Reinforcement-Driven Spread of Innovations and Fads

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We propose kinetic models for the spread of permanent innovations and transient fads by the mechanism of social reinforcement. Each individual can be in one of $M+1$ states of awareness $0, 1, 2, \ldots, M$, with state M corresponding to adopting an innovation. An individual with awareness $k < M$ increases to $k+1$ by interacting with an adopter. Starting with a single adopter, the time for an initially unaware population of size N to adopt a permanent innovation grows as $\ln N$ for $M = 1$, and as $N^{1-1/M}$ for $M > 1$. The fraction of the population that remains clueless about a transient fad after it has come and gone changes discontinuously as a function of the fad abandonment rate λ for $M > 1$. The fad dies out completely in a time that varies non-monotonically with λ.

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Disease propagation [1], spread of technological innovations [2–6], and outbreaks of social and political unrest [7, 8] are all driven by contagion. A variety of contagion-driven models [9–14] have been developed to account for their evolution. In this Letter, we introduce purely kinetic models to understand how permanent innovations, as well as transient fads, arise due to a small seed of aware individuals. The key feature of our models is that trend adoption is driven by *social reinforcement* — namely, multiple reinforcing prompts from acquaintances are needed to persuade an individual to adopt an innovation. Our models are motivated by a recent online social experiment of Centola [15], where reinforcement fostered the adoption of a desirable behavior in a controlled online social network.

In our models, each individual is endowed with a awareness variable $0, 1, 2, \ldots, M$. An individual in state 0 is ignorant, a larger awareness value corresponds to being closer to adopting the innovation, while attaining the highest awareness value M corresponds to adopting the innovation. The population evolves by repeated interactions between random pairs of individuals. In each interaction with an adopter, someone with awareness $k < M$ advances to awareness $k + 1$, while there are no state changes when two non-adopters interact. In our *innovation model*, an innovation is permanently adopted; in our *fad model*, the fad becomes passé and an adopter abandons the fad at a rate λ.

Innovation model: The simplest situation is a population with two classes of individuals: ignorant (state 0) and adopters (state 1). Whenever an ignorant individual and an adopter meet, the former is converted to an adopter via $0 + 1 \to 1 + 1$. The rate equations that describe the evolution of a homogeneous and well-mixed population (the mean-field limit) are:

$$\dot{n}_0 = -n_0 n_1, \qquad \dot{n}_1 = n_0 n_1. \tag{1}$$

We generically assume that the evolution begins with a small fraction of adopters in an otherwise ignorant population: $n_1(0) = \rho \ll 1$, $n_0(0) = 1 - \rho$. The solution to Eqs. (1) with this initial condition is (Fig. 1)

$$n_0 = \frac{(1-\rho)e^{-t}}{\rho + (1-\rho)e^{-t}}\ , \quad n_1 = \frac{\rho}{\rho + (1-\rho)e^{-t}}\ . \tag{2}$$

We define the *emergence time* t_* of the innovation by $n_0(t_*) = n_1(t_*) = \frac{1}{2}$. Using Eqs. (2) we get $t_* \simeq \ln(1/\rho)$. Ultimately everyone adopts the innovation. We estimate this *completion time* t^* from $n_1(t^*) = 1 - \frac{1}{N}$, corresponding to all but one individual in a population of size N adopting the innovation. Using this criterion we obtain $t^* \simeq \ln(N/\rho)$.

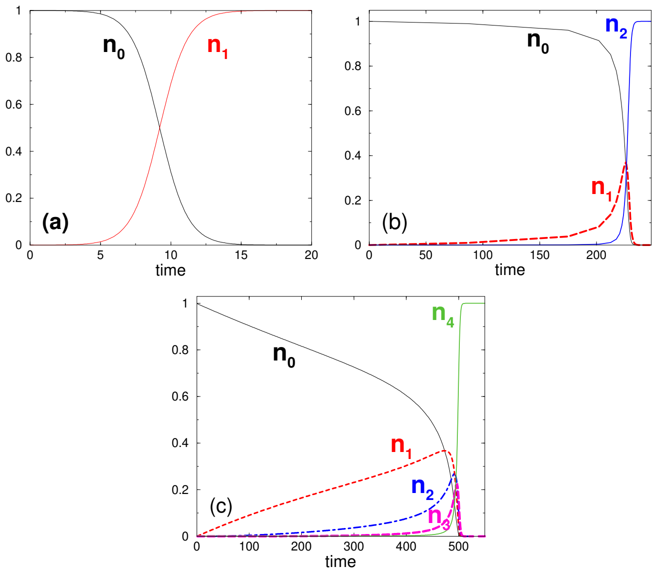

FIG. 1: The densities n_k as a function of time for (a) $M = 1$ (b) 2, and (c) 4 in a population of $N = 10^4$ individuals, in which $n_M(0) = \rho$ and $n_0(0) = 1 - \rho$, with $\rho = \frac{1}{N}$.

We now implement social reinforcement in the innovation model by allowing individuals to move to progressively higher awareness states. Possible examples of such progressions include: no TV, black & white TV, color TV [16], or no cell phone, dumb cell phone, smart phone, etc. The simplest such extension is a population that consists of three classes of individuals: ignorant (state 0), aware (state 1), and adopter (state 2), with respective densities n_0, n_1, and n_2. In an interaction with an adopter, an ignorant person becomes aware ($2+0 \to 2+1$), while an aware person adopts the innovation ($2+1 \to 2+2$). All other interactions do not change the states of individuals. When the rates of all processes are the same, the governing rate equations are:

$$\dot{n}_0 = -n_0 n_2, \quad \dot{n}_1 = n_0 n_2 - n_1 n_2, \quad \dot{n}_2 = n_1 n_2\,. \tag{3}$$

To solve these equations we introduce the internal time $\tau = \int_0^t dt'\, n_2(t')$ to simplify (3) to a linear system, whose solution, for the generic initial condition $n_2(0) = \rho$, $n_1(0) = 0$, $n_0(0) = 1-\rho$, is

$$\begin{aligned} n_0 &= (1-\rho)\, e^{-\tau}\,, \\ n_1 &= (1-\rho)\, \tau\, e^{-\tau}\,, \\ n_2 &= 1-(1-\rho)(1+\tau)\, e^{-\tau}\,. \end{aligned} \tag{4}$$

It is natural to define the emergence of the innovation as the point where where n_1 passes through a maximum (Fig. 1(b)). This yields $\tau_* = 1$, from which the corresponding value of the emergence time t_* is given by

$$t_* = \int_0^1 \frac{dx}{n_2(x)} = \int_0^1 \frac{dx}{1-(1-\rho)(1+x)e^{-x}}\,. \tag{5}$$

When $\rho \ll 1$, the asymptotic behavior of the integral is

$$t_* \simeq \frac{1}{\sqrt{\rho}} \int_0^{1/\sqrt{\rho}} \frac{dy}{1+y^2/2} \simeq \frac{\pi}{\sqrt{2\rho}}\,,$$

where $y = x/\sqrt{\rho}$ and the sub-leading term is of the order of one. For a single innovator in a population of size N (corresponding to initial density $\rho = \frac{1}{N}$), we conclude that the N dependence of the emergence time is

$$t_* = A_2 N^{1/2} + \mathcal{O}(1)\,, \qquad A_2 = \pi/\sqrt{2}\,. \tag{6}$$

Thus the existence an intermediate state changes the emergence time from a logarithmic to a power-law N dependence (Fig. 1). We can appreciate the mechanism for this slower adoption by eliminating n_1 from Eqs. (3) to give $\dot{n}_2 = n_2(1-n_0-n_2)$. In the early-time regime where $n_0 \approx 1$, the growth of $\ln n_2$ is negligible. In contrast, for direct innovation adoption, where $\dot{n}_1 = n_1(1-n_1)$, the initial growth rate of $\ln n_1$ is of the order of one. We estimate the completion time from the criterion $n_2(t^*) = 1-\frac{1}{N}$, which gives $t^* = A_2 N^{1/2} + \ln N$ to lowest order [17]. Thus once the innovation emerges, it takes little additional time before it is complete.

We generalize to an arbitrary number of intermediate states by assuming that an individual with awareness k increases to $k+1$ by interacting with a adopter, $[M]+[k] \to [M]+[k+1]$, with $k = 0, 1, \ldots, M-1$, while all other binary interactions do not change individual states. The corresponding rate equations are

$$\begin{aligned} \dot{n}_0 &= -n_M n_0\,, \\ \dot{n}_k &= n_M(n_{k-1} - n_k), \quad k = 1, \ldots, M-1\,, \\ \dot{n}_M &= n_M n_{M-1}\,. \end{aligned} \tag{7}$$

Introducing again the internal time $\tau = \int_0^t dt'\, n_M(t')$ reduces Eqs. (7) to a linear system whose solution is

$$\begin{aligned} n_j &= (1-\rho)\frac{\tau^j}{j!}\, e^{-\tau}, \qquad j = 0, , \ldots, M-1\,, \\ n_M &= 1-(1-\rho)\sum_{j=0}^{M-1} \frac{\tau^j}{j!}\, e^{-\tau}\,. \end{aligned} \tag{8}$$

In analogy with the case of $M = 2$, we define the innovation transition to occur at $\tau = 1$, where n_1 passes through a maximum (generally, each n_j passes through a maximum at $\tau = j$). To obtain explicit time dependences, we must recast the solution in terms of $t = \int_0^\tau dx/n_M(x)$. Applying the same steps as above and setting $\rho = 1/N$ we find that the emergence time scales as

$$t_* = A_M N^{1-1/M}\,, \tag{9}$$

where

$$A_M = \int_0^\infty \frac{dy}{1+y^M/M!} = (M!)^{1/M}\, \frac{\pi/M}{\sin(\pi/M)}\,.$$

Thus increasing the number of intermediate awareness states M progressively delays innovation emergence, as the exponent $1-\frac{1}{M}$ approaches 1 as M becomes large (Fig. 1(c)).

Transient fads: When adopters can independently abandon the innovation with rate $\lambda > 0$, the population at infinite time consists of adopters who ultimately abandoned the fad and individuals who are stuck in intermediate awareness states because of the absence of catalyzing adopters. Of particular interest are the *clueless* individuals who were never exposed to the fad while it was active. Their fraction $c_\infty(\lambda) \equiv n_0(t\!=\!\infty)$ characterizes the competing influences of contagion and fad abandonment [18]. In the thermodynamic $N \to \infty$ limit, c_∞ undergoes a continuous transition as a function of λ for $M = 1$, but a *discontinuous* transition for $M \geq 2$. We also find, surprisingly, that the time to reach the final state varies non-monotonically with λ.

For no intermediate states, the rate equations are now $\dot{n}_0 = -n_0 n_1$, $\dot{n}_1 = n_0 n_1 - \lambda n_1$, and have the solution

$$n_0 = (1-\rho)\, e^{-\tau}\,, \quad n_1 = 1 - \lambda\tau - (1-\rho)\, e^{-\tau}\,, \tag{10}$$

with $\tau = \int_0^t dt' n_1(t')$. The evolution ceases at an internal stopping time τ_∞ defined by $n_1(\tau_\infty) = 0$, corresponding

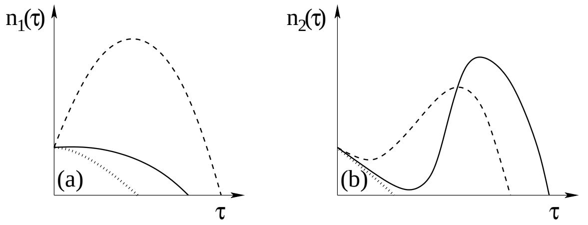

FIG. 2: Sketches (not to scale) of $n_M(\tau)$ versus τ for: (a) no intermediate states ($M = 1$), and (b) one intermediate state ($M = 2$). Shown dashed, solid, and dotted, are the cases $\lambda < \lambda_c$, $\lambda \approx \lambda_c$, and $\lambda > \lambda_c$.

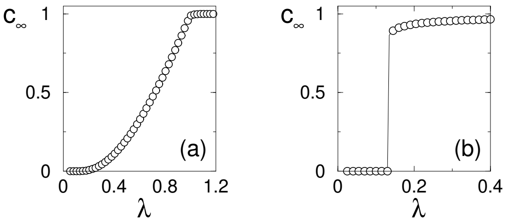

FIG. 3: Clueless fraction c_∞ versus abandonment rate λ for: (a) two-state and (b) three-state models. The initial adopter fraction is $n_1(0) = 10^{-4}$ in (a) and $n_2(0) = 10^{-2}$ in (b).

to physical time $t = \infty$. The condition $n_1(\tau_\infty) = 0$ (see Fig. 2(a)) defines three regimes of behavior for the fraction of clueless individuals c_∞. For $\lambda < 1$ (subcritical), adopters abandon the fad sufficiently slowly so that the fad spreads to a finite fraction of the population before dying out. For this subcritical case and also in the limit of a small initial fraction of adopters ρ, the stopping time is implicitly given by the root of $e^{-\tau_\infty} + \lambda\tau_\infty = 1$. Concomitantly, the clueless fraction is given by $c_\infty = e^{-\tau_\infty}$. The limiting behaviors for c_∞ are:

$$c_\infty = \begin{cases} e^{-1/\lambda} + \lambda^{-1}\, e^{-2/\lambda} + \ldots & \lambda \to 0\,, \\ 1 - 2(1-\lambda) + \frac{2}{3}(1-\lambda)^2 + \ldots & \lambda \to 1^-\,. \end{cases}$$

Conversely, for the supercritical case ($\lambda \geq \lambda_c = 1$), the fad dies out quickly, from which $c_\infty = 1$. At the next order of approximation, we keep the ρ term in the expression for n_1 in Eq. (10) and find, to leading order that $\tau_\infty = \rho/(\lambda - 1)$ and $c_\infty = 1 - \rho/(\lambda - 1)$. When λ equals the critical value $\lambda_c = 1$, the same approach gives $c_\infty = 1 - \sqrt{2\rho}$. Thus c_∞ undergoes a continuous transition (in the thermodynamic limit) as λ passes through the critical value $\lambda_c = 1$ (Fig. 3(a)).

For a transient fad with a single intermediate state ($M = 1$), n_0 and n_1 are again given by Eq. (4), while n_2 becomes

$$n_2 = 1 - (1-\rho)(1+\tau)e^{-\tau} - \lambda\tau\,. \tag{11}$$

Strikingly, the density of fad adopters $n_2(\tau)$ can first decrease with τ, then increase, and ultimately vanish for a certain range of λ (Fig. 2(b)). As a consequence, the stopping condition $n_2(\tau_\infty) = 0$ can have one, two, or three roots, depending on λ. This change in the number of roots is the mechanism for the discontinuity in the clueless fraction c_∞ as a function of λ.

To locate this transition, it is convenient to use the parameterization $\lambda = \mu\sqrt{\rho}$, because it leads to a transition at a value μ_c that is of the order of one. For $\mu > \mu_c$, we obtain the relevant root of $n_2(\tau_\infty) = 0$ by expanding $n_2(\tau_\infty)$ for small τ_∞ to lowest order. This gives $\rho + \frac{1}{2}\tau_\infty^2 - \mu\sqrt{\rho}\,\tau_\infty = 0$, with two solutions: $\tau_\infty = \sqrt{\rho}[\mu \pm \sqrt{\mu^2 - 2}]$. Using the relevant smaller solution, we find, for $\mu > \sqrt{2}$,

$$c_\infty = (1-\rho)\, e^{-\tau_\infty} \simeq 1 - \sqrt{\rho}\left(\mu - \sqrt{\mu^2 - 2}\right);$$

i.e., the clueless fraction is close to one (Fig. 3(b)). In the subcritical case, $\mu < \sqrt{2}$, the relevant root of $n_2(\tau_\infty) = 0$ is $\tau_\infty = 1/(\mu\sqrt{\rho})$ to leading order. Thus the clueless fraction

$$c_\infty = e^{-\tau_\infty} = e^{-1/(\mu\sqrt{\rho})} \tag{12}$$

vanishes extremely quickly as $\rho \to 0$.

Another striking aspect of our transient fad model is that the time for a fad to disappear has a complex dependence on the abandonment rate λ. The appropriate criterion for the disappearance of a fad in a finite population of size N is $n_M(\tau^*) = \frac{1}{N}$, which implies that the physical disappearance time is $t^* = \int_0^{\tau^*} d\tau/n_M(\tau)$. For the model with no intermediate states ($M = 1$), we compute $n_1(\tau^*)$ in the subcritical case of $\lambda < 1$ by Taylor expanding n_1 about the point τ_∞ and using the condition $e^{-\tau_\infty} + \lambda\tau_\infty = 1$ to obtain

$$t^* = \frac{1}{\lambda + \lambda\tau_\infty - 1}\int_{1/N} \frac{dy}{y} \tag{13}$$

where $y = (\lambda + \lambda\tau_\infty - 1)(\tau_\infty - \tau)$. The lower limit in (13) is just a consequence of relation $n_1(\tau^*) = \frac{1}{N}$, while the upper limit is immaterial for the asymptotic behavior. Thus we find

$$t^* = \frac{\ln N}{\lambda + \lambda\tau_\infty - 1} \qquad \lambda < \lambda_c\,. \tag{14}$$

In the supercritical regime of $\lambda > 1$, the density of adopters n_1 decreases almost linearly with internal time (Fig. 2(a)). In this case [19], the same expansion of n_1 about τ_∞ leads to the asymptotic behavior $t^* = \ln(\rho N)/(\lambda - 1)$. In the critical case of $\lambda = 1$, n_1 decreases quadratically with τ and the same expansion procedure as above shows that asymptotically $t^* = \ln(\rho N)/\sqrt{2\rho}$. While the $\ln N$ scaling of the completion time is natural, the non-monotonic λ dependence of the amplitude (Fig. 4) is defies an intuitive explanation.

Simulations of a finite population mirror our analytical predictions except near the first-order transition

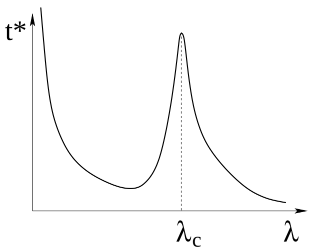

FIG. 4: Sketch (not to scale) of the completion time t^* as a function of λ for the transient fad model for any M. For $M = 1$, the width of the peak scales as $\sqrt{\rho}$.

(Fig. 3(b)), where large fluctuations arise. For example, near the transition the fad either quickly disappears and the population remains almost entirely clueless (Fig. 5), or nearly everyone adopts and then abandons the fad, in which case the fad is longer-lived. This apparent lack of self-averaging implies that at the transition point the deterministic rate equation description is inapplicable and stochastic effects must be considered. Preliminary simulations suggest that the $M = 2$ fad model average lifetime grows as $N^{1/4}$ rather than the logarithmic dependence predicted by the rate equation [17].

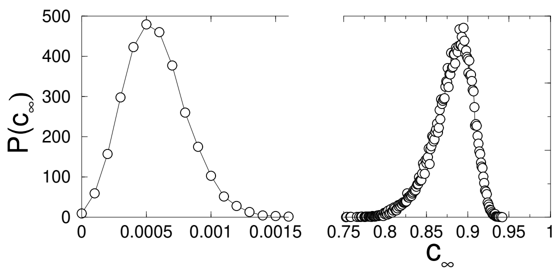

FIG. 5: Probability density $P(c_\infty)$ for the fraction of clueless individuals at infinite time at the critical point, $\lambda_c = \sqrt{2\rho}$, for the $M = 2$ fad model, with initial density $\rho = 10^{-2}$ and population size $N = 10^4$. (Note the horizontal scale break.)

An appealing extension of our innovation model is to allow individual heterogeneity (see also [11, 17]). In the situation of direct adoption (no intermediate states), each ignorant individual becomes an adopter with a distinct rate r chosen from a receptivity distribution. Then the emergence and completion times are asymptotically [17]

$$t_* \simeq \langle r \rangle^{-1} \ln N\,, \quad t^* \sim N^{1/(\mu+1)}.$$

The former result is valid for an arbitrary receptivity distribution with finite average $\langle r \rangle$, while the latter result applies to receptivity distributions which vary as r^μ in the $r \to 0$ limit. Thus heterogeneity changes the scaling of the completion time t^* from logarithmic to power-law in N. For $\mu < 0$, the completion time grows faster than linearly with N. That is, a preponderance of Luddites — those resistant to change — leads to an extremely long time for the innovation to be universally adopted.

To summarize, social reinforcement plays a crucial role in determining how permanent innovations are adopted in a population whose members evolve by progressively moving to states of increasing awareness. For transient fads, a subtle interplay between contagion and the fad abandonment rate determines whether a fad spreads globally or quickly dies out. The transition between these two limits is remarkably rich and the fad lifetime is maximal at the transition point.

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[19] Here we assume the natural situation of $\rho \ll 1$ and $\rho N \gg 1$. In the extreme case of $\rho = N^{-1}$, fluctuations are artificially large and the rate equation approach loses its usefulness.