Growth Inside a Corner: The Limiting Interface Shape

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We investigate the growth of a crystal that is built by depositing cubes onto the inside of a corner. The interface of this crystal evolves into a limiting shape in the long-time limit. Building on known results for the corresponding two-dimensional system and accounting for the symmetries of the three-dimensional problem, we conjecture a governing equation for the evolution of the interface profile. We solve this equation analytically and find excellent agreement with simulations of the growth process. We also present a generalization to arbitrary spatial dimension.

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Growing interfaces constitute a venerable subject, but the proper continuum framework to account for this growth has been developed not so long ago [1]. A detailed and beautiful description of fluctuations of *one-dimensional* growing interfaces has been proposed [2, 3], culminating in a recent solution of the KPZ equation [4]. For real applications, two-dimensional growing interfaces are more important, but in that setting the governing stochastic continuum equations [1] remain intractable. However, the analysis of two-dimensional growing interfaces is not hopeless. Indeed, although interface fluctuations have attracted the most attention, they become less important as the interface grows. The limiting shape — the average interface profile in the long-time limit — is the more primal characteristic. If growth begins from a flat substrate, the interface advances upward with a constant average speed, so only the fluctuations matter. However, in numerous applications the limiting shapes are curved, but known only in rare cases. For example, in one of the simplest two-dimensional growth models, the Eden-Richardson model [5], the statistics of its fluctuations are understood (and belong to the KPZ universality class) but the limiting shape is unknown.

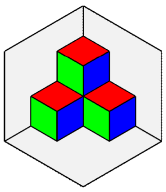

FIG. 1: (color online) 3d crystal of volume 4. The next elemental cube can be deposited at one of 6 inner corners.

Here we investigate the limiting shape of a crystal that grows from the inside of a corner. This process can be defined in arbitrary spatial dimension and on an arbitrary lattice (with an appropriately defined 'corner'). If not stated otherwise, we consider a cubic lattice, where the corner is the initially empty positive octant. Starting at $t = 0$, elemental cubes are deposited at a unit rate onto inner corners (Fig. 1). Initially, there is one inner corner and thus only one place where a cube can be placed. After the first deposition event, there are three available inner corners where the next cube can be deposited. In the long-time limit, the interface approaches a deterministic limiting shape shown in Fig. 2.

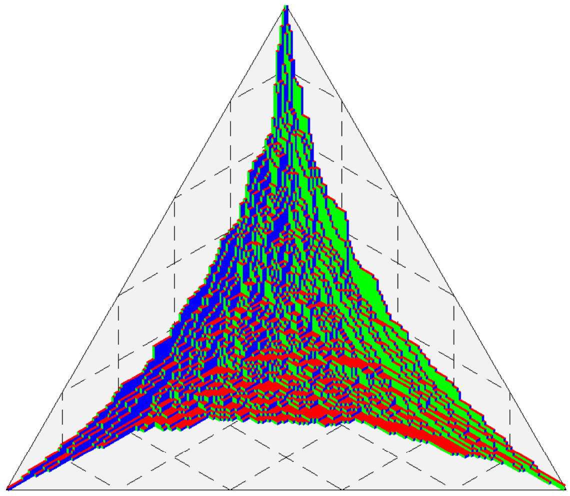

FIG. 2: (color online) The growth process at $t = 140$.

The corner growth model admits a dual interpretation as the melting of a three-dimensional cubic crystal by erosion from the corner. There is also a magnetic interpretation in which one initially assigns plus spins to each site inside the corner and minus spins to sites outside the corner, and endows the system with zero-temperature Glauber spin-flip dynamics [6] in a sufficiently weak negative magnetic field. This dynamics causes plus spins at inner corners to flip and thereby is isomorphic to the corner melting problem. The magnetic interpretation naturally suggests considering the system in zero magnetic field. This case indeed results in a different type of growing interface whose characteristic scale grows diffusively rather than ballistically.

In this work, we shall use the language of deposition dynamics; most importantly, we allow only deposition events and no evaporation. Growth inside a two-dimensional corner is well understood. The limiting shape was computed by Rost [7] by mapping the corner growth process onto the asymmetric simple exclusion process, and fluctuations were subsequently computed [8, 9]. Here we focus on the limiting shape in three (and higher) dimensional corners. Our analysis relies heavily on insights gleaned from the limiting two-dimensional shape [7].

For two-dimensional corner growth, the limiting shape $y(x;t)$ evolves according to the equation of motion [10–12]

$$y_t = \frac{y_x}{y_x - 1}\ , \tag{1}$$

from which the interface profile was found to be [7]

$$\sqrt{x} + \sqrt{y} = \sqrt{t}\,. \tag{2}$$

The parabolic limiting shape (2) describes the non-trivial part of the interface where $0 \leq x, y \leq t$. Outside this region, the original boundary remains undisturbed.

Two properties allow us to severely constrain the form of possible evolution equations for the growth inside a three-dimensional corner: (i) The governing equation for the interface $z(x, y;t)$ must reduce to the two-dimensional form (1) on the boundaries $x = 0$ or $y = 0$; (ii) The equation must be invariant under the interchange of any pair of coordinates.

Analogously to Eq. (1), we seek an evolution equation in three dimensions of the form $z_t = F(z_x, z_y)$ that involves only first derivatives (higher-order spatial derivative are asymptotically negligible). The simplest guess is the product $z_t = [z_x/(z_x - 1)]\,[z_y/(z_y - 1)]$. This equation reduces to (1) on the boundaries $x = 0$, where $z_x = -\infty$, and $y = 0$, where $z_y = -\infty$. The product ansatz is also invariant under the exchange $x \leftrightarrow y$, but it is *not* invariant under the exchanges $x \leftrightarrow z$ or $y \leftrightarrow z$, and therefore is wrong.

A simple modification to the product form that satisfies the necessary invariance requirements is

$$z_t = \frac{z_x}{z_x - 1}\,\frac{z_y}{z_y - 1}\left[1 - \frac{1}{z_x + z_y}\right]. \tag{3}$$

It is difficult to find any other equation that satisfies the invariance requirements. Indeed, if we seek a multiplicative correction factor to the product form as the Laurent series $\sum_{-\infty}^{\infty} \lambda_n (z_x + z_y)^{-n}$, coordinate interchange invariance gives $\lambda_0 = 1, \lambda_1 = -1$, while all other amplitudes must vanish [13]. Thus Eq. (3) is the only properly invariant choice among the family of evolutionary equations parameterized by λ_n.

Nevertheless, there are other evolution equations of the form $z_t = F(z_x, z_y)$ that satisfy coordinate interchange invariance. One example arises by replacing the factor in the square brackets in (3) with $\left[1 + (z_x z_y - z_x - z_y)^{-1}\right]$. This equation, which can be re-written more elegantly as

$$\frac{1}{z_t} = 1 - \frac{1}{z_x} - \frac{1}{z_y}\,, \tag{4}$$

and Eq. (3) are two independent evolution equations in three dimensions that satisfy the invariance requirements. We believe, but cannot prove, that other elemental evolution equations do not exist. Using these elemental equations, we can form two distinct one-parameter families of invariant equations [13]; an additive family

$$z_t = \frac{z_x}{z_x - 1}\,\frac{z_y}{z_y - 1}\left[1 - \frac{1 + c}{z_x + z_y} - \frac{c}{z_x z_y - z_x - z_y}\right]\ , \tag{5a}$$

and a multiplicative family

$$z_t = \left[\frac{1 - \frac{1}{z_x + z_y}}{\left(1 - \frac{1}{z_x}\right)\left(1 - \frac{1}{z_y}\right)}\right]^{1+c}\left[1 - \frac{1}{z_x} - \frac{1}{z_y}\right]^{c}\ . \tag{5b}$$

Our conjecture is that (3) is the correct evolution equation. Evidence in favor of this statement partly rests on the excellent agreement with simulation data. For this comparison, we solve Eq. (3) by the method of characteristics. Starting from an empty corner, we find [13] that the interface profile admits the following parametric representation (Fig. 3)

$$\frac{x}{t} = A(q, r), \quad \frac{y}{t} = B(q, r), \quad \frac{z}{t} = C(q, r) \tag{6}$$

where

$$\begin{aligned}
A =& \frac{1}{(q-1)^2}\,\frac{r}{r-1}\left[1 - \frac{1}{q+r}\right] - \frac{q}{q-1}\,\frac{r}{r-1}\,\frac{1}{(q+r)^2}\ ,\\
B =& \frac{q}{q-1}\,\frac{1}{(r-1)^2}\left[1 - \frac{1}{q+r}\right] - \frac{q}{q-1}\,\frac{r}{r-1}\,\frac{1}{(q+r)^2}\ ,\\
C =& \frac{q}{q-1}\,\frac{r}{r-1}\left[1 - \frac{1}{q+r}\right]\left[1 + \frac{1}{q-1} + \frac{1}{r-1}\right]\\
& - \frac{q}{q-1}\,\frac{r}{r-1}\,\frac{1}{q+r}\ ,
\end{aligned}$$

with $q = z_x$, $r = z_y$ and $-\infty < q, r \leq 0$. As a consistency check, note that for $r = -\infty$, we have $x/t = (q-1)^{-2}$, $y/t = 0$, and $z/t = q^2(q-1)^{-2}$. Eliminating q, we get $\sqrt{x} + \sqrt{z} = \sqrt{t}$, thereby recovering Eq. (2) for the intersection of the interface (6) with the $y = 0$ plane.

While it seems difficult to eliminate the parameters (q, r) from Eq. (6), the intersections of the interface (6) with certain planes admit simplified descriptions. For example, for the plane $x = y$, corresponding to $q = r$, we obtain

$$\frac{x}{t} = \frac{1}{2}\frac{z}{t} - \frac{3}{4}\left(\frac{z}{t}\right)^{2/3} + \frac{1}{4}\,, \tag{7}$$

which agrees quite well with simulations (Fig. 4).

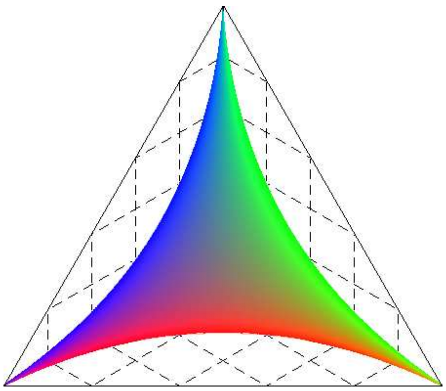

FIG. 3: The interface (6).

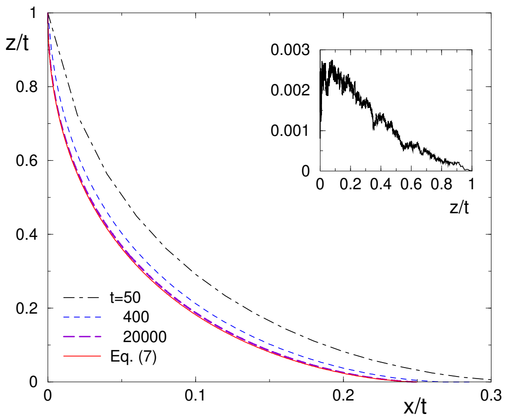

FIG. 4: Scaled interface profile z/t versus x/t along the diagonal ray $x = y$ for $t = 50$, 400 (10^4 realizations each), $t = 20000$ (10 realizations), and from Eq. (7). The inset shows the difference between x/t for the simulated interface at $t = 20000$ and (7).

Two additional tests suggest that the conjectured evolution equation (3) and its solution (6) describe the corner growth model accurately. Consider first how the interface advances along the ray $x = y = z$. The position of this point is given by [14]

$$x = y = z = wt, \qquad w = \tfrac{1}{8}\,. \tag{8}$$

Numerically, we measure $w \approx 0.1261(2)$, which agrees with our prediction $w = 0.125$ to within 0.9%. As a second test, we compute the total volume V beneath the growing interface at time t. Since the linear dimension of the interface grows linearly with time, $V = vt^3$; we want to determine the amplitude v. Using the parametric solution (6) and changing from the physical variables (x, y) to the parametric coordinates (q, r), the amplitude v reduces to the integral

$$v = \int_{-\infty}^{0}\int_{-\infty}^{0} dq\; dr\; C(q,r)\,\frac{\partial(A,B)}{\partial(q,r)}\,.$$

We compute the Jacobian $\frac{\partial(A,B)}{\partial(q,r)}$ and the integral using *Mathematica* to find

$$v = \frac{3\pi^2}{2^{11}} = 0.0144574283219\ldots \tag{9}$$

Numerically, we measure $v \approx 0.01472(3)$, which is within 1.8% of our prediction.

While Eq. (3) is quite accurate, small discrepancies between our measurements of the coefficients w and v, and their predicted values (8)–(9) persist. The alternative elementary evolution equation (4) leads to the interface profile

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{t}\,, \tag{10}$$

which generalizes the parabolic profile (2). The corresponding values $w = \frac{1}{9}$ and $v = \frac{1}{90}$ that arise from this profile substantially disagree with simulation results, suggesting that (4) is wrong.

One may also consider the composite forms given in Eq. (5). For the multiplicative class of equations (5b), the case $c \approx 0.074$ provides the best fit for the simulated value of v [15]. However, Nature should be described by a simple equation. Similarly, fitting simulation results to an evolution equation from the additive class, the optimal mixing parameter is $c \approx 0.079$. In this case, the evolution equation is analytic, but it still contains an anomalously small mixing parameter, which seems implausible. These results are suggestive that Eq. (3) describes corner interface evolution, but the approach to the asymptotic state is quite slow.

A similarly slow convergence to asymptotic behavior occurs in various well-understood $d = 1$ growth models (see e.g. Refs. [16, 17]). Less is known about fluctuations of the two-dimensional interfaces [2] and these are important to understand the approach to the asymptotic interface profile. For the two-dimensional corner problem, the intersection of the interface with the $(1, 1)$ direction evolves according to [3, 8, 9]

$$x(t) = \frac{t}{4} + t^{1/3}\,\xi\,, \tag{11}$$

where ξ is a stationary random variable with $\langle\xi\rangle > 0$. Thus averaging over many realizations gives an effective velocity $w_{\rm eff} - \frac{1}{4} \sim t^{-2/3}$.

For the three-dimensional corner problem, we similarly expect $w_{\rm eff} - \frac{1}{8} \sim t^{-\alpha}$, with a still-unknown exponent α. The best simulation results for flat interfaces indicate that α is close to 0.77 [18, 19]. From our measurements over the time range $t \le 20000$, the best extrapolation is provided by using $\alpha \approx 0.74$, which indicates that we are still outside the long-time regime. Although we find discrepancies between extrapolations of simulation results and theoretical predictions for the interface profile based on (3) that are an order of magnitude larger than in two dimensions [13], these persistent discrepancies may result from the slower convergence of the three-dimensional corner problem to the asymptotic behavior.

In four dimensions, we use coordinate interchange invariance and symmetry to conjecture that the height $h(x, y, z; t)$ obeys the following generalization of the two- and three-dimensional evolution equations

$$h_t = \frac{\left(1 - \frac{1}{h_x + h_y}\right)\left(1 - \frac{1}{h_y + h_z}\right)\left(1 - \frac{1}{h_z + h_x}\right)}{\left(1 - \frac{1}{h_x}\right)\left(1 - \frac{1}{h_y}\right)\left(1 - \frac{1}{h_z}\right)\left(1 - \frac{1}{h_x + h_y + h_z}\right)} .$$

In d dimensions, this same line of reasoning suggests that the height $h(x_1, \ldots, x_{d-1}; t)$ satisfies

$$h_t = \prod_{1 \le i_1 < \ldots < i_p \le d-1} \left(1 - \frac{1}{h_{i_1} + \ldots + h_{i_p}}\right)^{(-1)^p} \tag{12}$$

where $h_i \equiv \frac{\partial h}{\partial x_i}$. All these equations are solvable using the method of characteristics [13].

The computation of the limiting shape — the primary characteristic of the growing interface — does not close the problem. There are many other interesting features of growing interfaces that are ripe for exploration. One challenging problem, given that interface fluctuations are unknown even for flat interfaces, is to generalize Eq. (11) to account for fluctuations of the corner interface in three dimensions. Fluctuations of integral characteristics, such as the crystal volume, may be more tractable and give rise to new phenomena. Consider, for example, the total number of sites of various fixed degrees (number of adjacent vertices). Sites of degree 3, in particular, can be categorized as either inner or outer corners. The number of inner corners grows as $N_{\text{inner}} = \frac{dV}{dt} = 3vt^2$, with $v = 3\pi^2/2^{11}$ to leading order. One might anticipate the same asymptotic growth for outer corners, but simulations indicate that the latter grows slightly faster [13]:

$$N_{\text{outer}}/N_{\text{inner}} \approx 1.04 .$$

Strangely, even though the interface is globally concave, the excess of outer corners indicates that at the local scale the interface is slightly convex.

It would be interesting to generalize from strict growth to Ising growth, where deposition at inner corners and desorption from outer corners occur with equal rates. A related extension is to the situation when the desorption rate slightly exceeds the deposition rate to give a large equilibrium crystal. The corresponding *equilibrium* shape has been determined both in two dimensions [20–22] and in three dimensions [23, 24]. Similar to the conjectured exact evolution equations (12) for the corner growth problem, it seems feasible to conjecture an exact generalization of equilibrium limiting shapes [20–24] in higher dimensions.

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[14] The position (8) follows from (6) by setting $q = r = -1$. One can also derive (8) directly from the governing equation (3) without using the solution (6) for the interface shape, it suffices to note that by symmetry $z_x = z_y = -1$ at the middle point.

[15] Generally for Eq. (5b), one finds $v = 9^c/8^{1+c}$, which is best fit to the simulation data by choosing $c \approx 0.074$; similarly for equations from the additive class $8v = 1 + \frac{c}{9}$.

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