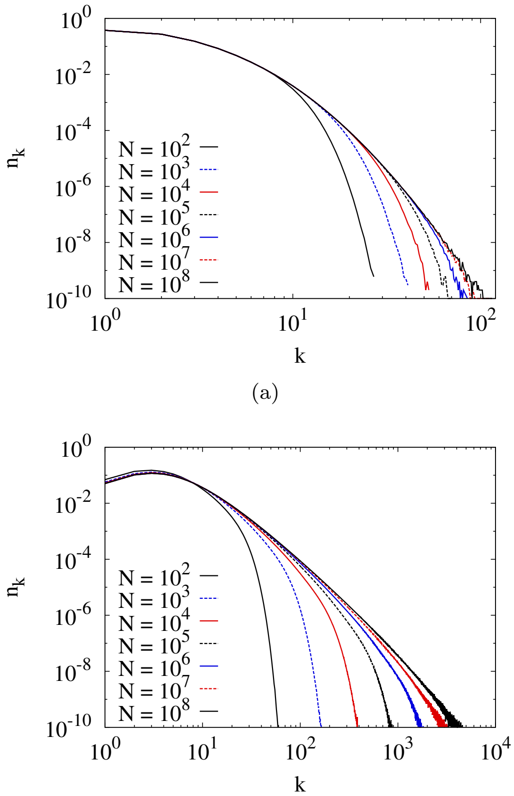
In the sparse regime, we make the standard assumption [21] that the fractions $n_k = N_k/N$ are independent of N for $N \gg 1$. With this ansatz, we recast Eq. (18) as

$$[2 + p(k+1)]n_{k+1} = [1 + pk]n_k + \sum_{s\geq k} n_s \binom{s}{k} p^k(1-p)^{s-k}\,. \tag{19}$$

While this is *not* a recurrence, we can use this equation to determine the behavior of low-order moments of the degree distribution. For instance, multiplying (19) by $k(k+1)$ and summing over all $k \geq 0$ gives, after some straightforward steps,

$$\langle k^2\rangle = \sum_{k\geq 1} k^2 n_k = \frac{2}{1-2p}\,\frac{3+2p-p^2}{1-2p-p^2}\,. \tag{20}$$

Thus $\langle k^2\rangle$ is finite for $p < p_2$, where $p_2 = \sqrt{2} - 1$ is the positive root of the polynomial $1 - 2p - p^2 = 0$. We used (20) in deriving (15)–(16) and establishing (17) for $p \leq \frac{1}{4}$.



(a)

(b)

FIG. 3: The scaled degree distributions in the sparse regime for (a) $p = 0.1$ and (b) $p = 0.4$, For each N, the number of realizations is $10^{10}/N$.

We also note that (20) reduces to $\langle k^2\rangle = 6$ for $p = 0$. This last result can be verified by recalling that the copying model reduces to random recursive trees when $p = 0$, and the degree distribution for random recursive trees is $n_k = 2^{-k}$.

To extract the asymptotics of n_k from Eq. (19), first notice that for large k, the summand on the right is sharply peaked around $s \approx k/p$ and thus reduces to [12, 15]

$$n_{k/p}\sum_{s\geq k}\binom{s}{k}p^k(1-p)^{s-k} = p^{-1}n_{k/p}\,,$$

where we use a binomial identity [31] to compute the sum. Thus the equation for the degree distribution reduces to

$$[2 + p(k+1)]n_{k+1} = [1 + pk]n_k + p^{-1}n_{k/p}\,. \tag{21}$$

This is now a non-local recurrence, as the value of n_{k+1} depends both on n_k and $n_{k/p}$, where the index k/p is generally much larger than k itself.

While we have not found a systematic way to solve such a recurrence, we make the assumption (justifiable a posteriori) that n_k decays slower than exponentially in k. This allows us to replace differences by derivatives in

(21) to give

$$\frac{d}{dk}\,[1+pk]n_k = p^{-1}\,n_{k/p} - n_k\,. \tag{22}$$

This ordinary differential equation is still non-local, but it nevertheless admits the algebraic solution $n_k \sim k^{-\gamma}$ for $k \gg 1$. Substituting this ansatz in (22) gives the following transcendental relation for the degree distribution exponent

$$\gamma = 1 + p^{-1} - p^{\gamma-2}\,. \tag{23}$$

Equation (23) has two solutions in the (γ, p) plane. One, $\gamma = 1$, is unphysical because it violates the sum rule $\sum_{k\geq 1} n_k = 1$. The other applies for $0 \leq p < \frac{1}{2}$. In this case, the exponent $\gamma = \gamma(p)$ is a monotonically decreasing function of p, with $\gamma(0) = \infty$ and $\gamma(\frac{1}{2}) = 2$. The feature that γ is always greater than 2 is consistent with the sparseness of the network, in which $\langle k\rangle = \sum_{k\geq 1} kn_k$ is finite.

Numerical results for the degree distribution in the sparse regime show that for small k, the n_k quickly converge to a stationary limit as a function of N (Fig. 3). For larger k, the degree distribution slowly converges to a power-law asymptotic tail whose exponent is consistent with the prediction given in (23). This convergence becomes progressively slower as p approaches $\frac{1}{2}$. This slow approach to the asymptotic behavior was previously observed in a related model for protein interaction networks [12], and seems to stem from the non-locality of the equation for the degree distribution.

B. Dense regime

The degree distribution has a very different nature in the dense regime. Instead of a power-law tail, the degree distribution has a well-defined peak (Fig. 4) whose location is determined by the mean degree, which grows as N^{2p-1}, see Eq. (2a). An important feature of the degree distribution in the dense regime is that the fractions of nodes of degree k, n_k, are no longer stationary. To show that the distribution is not a power law as well as the lack of stationarity, let us assume the converse and derive a contradiction. We thus assume that $n_k \sim k^{-\gamma}$ and that n_k is independent of N. Using this form for n_k, the number of links in a finite network is given by

$$L = \frac{N}{2}\,\langle k\rangle = \frac{N}{2}\sum_{k=1}^{k_{\rm max}} kn_k \sim Nk_{\rm max}^{2-\gamma}\,, \tag{24}$$

where $k_{\rm max}$ denotes the largest expected degree in a network of N nodes. We estimate this maximal degree by the standard extremal condition (see, e.g., [32]) $N\sum_{k\geq k_{\rm max}} n_k = 1$; namely, that there is of the order of a single node whose degree is $k_{\rm max}$ or greater. This relation gives $k_{\rm max} \sim N^{1/(\gamma-1)}$, so that (24) reduces to

$$L \sim Nk_{\rm max}^{2-\gamma} \sim N^{1/(\gamma-1)}\,. \tag{25}$$

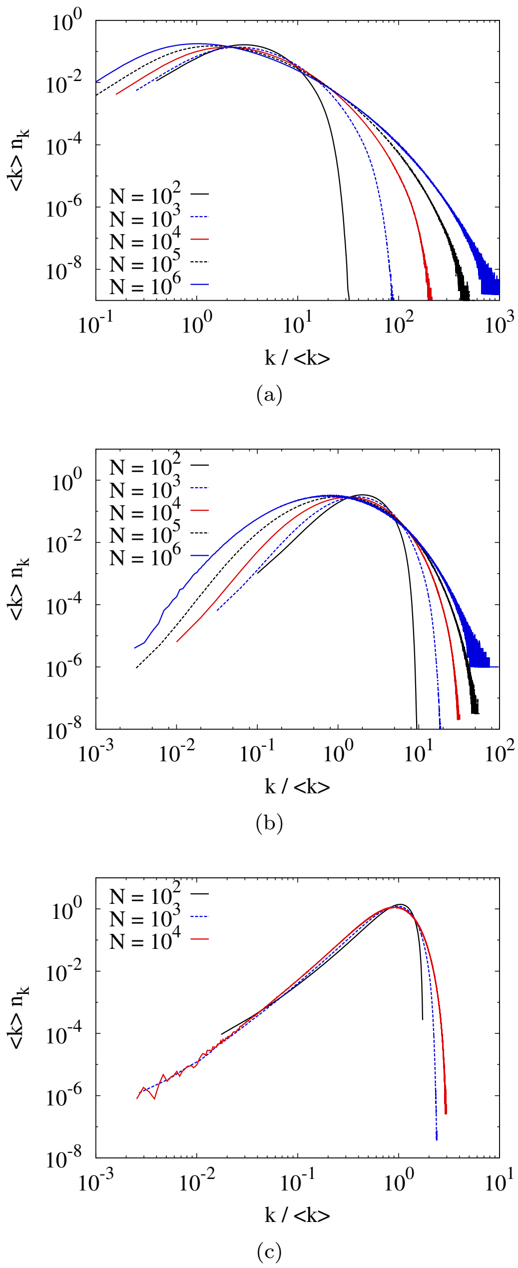


FIG. 4: The degree distributions in the dense regime for: (a) $p = 0.6$, (b) $p = 0.75$ and (c) $p = 0.9$. For each N, the number of realizations is $10^{10}/N$.

On the other hand, Eq. (3) gives $L \sim N^{2p}$. These two results are consistent only when $2p(\gamma - 1) = 1$, and this consistency condition agrees with (23) only at $p = \frac{1}{2}$. Thus we conclude that for $p > \frac{1}{2}$, the degree densities n_k must depend on N, and further, that the degree distribution is not algebraic in k.

Because of the non-locality of Eq. (18) and the non-stationary nature of the solution, we have not found an analytical solution for the degree distribution in the dense regime. We therefore report on simulation results. Fig-

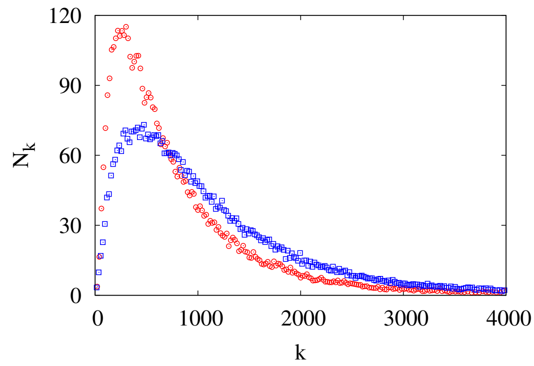


FIG. 5: The degree distributions for two representative realizations of the copying model for $p = 0.75$ for a network of $N = 10^5$ nodes. The data are averaged over a 20-point range.

ure 4 shows the degree distribution, averaged over many realizations, for representative values of p, with N ranging between 10^2 and 10^6. For each N, the number of realizations is $10^{10}/N$. These data clearly show that the degree densities are not stationary and that scaling the degree by the average degree $\langle k\rangle$ does not collapse the data onto a single universal curve for networks with 10^6 nodes or less. It is also worth noting that the degree distributions all exhibit a single peak, so that nodes of small degrees do not exist for $N \to \infty$. This behavior contrasts sharply with the sparse regime where the degree distribution is dominated by the smallest-degree nodes. Finally, the degree distribution is non self-averaging in the dense regime, as there is a wide disparity in the degree distributions of individual network realizations (Fig. 5).

IV. CLIQUES AND OTHER MOTIFS

As p is increased, it becomes increasingly likely that triangles are generated when each node is introduced. With this increased frequency for triangles, there is a concomitant increased propensity for the appearance of m-cliques — complete subgraphs of m nodes. To investigate this feature, we extend the approach of Sec. II for the number of links, to first account for the average number of triangles, and then the average number of m-cliques for general m.

A. Triangles

We begin by giving a (trivial) lower bound for the number of triangles T_N in a network of N nodes. If there was no copying, the number of links L_N would equal $N-1$ in the resulting tree network, so that no triangles would exist. For each copying event, the number of links increases by 1 while the number of nodes remains fixed, and at least one triangle is created. This reasoning gives the bound

$$T_N \geq L_N - (N-1)\,. \tag{26}$$

For $p < \frac{1}{2}$, this bound, together with (2a), gives, for the average number of triangles,

$$T(N) \equiv \langle T_N\rangle \geq \frac{2pN}{1-2p}\,.$$

We will see that the average number of triangles grows linearly with N when $p < \frac{1}{2}$, while for $p > \frac{1}{2}$, the growth of $T(N)$ is superlinear in N.

In each successful copying event a triangle is generated that consists of the new node, the target node and the neighbor that receives a copied link. We term this triangle-generating mechanism as *direct linking*. If links to two neighbors of the target are created, then two triangles necessarily arise by direct linking. Additional triangles may be created by a process that we term *induced linking*: when links to two neighbors of the target are created and these neighbors were previously linked, then a third triangle is created (shaded in Fig. 6).

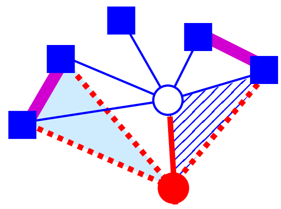


FIG. 6: Counting triangles. The target node (open circle) has five neighbors (squares), two of which are joined by 'clustering' links (heavy lines). When a new node (filled circle) is introduced, three copying links (dashed) create three new triangles (one is hatched for illustration) and one new triangle by induced linking (shaded).

To determine $T(N)$, we need to account for both of these mechanisms. Suppose that the target node has degree k and that its neighbors are connected via c 'clustering' links (Fig. 6). If a links to the neighbors are made by copying, the number of triangles increases on average by

$$\Delta T = a + \frac{a(a-1)}{2}\,\frac{c}{k(k-1)/2}\,. \tag{27}$$

The first term on the right accounts for direct linking and the second for induced linking. For the latter, we count how many of $a(a-1)/2$ possible links between a neighbors of the target, which also connect to the new node, are actually present. We now average (27) with respect to the binomial distribution (5) for a. This elementary calculation, together with the already-known result $\overline{a} = pk$ from Eq. (7), gives

$$\overline{a(a-1)} = \sum_{a=0}^{k} a(a-1)Q(a|k) = p^2k(k-1),$$

from which we obtain the compact result

$$\overline{\Delta T} = pk + p^2 c. \tag{28}$$

The term p^2c in Eq. (28) can be understood by noting that two previously connected neighbors also get connected to the new node with probability p^2 since linking to each node occurs independently.

We now express the average degree $\langle k\rangle$ via $L(N)$ and the average number of clustering links $\langle c\rangle$ via $T(N)$. The former relation is known, while to determine the latter we note that c equals the number of triangles that contain the target node. Thus

$$\langle k\rangle = \frac{2L(N)}{N}\,, \qquad \langle c\rangle = \frac{3T(N)}{N}\,. \tag{29}$$

Using (29), we average the increment of the number of triangles in (28) to obtain $\langle\overline{\Delta T}\rangle = 2pL/N + 3p^2T/N$ each time a new node is added. Therefore the number of triangles evolves according to

$$T(N+1) = \left(1 + \frac{3p^2}{N}\right) T(N) + 2p\frac{L(N)}{N}\,. \tag{30}$$

Solving this recurrence equation (see Appendix B) gives the asymptotic behaviors

$$T(N) = \begin{cases} \dfrac{2p}{(1-2p)\,(1-3p^2)}\, N & p < \frac{1}{2}\,, \\ 4N\,\ln N & p = \frac{1}{2}\,, \\ \dfrac{A(p)}{1-3p/2}\, N^{2p} & \frac{1}{2} < p < \frac{2}{3}\,, \\ \frac{4}{3}A\big(\frac{2}{3}\big)\, N^{4/3}\ln N & p = \frac{2}{3}\,, \\ C(p)\, N^{3p^2} & \frac{2}{3} < p \le 1, \end{cases} \tag{31a}$$

with $A(p)$ given by (2b) and

$$C(p) = \frac{2}{(3p-2)\,(3p^2-1)\,\Gamma(3p^2+1)}\,. \tag{31b}$$

Notice that for $N \gg 1$, the recursion (30) reduces to the differential equation

$$\frac{dT}{dN} = 3p^2\frac{T}{N} + 2p\frac{L}{N}\,,$$

whose solution coincides with (31a), except for the amplitude in the regime $p > \frac{2}{3}$, which cannot be determined within the continuum approach.

Equation (31a) exhibits several striking features. First, the triangle density (the average number of triangles per node) converges to a non-vanishing value for all $0 < p < \frac{1}{2}$, as observed in many empirical complex networks. This linearity arises because for any $p > 0$ a non-zero number of triangles are typically created when each node is added. Second, the average number of triangles $T(N)$ undergoes phase transitions at $p = \frac{1}{2}$ and at $p = \frac{2}{3}$. Although there is change in the N dependence at $p = \frac{1}{2}$, the average number of triangles continues to scale linearly with the number of links for any $p < \frac{2}{3}$. However, beyond $p = \frac{2}{3}$, the number of triangles grows *faster* than the number of links.

B. Cliques

We can extend the above considerations to treat complete subgraphs, or motifs, of arbitrary size m (with links and triangles corresponding to motifs of size 2 and 3 respectively). Let $K_m(N)$ be the average number of such motifs in a network of N nodes, with $K_2(N) \equiv L(N)$ and $K_3(N) \equiv T(N)$.

To determine the number of quartets—cliques of size four—we use similar reasoning that led to Eq. (28). We thus find that adding a node gives, for the average increase $\overline{\Delta K_4}$ in the number of quartets:

$$\overline{\Delta K_4} = p^2 c + p^3 d. \tag{32}$$

Here d is the number of triangles whose vertices are all neighbors of the target node. Using $\langle c\rangle = 3T/N$ and $\langle d\rangle = 4K_4/N$, we find that in the large-N limit the average number of quartets evolves according to

$$\frac{dK_4}{dN} = 3p^2\frac{T}{N} + 4p^3\frac{K_4}{N}\,, \tag{33}$$

whose solution is

$$K_4(N) \sim \begin{cases} N & 0 < p < \frac{1}{2}, \\ N^{2p} & \frac{1}{2} < p < \frac{2}{3}, \\ N^{3p^2} & \frac{2}{3} < p < \frac{3}{4}, \\ N^{4p^3} & \frac{3}{4} < p \le 1. \end{cases} \tag{34a}$$

At the transition points $p = \frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, the corresponding algebraic factor is multiplied by $\ln N$.

We can refine the above results by incorporating the exact asymptotic behaviors about triangles from (31a), to obtain the exact amplitudes in the range $0 \le p \le \frac{3}{4}$:

$$K_4(N) = \begin{cases} \dfrac{6p^3}{(1-2p)(1-3p^2)(1-4p^3)}\, N & p < \frac{1}{2}, \\ 6N\,\ln N & p = \frac{1}{2}, \\ \dfrac{3pA(p)}{(2-3p)(1-2p^2)}\, N^{2p} & \frac{1}{2} < p < \frac{2}{3}, \\ 12A\big(\frac{2}{3}\big)\, N^{4/3}\ln N & p = \frac{2}{3}, \\ \dfrac{C(p)}{1-4p/3}\, N^{3p^2} & \frac{2}{3} < p < \frac{3}{4}, \\ \frac{27}{16}C\big(\frac{3}{4}\big)\, N^{27/16}\ln N & p = \frac{3}{4}, \\ \sim N^{4p^3} & \frac{3}{4} < p \le 1. \end{cases} \tag{34b}$$

To obtain the amplitude in the range $\frac{3}{4} < p < 1$ range requires an analysis of an exact recurrence for $K_4(N)$.

More generally, the average number of cliques of m nodes, $K_m(N)$, satisfies

$$\frac{dK_m}{dN} = (m-1)p^{m-2}\frac{K_{m-1}}{N} + mp^{m-1}\frac{K_m}{N}\,. \tag{35}$$

Solving (35) recursively gives

$$K_m = \frac{N}{1-mp^{m-1}}\prod_{j=1}^{m-2}\frac{(j+1)p^j}{1-(j+1)p^j} \tag{36a}$$

in the sparse phase ($p < \frac{1}{2}$), while in the dense phase

$$K_m \sim N^{(j+1)p^j} \qquad \text{for} \quad \frac{j}{j+1} < p < \frac{j+1}{j+2}\,, \tag{36b}$$

with $j = 0, 1, 2, \ldots, m-1$ (the last asymptotic for $j = m-1$ holds when $1 - m^{-1} < p < 1$). The N dependence of the average number of cliques of size m therefore undergoes transitions at $p = 1 - 1/n$ with $n = 2, \ldots, m$. Thus the dense regime of the copying model can be partitioned into progressively finer subintervals where there are distinct N dependences for the number of m-cliques.

C. Star Subgraphs

Another simple motif within a complex network is a star graph. Part of the reason to study star graphs is that they are simply related to the degree distribution itself. Let S_j denote the number of star graphs with j leaves (nodes of degree 1). A node of degree k is thus a central node in $\binom{k}{j}$ subgraphs of type S_j. As a consequence, the number of star graphs and the degree distribution in a given network are related by

$$S_j = \sum_{k\geq j}\binom{k}{j}N_k\,. \tag{37}$$

We denote by $S_j(N)$ the average number of subgraphs of type S_j in a network of N nodes. From (37), there is a simple relation between the average number of stars and the falling factorial moments of the degree distribution:

$$S_j(N) = \frac{1}{j!}N\mu_j, \quad \mu_j = \langle k(k-1)\ldots(k-j+1)\rangle\,. \tag{38}$$

Using the evolution equation for the degree distribution, Eq. (18a), the falling factorial moment, which is a function of p and N, evolves according to

$$N\frac{d\mu_j}{dN} = (p^j + jp - 1)\mu_j + j[1 + (j-1)p + p^{j-1}]\mu_{j-1}\,. \tag{39}$$

From (39), each factorial moment $\mu_j \equiv \mu_j(p, N)$ remains finite, $\lim_{N\to\infty}\mu_j(p, N) \equiv \mu_j(p)$, when $p < p_j$, where p_j is the positive root of

$$p^j + jp - 1 = 0\,. \tag{40}$$

When $p < p_j$, Eq. (39) yields the recurrence

$$\mu_j(p) = j\,\frac{1 + (j-1)p + p^{j-1}}{1 - jp - p^j}\,\mu_{j-1}(p)\,,$$

from which

$$\mu_j(p) = j!\,\lambda_j(p), \tag{41}$$

where we define the shorthand notation

$$\lambda_j(p) \equiv \prod_{i=1}^{j}\frac{1 + (i-1)p + p^{i-1}}{1 - ip - p^i}\,.$$

Generally

$$\mu_j = \begin{cases} j!\,\lambda_j(p) & p < p_j\,,\\ j!\,\Lambda_j \ln N & p = p_j\,,\\ \sim N^{jp+p^j-1} & p_j < p \leq 1\,, \end{cases} \tag{42}$$

where $\Lambda_j = [1 + (j-1)p_j + p_j^{j-1}]\lambda_{j-1}(p_j)$. As a consistency check, notice for the case $j = 1$, equation (42) reproduces the average degree $\mu_1 = \langle k\rangle = 2L(N)/N$.

Combining (38) and (42), the number of stars asymptotically behaves as

$$S_j(N) = \begin{cases} \lambda_j(p)N & p < p_j\,,\\ \Lambda_j N \ln N & p = p_j\,,\\ \sim N^{jp+p^j} & p_j < p \leq 1\,. \end{cases} \tag{43}$$

Overall, the numbers of star subgraphs have a simpler N dependence than cliques because the former undergo a single transition for each j at an irrational value of p_j whose first few values are:

$$\begin{aligned} p_2 &= \sqrt{2} - 1\,,\\ p_3 &= -\big[2/(1+\sqrt{5})\big]^{1/3} + \big[(1+\sqrt{5})/2\big]^{1/3}\,,\\ p_4 &= \left[\sqrt{2\sqrt{2}-1} - 1\right]/\sqrt{2}\,, \end{aligned}$$

etc. From (40), the asymptotic behavior of the threshold values are given by $p_j \to 1/j - 1/j^{j+1}$ for $j \gg 1$. In contrast, the phase transition points for complete-graph motifs $K_m(N)$ are all rational and at the same location for every m—only the number of transition points is variable, with $m - 1$ transition points.

V. CLUSTERING

For intermediate values of p, we have seen that the copying model gives rise to non-trivial motifs, and we now investigate whether their appearance corresponds to the emergence of significant network clustering, as might be surmised visually in Fig. 2. There are two popular measures of network clustering: (i) the transitivity, or global clustering coefficient, and (ii) the local clustering coefficient (see e.g., [33]). The transitivity τ_G for a connected, undirected, and simple (no multiple links between two nodes) graph G is defined as

$$\tau_G = 3 \times \frac{\#(\text{triangles in } G)}{\#(\text{twigs in } G)}\,. \tag{44}$$

Here, a twig is a node with two neighbors and thus looks like: •–•–•. By definition, the transitivity is already averaged over all network nodes.

To define the local clustering coefficient, first consider an arbitrary node $\mathbf{n}$ of degree k in the network. The k neighbors of $\mathbf{n}$ could potentially be connected by up to $\binom{k}{2}$ edges. The clustering coefficient of node $\mathbf{n}$ is then defined as $c(\mathbf{n})/\binom{k}{2}$, where $c(\mathbf{n})$ denotes the actual number of connections between the neighbors of $\mathbf{n}$. Finally, the local clustering coefficient $CC(N)$ is obtained by averaging the node clustering coefficient over all nodes:

$$CC(G) = \frac{1}{N}\sum_{\mathbf{n}\in G}\frac{c(\mathbf{n})}{\binom{k}{2}}\,. \tag{45}$$

If G is a tree, the above clustering coefficients vanish, while if G is the complete graph, both clustering coefficients equal one (which explains the choice of the numerical factor in the definition (44)). We now examine the dependence of the clustering coefficients on the copying probability. Each network realization leads to distinct values for the clustering coefficients. In fact, the transitivity is non-self-averaging when $p > \frac{1}{2}$. In this dense region, however, the transitivity vanishes as $N \to \infty$ so that the lack of self averaging does not pose any difficulties. Conversely, for sufficiently small p, where the transitivity is non-zero in the $N \to \infty$ limit, the transitivity is self-averaging and is determined from

$$\tau(N) \equiv \langle \tau_G \rangle = \frac{3T(N)}{S_2(N)}\,, \tag{46}$$

where $T(N)$ is the average number of triangles and $S_2(N)$ is the average number of twigs.

To determine the transitivity in the limit $N \to \infty$, we need the average number of triangles $T(N)$, which is given by Eq. (31a) and the average number of twigs $S_2(N)$. The latter is given by specializing (43) to $j = 2$:

$$S_2(N) = \begin{cases} \dfrac{2(1+2p)}{(1-2p)(1-2p-p^2)}\,N & p < p_2,\\[2mm] 2\,\dfrac{1+2p_2}{1-2p_2}\,N\ln N & p = p_2,\\[2mm] \sim N^{2p+p^2} & p_2 < p \le 1. \end{cases} \tag{47}$$

With these results, the transitivity is (Fig. 7)

$$\tau(\infty) = \begin{cases} \dfrac{3p(1-2p-p^2)}{(1+2p)(1-3p^2)} & 0 \le p \le p_2,\\[2mm] 0 & p_2 < p < 1,\\[2mm] 1 & p = 1, \end{cases} \tag{48}$$

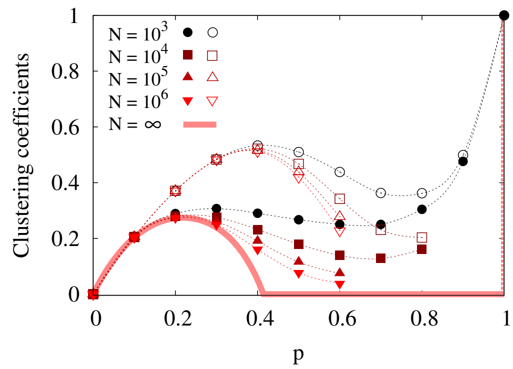


FIG. 7: The transitivity $\tau(N)$ (solid symbols) and the local clustering coefficient $CC(N)$ (open symbols) versus the copying probability p for networks of different sizes. The solid smooth curve is the analytical expression (48). The dotted curves are guides to the eye.

where $p_2 = \sqrt{2} - 1$ is again the positive root of the quadratic equation $p^2 + 2p - 1 = 0$.

A perplexing feature of the transitivity is its non-monotonic dependence on p, with a maximum deep in the sparse regime (at $p \approx 0.2181$). We also emphasize that when $p_2 \le p < 1$, the transitivity vanishes in the thermodynamic limit $N \to \infty$. However, the simulations show that even for large networks the transitivity is non-zero and approaches zero very slowly as N increases (Fig. 7). This features can be understood theoretically. For instance, in the marginal case of $p = p_2$, Eq. (46), in conjunction with (31a) and (47), shows that the transitivity exhibits a slow inverse logarithmic decay: $\tau(N) \sim (\ln N)^{-1}$.

VI. DISTRIBUTION OF LINKS AND CLIQUES

Because a varying number of links are added to the network each time a new node is introduced, the *distributions* of the number of links and the number of cliques are non-trivial quantities. Here we investigate the asymptotic properties of these link and clique distributions by numerical simulations, as well as basic probabilistic and extreme statistics arguments.

A. Link Distribution

Let $P(L, N)$ be the probability that a network of N nodes contains L links: $P(L, N) = \text{Prob}(L_N = L)$. As a function of p, this distribution exhibits a wide range of behaviors (Fig. 8). For $p \ll 1$, the distribution $P(L, N)$ is visually symmetric and Gaussian in appearance. As p is increased, $P(L, N)$ broadens considerably and is enhanced at large argument. Visually, $P(L, N))$ is maximally broad for $p \approx 0.7$, while for larger p, the distribution progressively narrows and develops an enhancement at small argument.

Each time a new node is introduced, the number of links increases by $1 + a$, where the random variable a is the number of copying links that are created (Eq. (4)). In the sparse phase, where the degree distribution reaches a stationary limit with the algebraic tail $k^{-\gamma}$ (Eq. (23)), the increment in the number of links $1+a$ is also drawn from this same distribution. For $\gamma > 3$, which occurs when $p < \sqrt{2} - 1$, the second moment $\langle a^2 \rangle$ is finite. Because the first two moments of the link increment are finite, one might anticipate that the central limit theorem applies, from which $P(L, N)$ would asymptotically be Gaussian.

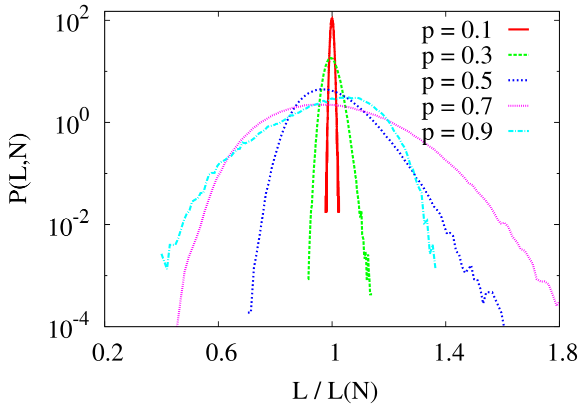


FIG. 8: Semi-logarithmic plot of the distribution $P(L, N)$ versus the scaled number of links for $N = 10^4$ and representative values of p. Data collected over 10^6 realizations for p up to 0.7 and 10^5 realizations for $p = 0.9$.

However, the increments $1+a$ when each node is introduced are not statistically independent. A particularly fruitful copying event for a high-degree target node increases the degrees of many neighboring nodes, which, in turn, affects the increment in the number of links in later node additions. Thus the growth in the number of links is governed by a correlated random-walk process and the central-limit theorem is not applicable to infer the asymptotic form of $P(L, N)$.

From Eqs. (2a) and (17), the ratio of the square root of the variance to the average number of links, $\sqrt{V(N)}/L(N)$ decays as $N^{-1/2}$ for $p < \frac{1}{4}$ and slower than $N^{-1/2}$ for larger p. This behavior suggests that $p = \frac{1}{4}$ might be the point where $P(L, N)$ changes in character from Gaussian to non Gaussian. We also test the Gaussianity of $P(L, N)$ by measuring its skewness, μ_3/σ^3 where μ_n is the n^{th} central moment and σ is the standard deviation of the probability distribution, and excess kurtosis, $\mu_4/\sigma^4 - 3$. Both these quantities are zero for the Gaussian distribution. Numerically, we find that for $p < \frac{1}{4}$, the skewness and excess kurtosis do approach zero as $N \to \infty$, while for $p > \frac{1}{4}$, these quantities are both non-zero as $N \to \infty$ (Fig. 9). These results indicate that the distribution $P(L, N)$ is non Gaussian for $p > \frac{1}{4}$.

When $p > \frac{1}{2}$, the standard deviation in the number of

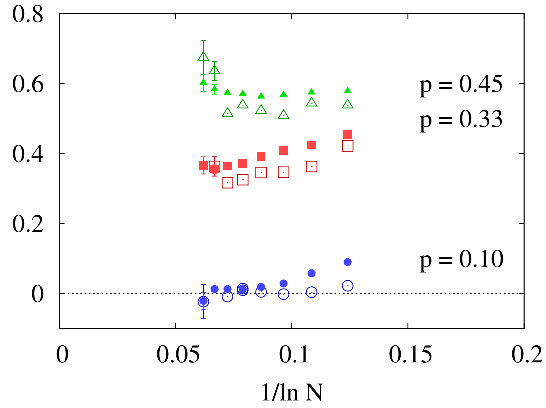


FIG. 9: Skewness (solid symbols) and excess kurtosis (open symbols) of the link distribution as a function of N.

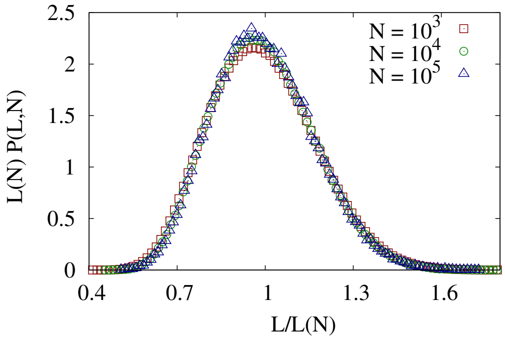


FIG. 10: The scaled distribution of the number of links for the copying model with $p = 0.7$.

links $\sqrt{V(N)}$ grows as L, and this suggests that $P(L, N)$ approaches the single-parameter scaling form,

$$P(L, N) \simeq \frac{1}{L(N)}\,\Phi(\mathsf{L}) \quad \text{with} \quad \mathsf{L} = L/L(N)\,, \tag{49}$$

as confirmed in Fig. 10. In many processes that are generated by a random-walk-like process, the scaling function $\Phi(\mathsf{L})$ has the limiting forms [48–50]

$$-\ln \Phi(\mathsf{L}) \sim \begin{cases} \mathsf{L}^{\delta_+} & \mathsf{L} \gg 1, \\ (1/\mathsf{L})^{\delta_-} & \mathsf{L} \ll 1\,. \end{cases} \tag{50}$$

We now give heuristic arguments for the tail exponents $\delta_\pm$ by considering the extreme cases where L is: (i) as large as possible, and (ii) as small as possible, and matching the distribution $P(L, N)$ in these extreme cases to the hypothesized limiting form of the full distribution.

The maximal number of links $L_{\max} = N(N-1)/2$ corresponds to generating a complete graph. The probability $C(N)$ to construct a complete graph is

$$C(N) = p\, p^2\, p^3 \ldots p^{N-2} \simeq \exp\left(\tfrac{1}{2} N^2 \ln p\right). \tag{51}$$

Each factor p^n gives the probability that the addition of a node to a complete graph of $n+1$ nodes leads to a complete graph of $n+2$ nodes. In the dense regime $L(N) = A(p)N^{2p}$ [see Eq. (2a)], so that the maximal value of the scaling variable is $\mathsf{L}_{\max} = L_{\max}/L(N) \sim N^{2(1-p)}$. Using this value of $\mathsf{L}_{\max}$ and matching (50) with (51), we obtain $[N^{2(1-p)}]^{\delta_+} \sim N^2$, from which we extract the large-L tail exponent

$$\delta_+ = \frac{1}{1-p}\,.$$

Conversely, the smallest possible L arises if no copying connections are made, so that the resulting network is a tree with $L = L_{\min} = N-1$. The probability that no copying connections are made when a new node attaches to a node of degree k is $(1-p)^k$. Thus the probability to generate a tree is $(1-p)^{\sum k}$, where the sum runs over the degrees of all selected target nodes. The upper bound $(1-p)^{N-1}$ arises in the situation when only leaves (nodes of degree 1) have been selected during the network creation. Generally one still anticipates that $\sum k \sim N$ and hence $\ln \Phi(\mathsf{L}_{\min}) \sim N \ln(1-p)$. Since $\mathsf{L}_{\min} = L_{\min}/L(N) \sim N^{1-2p}$ the matching gives $[N^{2p-1}]^{\delta_-} \sim N$ leading to the left tail exponent

$$\delta_- = \frac{1}{2p-1}\,.$$

To summarize, the tails of the distribution of the number of links are given by

$$\ln \Phi(\mathsf{L}) \sim -\begin{cases} \mathsf{L}^{1/(1-2p)} & \mathsf{L} \ll 1\,,\\ \mathsf{L}^{1/(1-p)} & \mathsf{L} \gg 1\,. \end{cases} \tag{52}$$

B. Triangle Distribution

One can also investigate the distributions of other cliques. For triangles, for example, the corresponding probability distribution is $P(T,N) \equiv \mathrm{Prob}(T_N = T)$. We make that ansatz that in the dense phase the distribution $P(T,N)$ approaches the single-parameter scaling form,

$$P(T,N) \simeq \frac{1}{T(N)}\,\Psi(\mathsf{T}) \quad \text{with} \quad \mathsf{T} = T/T(N)\,. \tag{53}$$

As in the case of the link distribution, we postulate that the large-argument tail of the scaled distribution has the form $\ln \Psi(\mathsf{T}) \sim -\mathsf{T}^{\delta}$ for $\mathsf{T} \gg 1$, which we expect will be valid in the dense phase $p > \frac{1}{2}$. We now estimate the large-argument tail of the triangle distribution by again considering the extreme case where the number of triangles is as large as possible. The largest possible value of T arises when a complete graph is generated. In this case, $T = T_{\max} = \binom{N}{3}$ and using Eq. (31a) the scaling variable T is given by

$$\mathsf{T} \sim \begin{cases} N^{3-2p} & \frac{1}{2} < p < \frac{2}{3}\,,\\ N^{3-3p^2} & p > \frac{2}{3}\,. \end{cases} \tag{54}$$

On the other hand, from Eq. (51), the probability to construct a complete graph is given by $\exp(\frac{1}{2}N^2 \ln p)$. This form matches (53) if the $\mathsf{T} \gg 1$ tail of the triangle distribution is given by

$$\ln \Psi(\mathsf{T}) \sim -\begin{cases} \mathsf{T}^{2/(3-2p)} & \frac{1}{2} < p < \frac{2}{3}\,,\\ \mathsf{T}^{2/(3-3p^2)} & \frac{2}{3} < p < 1\,. \end{cases} \tag{55}$$

This same line of reasoning can be straightforwardly adapted to obtain the large-argument tail of the distribution of m-cliques.

VII. SECOND-NEIGHBOR CONNECTIONS

Suppose that in addition to connecting to the neighbors of the target with probability p, a new node *also* connects to the second neighbors of the target with probability q. Such a mechanism naturally arises in social media, such as Facebook, where we are sporadically encouraged to make connections to friends of our friends. The surprising outcome of second-order linking is that the probability that the network is complete with non-zero, albeit may be very small, for any $q > 0$ probability.

To estimate this completeness probability, suppose that as the network is complete when it contains N nodes. Then the probability that the network remains complete when the $(N+1)^{\text{st}}$ is introduced is

$$C(N) = \sum_{k=0}^{N} B(N,k,p)\left[1-(1-q)^k\right]^{N-k}\,, \tag{56}$$

where $B(r,k,p) = \binom{r}{k}p^k(1-p)^{r-k}$ is the binomial probability. The factor $B(r,k,p)$ gives the probability that there are k first-neighbor connections from the new node, while the remaining factor $\left[1-(1-q)^k\right]^{N-k}$ gives the probability that the remaining $N-k$ nodes are linked to the new node by second-neighbor connections.

We now argue that $C(N)$ approaches to 1 sufficiently quickly as N increases, so that the product of these factors converges to a non-zero value. In the large-N limit, the binomial factor becomes a Gaussian distribution that is sharply peaked about $k = Np$, with a width that is of the order of $\sqrt{N}$. Over this range of k, the factor $\left[1-(1-q)^k\right]^{N-k}$ in Eq. (56) is nearly constant. We therefore replace k by its most probable value Np in the above expression. After doing so, this factor can be written as

$$\begin{aligned} C(N) &\simeq \left[1-(1-q)^{Np}\right]^{N(1-p)}\\ &\simeq \exp\Big\{-N(1-p)\exp\left[Np\ln(1-q)\right]\Big\}\,. \end{aligned}$$

The probability that the network of N nodes is com-

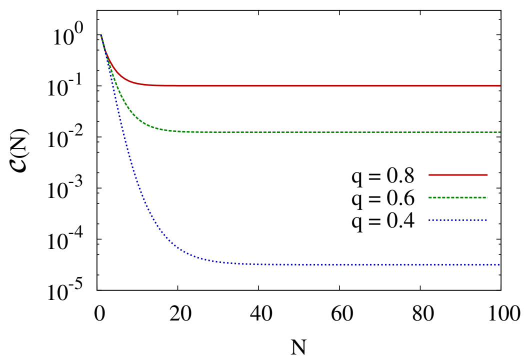


FIG. 11: Numerical evaluation of the probability for network completeness, $\mathcal{C}(N)$ from Eq. (57), for fixed $p = \frac{1}{2}$ and representative values of q. The saturation is obvious, $\mathcal{C}(\infty) > 0$ for all $q > 0$. The ultimate values of $\mathcal{C}(\infty)$ can be very small for small q.

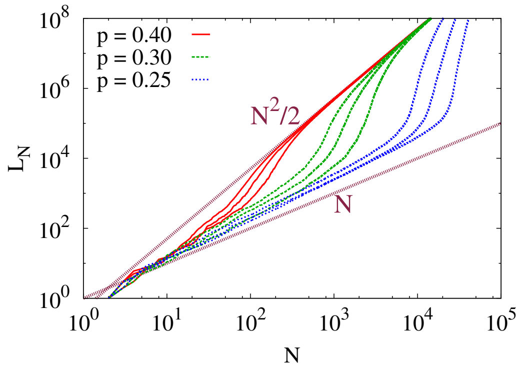


FIG. 12: The N dependence of the number of links for second-neighbor copying with $q = p^2$.

plete, $\mathcal{C}(N)$, is then given by

$$\mathcal{C}(N) = \prod_{j \le N-1} C(j)$$
$$\simeq \exp\left\{-\int^N j(1-p)\exp\big[j\,p\ln(1-q)\big]\,dj\right\}. \quad (57)$$

Because the integral in the exponent converges as $N \to \infty$, the completion probability is necessarily non-zero.

Numerical numerical evaluation of (57) shows that the completion probability $\mathcal{C}(N)$ indeed converges to a non-zero, albeit extremely small, value as $N \to \infty$. Figure 11 shows this evaluation for the case of $p = \frac{1}{2}$ and various q. A more relevant criterion is not defect-free completeness, but whether the number of links eventually scales as $N^2/2$, as in the complete graph. Simulations show that for representative values of p and q, the average number of links $L(N)$ initially grows linearly with N but then crosses over to growing as $N^2/2$ (Fig. 12). Thus second-order copying generically leads to networks that are effectively complete—eventually each individual knows almost everybody. Moreover, Fig. 12 illustrates the macroscopic differences between individual network realizations. Thus copying leads to non-self-averaging in the dense regime—unpredictable outcomes when starting from a fixed initial state. This intriguing feature also arises in empirical networks and related systems [41–43], and intellectually originates with the classic Pólya urn model [44–46].

VIII. OUTLOOK

We introduced and investigated the properties an exceedingly simple growing network model that is based on the mechanism of node copying. Each new node that joins the network attaches to a randomly selected target node and also to each of the neighbors of the target with an independent copying probability p. In spite of its deceptive simplicity, the structure of the network that results from this growth mechanism is extremely rich. One of the fundamental outcomes of our copying model is that a transition from a sparse to dense regime occurs as the copying probability p increases beyond $\frac{1}{2}$. Dense networks are characterized by a mean degree that increases with the number N of nodes in the network, a feature that appears in a variety of empirical networks [3], as well as by large fluctuations between individual network realizations. For these reasons alone, it is important to understand this densification process.

The bulk of our analysis focused on the analytical description of various global network quantities, such as the N dependence of the average number of links $L(N)$ and the number of m-cliques, $K_m(N)$. We found that $L(N) \sim N$ for $p < \frac{1}{2}$ and $L(N) \sim N^{2p}$ for $p > \frac{1}{2}$. Analogously, for triangles, we found that $K_3(N) \sim N$ for $p < \frac{1}{2}$, $K_3(N) \sim N^{2p}$ for $\frac{1}{2} < p < \frac{2}{3}$, and $K_3(N) \sim N^{3p^2}$ for $p > \frac{2}{3}$. For general m, there are $m-1$ transitions points where the N dependence of the m-clique density suddenly changes. Given the richness of our predictions, it would be worthwhile to reanalyze the densifying networks have have been observed empirically [3] to test whether they can be accounted for within the framework of the copying model.

The possibility to generate networks with controllable densities of specific motifs might help in the design of controlled environments to explore how the network topology affects the diffusion of an innovation or the spread of a virus in a social system. The copying model could also serve as benchmark to test the veracity and the robustness of various types of algorithms for characterizing networks, such as community detection [39, 40], by generating more realistic structural properties than those of random benchmarks [51, 52]. Another basic unanswered question is: What are the spectral properties of networks generated by the copying model? This question is particular intriguing in the dense regime where there are large fluctuations between individual network realizations.

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Appendix A: Exact behavior of $L(N)$

To determine the exact solution of Eq. (1), we first solve the homogeneous version of this equation and use this solution as an integrating factor. The homogeneous solution is

$$\prod_{j=1}^{N-1}\left(1+\frac{2p}{j}\right)=\frac{\Gamma(2p+N)}{\Gamma(2p+1)\,\Gamma(N)}\,.$$

We thus we seek a solution to Eq. (1) of the form

$$L(N)=U(N)\,\frac{\Gamma(2p+N)}{\Gamma(2p+1)\,\Gamma(N)}\,.$$

This ansatz allows us to recast Eq. (1) into the recurrence

$$U(N+1)=U(N)+\frac{\Gamma(2p+1)\,\Gamma(N+1)}{\Gamma(2p+N+1)}\,. \tag{A1}$$

Solving Eq. (A1) subject to the initial condition $U(1)=0$ (recall that $L_1=0$), we find

$$L(N)=\frac{\Gamma(2p+N)}{\Gamma(N)}\sum_{j=2}^{N}\frac{\Gamma(j)}{\Gamma(2p+j)}. \tag{A2}$$

To determine asymptotic properties, we will often use the well-known feature of the gamma function

$$\frac{\Gamma(2p+x)}{\Gamma(x)}\to x^{2p} \qquad x\gg 1. \tag{A3}$$

When $p<\frac{1}{2}$, the sum on the right-hand side of (A2) diverges. Thus we can use (A3) to give

$$L(N)\to N^{2p}\sum_{j\leq N}j^{-2p}\to\frac{N}{1-2p}\,,$$

leading to the result quoted in (2a). For $p=\frac{1}{2}$ the exact solution to (A2) is

$$L(N)=N(H_N-1)\,, \tag{A4}$$

where $H_N=\sum_{1\leq j\leq N}j^{-1}$ is the N^{th} harmonic number. From the asymptotics of the harmonic numbers [31] we obtain

$$L(N)=N(\ln N+\gamma-1)+\frac{1}{2}-\frac{1}{12N}+\frac{1}{120N^3}+\ldots \tag{A5}$$

where $\gamma=0.57721566\ldots$ is the Euler-Masceroni constant. Keeping only the leading term in Eq. (A5) gives the result quoted in (2a). For $p>\frac{1}{2}$, the sum on the right-hand side of Eq. (A2) converges. Hence

$$L(N)\to N^{2p}\sum_{j=2}^{\infty}\frac{\Gamma(j)}{\Gamma(2p+j)}\equiv A(p)\,N^{2p} \tag{A6}$$

with $A(p)$ given by Eq. (2b). The sum on the right-hand side of Eq. (A6) is found by specializing the identity [31]

$$\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)}=\frac{\Gamma(a)}{(c-a-1)\,\Gamma(c-1)} \tag{A7}$$

to $a=2,\ c=2p+2$.

Appendix B: Exact Behavior of $T(N)$

To find the amplitude $C(p)$ quoted in (31a), we need to solve the recurrence (30). Following the same approach as that used for the number of links, we first solve the homogeneous version of (30) and use this the homogeneous solution as an integrating factor

$$T(N)=R(N)\,\frac{\Gamma(3p^2+N)}{\Gamma(N)} \tag{B1}$$

We use this substitution together with the exact solution (A2) to recast (30) into recurrence

$$R(N+1)=R(N)+2p\frac{\Gamma(2p+N)}{\Gamma(3p^2+1+N)}\sum_{j=2}^{N}\frac{\Gamma(j)}{\Gamma(2p+j)}$$

which is solved to give

$$R(N)=2p\sum_{j=2}^{N-1}\frac{\Gamma(j)}{\Gamma(2p+j)}\sum_{n=j}^{N-1}\frac{\Gamma(2p+n)}{\Gamma(3p^2+1+n)}\,. \tag{B2}$$

When $p>\frac{2}{3}$, both sums in (B2) are convergent. Hence (B1) asymptotically becomes $T(N)=R(\infty)N^{3p^2}$, where we additionally used (A3). Thus the amplitude $C(p)$ in Eq. (31a) is equal to $R(\infty)$; that is,

$$C(p)=2p\sum_{j=2}^{\infty}\frac{\Gamma(j)}{\Gamma(2p+j)}\sum_{n=j}^{\infty}\frac{\Gamma(2p+n)}{\Gamma(3p^2+1+n)}\,. \tag{B3}$$

Using the identity (A7) twice, we compute the sums in (B3) and arrive at the result for $C(p)$ quoted in Eq. (31b).

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