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J. Phys. A: Math. Gen. **31** (1998) 9229–9240. Printed in the UK PII: S0305-4470(98)94937-X

Slow coarsening in an Ising chain with competing interactions

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Received 8 June 1998, in final form 27 August 1998

Abstract. We investigate the zero-temperature coarsening dynamics of a chain of Ising spins with a nearest-neighbour ferromagnetic and an nth-neighbour antiferromagnetic interactions. For sufficiently large antiferromagnetic interaction, the ground state consists of n consecutive up spins followed by n down spins, etc. We show that the asymptotic coarsening into this ground state can be described in terms of a multispecies reactive gas of elementary excitations. The basic elementary excitations are identified and each decays at a different power-law rate in time. The dominant excitations are domains of $n+1$ spins which diffuse freely and disappear through processes which are effectively governed by $(n+1)$-particle annihilation. This leads to a slow $t^{-1/n}$ temporal approach to the ground state.

1. Introduction

Ising models with nearest-neighbour ferromagnetic and more distant antiferromagnetic interactions exhibit rich magnetic ordering [1–3]. The competition between ferromagnetism and longer-range antiferromagnetism leads to different ordered states and an associated sequence of phase transitions as a function of these two interaction strengths. Such models were originally formulated to help describe the complex magnetism of the rare earths [4]. Their unusual magnetic ordering is believed to arise from the so-called Ruderman–Kittel–Kasuya–Yosida (RKKY) interaction [5], in which the exchange interaction between localized magnetic moments oscillates between ferromagnetic and antiferromagnetic as a function of their separation. Some of the essential consequences of this situation seem to be captured by Ising models with competing interactions.

One of the simplest versions is the axial next-nearest-neighbour Ising (ANNNI) model in which there is an isotropic nearest-neighbour ferromagnetic interaction and a next-nearest-neighbour antiferromagnetic interaction along a single axis [2]. Even in one dimension, intriguing magnetic properties arise. For weak antiferromagnetic interaction, the ground state is ferromagnetic, while for strong antiferromagnetic interaction there is an 'antiphase' ground state which consists of two spins up, followed by two spins down, etc. For a specific ratio of these two interactions, an infinitely degenerate ground state arises in which each spin domain is of length 2 or greater [6].

Given the disparate natures of these ground states, one might expect that dynamical behaviour is also strongly affected by such competing interactions. Our goal is to understand the kinetics of an Ising chain with nearest-neighbour ferromagnetic interaction J_1 and nth-

0305-4470/98/469229+12$19.50 © 1998 IOP Publishing Ltd

neighbour antiferromagnetic interaction $-J_n$, when the system is endowed with single spin-flip Glauber dynamics. The Hamiltonian of the system is

$$\mathcal{H} = -J_1 \sum_i s_i s_{i+1} + J_n \sum_i s_i s_{i+n}. \tag{1}$$

For $J_n > J_1/n$, the ground state is a sequence of n consecutive up spins followed by n down spins, etc. Our basic result is that for $J_n > J_1$ the asymptotic approach of the system to this alternating ground state can be described in terms of a reactive gas of elementary excitations. The rate limiting step of this process is governed by $(n+1)$-particle annihilation of the dominant excitations; this implies that the system approaches the ground state in time as $t^{-1/n}$.

To provide a context for this work, let us recall the well known example of zero-temperature ($T = 0$) coarsening in the Ising–Glauber chain with nearest-neighbour ferromagnetic interactions only [7]. Spin flips inside ferromagnetic domains are forbidden at $T = 0$ because they cost energy, while spins at domain interfaces can flip freely, as indicated by

$$\begin{array}{c} - - - - - - + + + + + + \\ \Downarrow \\ - - - - - + + + + + + + \end{array}$$

since no energy cost is involved. This spin flip is equivalent to the hopping of a domain wall 'excitation' which lies between the neighbouring misaligned spins. Two domain walls can annihilate when they meet, a process which reduces the energy of the system. This event is equivalent to a domain which shrinks to zero size via the process

$$\begin{array}{c} + + + + + + - + + + + + + \\ \Downarrow \\ + + + + + + + + + + + + + + . \end{array}$$

The correspondence between the ferromagnetic Ising–Glauber chain and a gas of domain walls which undergo nearest-neighbour hopping and single-species two-particle annihilation provides a simple way to understand the $t^{-1/2}$ coarsening dynamics of the spin system [8]. In the following sections we show that the Ising–Glauber chain with competing interactions can be understood through a much richer picture of reactive excitations.

2. Second-neighbour interaction

2.1. Strong antiferromagnetism: $J_2 > J_1$

Let us first treat the Ising–Glauber chain with competing interactions at $T = 0$ for the case $n = 2$, i.e. a near-neighbour ferromagnetic interaction J_1 and a second-neighbour antiferromagnetic interaction $-J_2$. We focus on the case of strong antiferromagnetic interaction, $J_2 > J_1$, as this is the case which leads to interesting dynamics. For simplicity, let us consider the dynamics starting from an initial ferromagnetically ordered up state. Subsequently, we will consider the evolution when starting from an arbitrary initial state.

An initially ferromagnetic system evolves to the alternating $\cdots + + - - + + - - \cdots$ ground state by a two-stage process. First, there is initial nucleation of down domains. Within a ferromagnetic domain of up spins, there is an energy loss $\Delta E = 4(J_1 - J_2)$ when a single spin flips to create an isolated down spin. This same energy loss arises for any nucleation event which occurs within a domain of length $\geqslant 5$ when the flipped spin is at

least two lattice spacings from both domain walls. After this nucleation, the single-spin domain can grow to length 2 with an energy loss $\Delta E = -4J_2$, as long as the length of the neighbouring domain into which this inflation occurs is greater than 2. At the end of this nucleation stage, therefore, the system consists of ordered $\cdots++--++\cdots$ regions, as well as domains of size 1, 3, and 4. Further, domains of size unity exist only if both neighbouring domains are of length 2.

These remaining domains now undergo a sequence of reactions which ultimately leads to the system reaching the ground state. To determine this evolution, first consider an isolated 3-domain within an otherwise stable array to 2-domains. Since the spin on either edge of the 3-domain can flip without energy cost, the 3-domain can hop isotropically by two lattice sites, as indicated by

$$\begin{array}{c} ++---++---++--\\ \Downarrow\\ ++--++++--++--. \end{array}$$

Similarly, an isolated 1-domain within a sea of 2-domains also diffuses freely, since either neighbouring spin of the 1-domain can flip with no energy cost, as indicated by

$$\begin{array}{c} --+-+++--+++--\\ \Downarrow\\ --++---+--++--. \end{array}$$

These freely diffusing 1- and 3-domains are the elementary excitations of the system.

To appreciate the consequences of this statement, consider an isolated 4-domain within an otherwise stable array of 2-domains. Since there is an energy cost associated with flipping either spin in the interior of a 4-domain, this process does not occur at zero temperature. If, however, either spin at the end of the 4-domain flips, for example,

$$\begin{array}{c} --++----++--++\\ \Downarrow\\ --++---+++--++ \end{array}$$

then the configuration becomes two adjacent 3-domains within the stable sea of 2-domains. Each of these 3-domains can then diffuse freely; the first step of this process is indicated by

$$\begin{array}{c} --++---++--++\\ \Downarrow\\ --+++--++--++. \end{array}$$

One can thus regard an isolated 4-domain as a [3, 3] resonant state which is formed whenever two 3-domains collide. This resonance is short-lived, however, since its binding energy is zero.

Continuing this reasoning, consider the evolution of a 4, 3 pair within a stable sea of 2's. Since both the 3 and 4 diffuse freely, these two excitations could move apart with zero energy cost. On the other hand, the 3-domain can shrink and the 4-domain can grow to size 5, also with zero energy cost. If this occurs, the 5 is unstable to the nucleation event: $5 \to 2+1+2$. Due to the fact that the 4 can be viewed as a [3, 3] resonance, the resultant stoichiometry of this process is $3+3+3 \to 1$, i.e. 3's annihilate through triple collisions.

Consider now how isolated 1's evolve. Since 1's diffuse freely they react only upon meeting another 1 or a 3. In the former collision, a 4 is formed,

$$\begin{array}{c} ++--++-+--++-- \\ \Downarrow \\ ++--++++--++-- \end{array}$$

with an associated energy loss $\Delta E = -4J_1$. Because the 4 is equivalent to a pair of 3's, the ultimate stoichiometry of this process is $1 + 1 \to 3 + 3$. It is also possible for three 1's to diffuse to adjacent lattice sites. If this occurs, these domains can evolve via the 'anti-nucleation'

$$\begin{array}{c} ++-+-++ \\ \Downarrow \\ ++---++ \end{array}$$

with an associated energy decrease $\Delta E = -4(J_1 + J_2)$.

Finally if a 1 meets a 3, they react to form a stable pair of 2's,

$$\begin{array}{c} ++--+---++-- \\ \Downarrow \\ ++--++--++-- \end{array}$$

with an associated energy loss $\Delta E = -4J_2$. This can be viewed as the two-species annihilation process $1 + 3 \to 0$, since the pair of 2's that are formed become part of the stable ground state.

The underlying stoichiometry of these processes can therefore be summarized by the four reactions

$$\begin{array}{ll} 1 + 1 \to 3 + 3 & \Delta E = -4J_1 \\ 1 + 3 \to 0 & \Delta E = -4J_2 \\ 3 + 3 + 3 \to 1 & \Delta E = +4(J_1 - J_2) \\ 1 + 1 + 1 \to 3 & \Delta E = -4(J_1 + J_2). \end{array} \tag{2}$$

Since these processes all lower the energy, they each occur at the same rate when $T = 0$. The overall effect of these reactions is that the density of these elementary excitations ultimately vanish. Given that 1's disappear through single-species binary collisions, the role of triple collisions of 1's is asymptotically negligible. In contrast, triple collisions of 3's continue to play a role asymptotically, as we shall see below.

Let us now determine the time dependence for the densities of these excitations in the mean-field limit. Using A to denote both domains of length 3 and their density, and similarly using B for 1's, the rate equations associated with the reactions of equation (2) are

$$\dot{A} = -3A^3 + 2B^2 + B^3 - AB \tag{3}$$

$$\dot{B} = A^3 - 2B^2 - 3B^3 - AB. \tag{4}$$

A naive qualitative analysis of these equations indicates that A's are asymptotically dominant. Thus using $A \gg B$, the above rate equations simplify to

$$\dot{A} \simeq -3A^3 - AB \tag{5}$$

$$\dot{B} \simeq A^3 - AB. \tag{6}$$

Subtracting equation (6) from (5) gives $\dot{A} - \dot{B} \simeq -4A^3$. Given $A \gg B$, we neglect $\dot{B}$ to give the closed equation $\dot{A} \simeq -4A^3$. Comparing this with equation (5) yields $A^3 \simeq AB$. This implies $B \simeq A^2$, which agrees with $B \ll A$. The results of these considerations are

$$A(t) \simeq \frac{1}{\sqrt{8t}} \qquad \text{and} \qquad B(t) \simeq \frac{1}{8t} \tag{7}$$

which are confirmed by numerical integration of the rate equations.

Let us now adapt the rate equations to the case of one dimension. Quite generally, we write

$$\dot{A} \simeq -3R - r \tag{8}$$

$$\dot{B} \simeq R - r. \tag{9}$$

Here R is the rate at which A's disappear due to the triple collisions and r is the rate at which both A's and B's disappear due to their mutual annihilation. These equations are at the same level of approximation as equations (5) and (6), as we again neglect the effect of interactions between two B's. The rate R can be shown to scale as $A^3/\ln(1/A)$ [9]. The cubic term is just the mean-field rate of triple collisions between A's, and the logarithmic correction arises because one dimension is the critical dimension for the three-particle reaction–diffusion processes [10]. For three-particle annihilation, this reactivity leads to the rate equation $\dot{A} \sim -A^3/\ln(1/A)$, which predicts $A \sim \sqrt{\ln t/t}$, in agreement with previous numerical and theoretical treatments [11]. The two-species annihilation rate r can be estimated as $r \sim B/\tau$, where τ is the reaction time for an A and a B to meet by diffusion. This reaction time is proportional to the square of the distance between a B and its nearest A; therefore, $\tau \sim 1/A^2$ which gives $r \sim BA^2$.

Subtracting equation (9) from (8) gives $\dot{A} - \dot{B} \simeq -4R$. Since we again anticipate that $A \gg B$, we ignore $\dot{B}$ to obtain $\dot{A} \simeq -4R$. This relation implies that triple collisions are the dominant kinetic mechanism for elimination of A's, so that this density decays as in three-particle annihilation. The factor of 4 in $\dot{A} \simeq -4R$ indicates that four A's eventually disappear after a triple collision—three particles are eliminated in the process $A+A+A \to B$, and then the newly-formed B will eliminate another A. To determine $B(t)$, note that equation (8), together with $\dot{A} \simeq -4R$, imply that $r \simeq R$; that is, the gain and loss terms in equation (9) cancel. The relation $r \simeq R$ can be rewritten as $B \sim R/A^2 \sim -\dot{A}/A^2$. We therefore conclude that the density of elementary excitations are

$$A(t) \sim \sqrt{\frac{\ln t}{t}} \qquad \text{and} \qquad B(t) \sim \frac{1}{\sqrt{t \ln t}}. \tag{10}$$

Monte Carlo simulations of the $T = 0$ Ising–Glauber chain yield results which are qualitatively consistent with these predictions. As shown in figure 1, the densities of the various elementary excitations decay at different temporal rates. Over the last two decades of data, linear least-squares fits give 0.40 and 0.56 for the exponent associated with the density of 3's and 1's, respectively. However, curvature in the data is clearly evident and the linear fit is not indicative of the asymptotic behaviour. In fact, the data for the density of 1's is curved upward while that for the density of 3's is curved downward; these are suggestive of an asymptotic exponent of $\frac{1}{2}$ with slowly vanishing pre-asymptotic corrections which are qualitatively consistent with equations (10). A serious quantitative test of (10) would require extensive simulation, since the corrections to the local exponents will vanish only as $1/\ln t$. Similar least-squares fits give exponents for the density of 4's and 5's as 0.84 and 1.42 with both both data sets curving downward. These features are qualitatively consistent with the hypothesis that 4's and 5's are equivalent to [3, 3] and [3, 3, 3] resonances, respectively, so that their densities should scale as A^2 and $A^3/\ln(1/A)$.

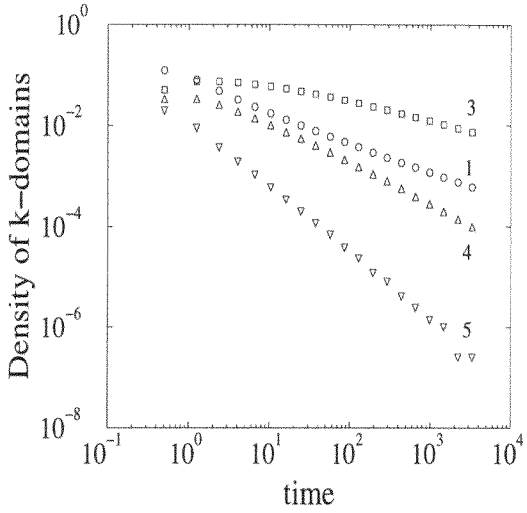

Figure 1. Density of domains of length 1 (○), 3 (□), 4 (△) and 5 (▽). Simulation results are based on 1000 realizations of a chain of 20 000 sites with $J_2/J_1 > 1$.

To provide an additional insight into the basic nature of this coarsening process it is helpful to consider the case $J_1 = 0$, where the chain breaks up into two independent antiferromagnetic sublattices. For each sublattice, the dynamics must coincide with the usual $t^{-1/2}$ Glauber coarsening. It is instructive to see how this behaviour arises within our picture of elementary excitations. The crucial feature for the case $J_1 = 0$ is that the process $1 + 1 \to 3 + 3$ now involves no energy change (see equation (2)) and the reverse process $3 + 3 \to 1 + 1$ occurs freely. This additional process leads now to the rate equations

$$\dot{A} = -3A^3 - 2A^2 + 2B^2 + B^3 - AB \tag{11}$$

$$\dot{B} = A^3 + 2A^2 - 2B^2 - 3B^3 - AB. \tag{12}$$

The cubic term turns out to be negligible and the resulting asymptotic behaviour is $A(t) = B(t) \sim 1/t$. Similarly, for the one-dimensional system, the density of 1's and 3's become identical and both decay as $t^{-1/2}$. The role of the $3 + 3 \to 1 + 1$ reaction and its influence on the rate equations shows clearly that the dynamics of the Ising–Glauber chain with competing interactions is in a different universality class than that of the purely antiferromagnetic spin chain.

2.2. Weak antiferromagnetism: $J_2 < J_1$

To complete the discussion of the competing interaction Ising–Glauber chain, let us consider weak antiferromagnetism, namely, $J_2 < J_1$. The basic new dynamical features are that nucleation of isolated single-spin domains and also the coalescence $3 + 3 + 3 \to 1$ are energetically forbidden at $T = 0$, since for both processes $\Delta E = 4(J_1 - J_2)$ is now positive. We shall argue that the combination of these two features leads to the system evolving to a trivial non-equilibrium steady state. For the case $J_2 < J_1/2$ the ferromagnetic ground state cannot be reached because of the repulsion of domain walls which forces them to always be at least two lattice spacings apart. On the other hand, for $J_1/2 < J_2 < J_1$, the $++--++--$ ground state cannot be reached because there is no mechanism for isolated domains of length $\geqslant 3$ to break up. In both the cases $J_2 < J_1/2$ and $J_1/2 < J_2 < J_1$

the local spin dynamics is the same, but their manifestation in terms of domains is rather different.

The absence of domain nucleation means that a ferromagnetic initial state does not evolve; thus an initial state with many domains is needed to have interesting dynamics. In this case, finite size domains of length $\geqslant 2$ can evolve by the diffusion of domain walls between oppositely-oriented spins, just as in the case of the classical Ising–Glauber model with $J_2 = 0$. However, the absence the process $3 + 3 + 3 \to 1$ means that 1's and 3's evolve quite differently. Within the rate equation approximation, there is no A^3 term on the right-hand sides of equations (3) and (4). An asymptotic analysis of these equations in the spirit of the previous subsection shows that the density of 1's decays exponentially in time, while the density of 3's saturates to a finite value. Additionally, while resonances, such as [3, 3], [3, 3, 3], etc can form freely by collisions between 3's, such high-order resonances cannot transmute to another form, but merely re-fragment into lower-order resonances of length $\geqslant 3$. Because of this persistence of 3's, the ground state, namely, ferromagnetic for $J_2 < J_1/2$, and the configuration $\cdots + + - - + + - - \cdots$ for $J_1/2 < J_2 < J_1$, is never reached. However, for the case $J_2 = J_1/2$, the ground state is an array of domains which all have lengths $\geqslant 2$ [6], and this can be reached by Glauber dynamics.

Another intriguing aspect of the weakly antiferromagnetic system is that for all $J_2 < J_1$, repulsion of domain walls ensures that domains of length $\geqslant 2$ can never be eliminated, while initial domains of length 1 can disappear by anti-nucleation. The flipping of a single spin within a string of $\geqslant 3$ consecutive antiferromagnetic spins can be viewed as the replacement of the antiferromagnetic triplet by a ferromagnetic 'trimer'. This replacement of antiferromagnetic triplets by a ferromagnetic trimer continues until an antiferromagnetic region is converted into a ferromagnetic domain. Once all antiferromagnetic strings are eliminated the subsequent dynamics cannot change the number of domains. This provides an invariant to classify each initial state.

The replacement dynamics is equivalent to the random sequential adsorption of trimers which can overlap. The evolution of this adsorption process can be solved according to well known procedures [12]. Let $P_n(t)$ denote the probability that exactly n consecutive sites are occupied by 1-domains, that is, an antiferromagnetically ordered string of length n. The probabilities $P_n(t)$ evolve according to the rate equations

$$\dot{P}_n = -(n+2)P_n + 2\sum_{j=n+1}^{\infty} P_j. \tag{13}$$

The loss term arises because the adsorption of a trimer whose centre coincides with any of the n sites of the string or the two sites adjacent to the string destroys the n-string. Similarly, the gain term arises from processes in which the adsorption of a trimer onto a larger string leads to the creation of an n-string.

To solve the rate equations, the ansatz

$$P_n(t) = F(t)f^{n+2}(t) \tag{14}$$

transforms the infinite set of differential equations (13) into the pair of equations

$$\dot{f} = -f \qquad \dot{F} = \frac{2Ff^2}{1-f}. \tag{15}$$

Consider now an initially disordered Ising chain (corresponding to a quench from $T = \infty$ to $T = 0$). An antiferromagnetic string of length n occurs with probability $P_n(0) = 2^{-n-3}$, which implies

$$F(0) = f(0) = \tfrac{1}{2}. \tag{16}$$

Solving equations (15) subject to (16) yields

$$f(t) = \tfrac{1}{2}\mathrm{e}^{-t} \tag{17}$$

$$F(t) = 2(1 - \tfrac{1}{2}\mathrm{e}^{-t})^2 \exp[\mathrm{e}^{-t} + \tfrac{1}{4}\mathrm{e}^{-2t} - \tfrac{5}{4}]. \tag{18}$$

Therefore domain walls inside antiferromagnetic strings quickly disappear and the total density of domain walls reaches a saturation level. Since the subsequent dynamics does not allow for processes which change the number of domain walls, the system continues to explore a sector of the phase space which contains all states with the same number of domain walls. The system decays into its asymptotic sector exponentially in time. Different sectors are mutually disconnected, and the sector eventually reached by the system depends on initial conditions. This initial condition dependence and exponential relaxation are outcomes of the lack of ergodicity in the Glauber dynamics of the Ising chain with competing interactions for all $J_2 < J_1$.

2.3. Marginal antiferromagnetism: $J_2 = J_1$

In the marginal case $J_2 = J_1$, the crucial difference with the case of strong antiferromagnetism, $J_2 > J_1$, is that the process $3+3+3 \to 1$ now involves no energy gain and thus the reverse process $1 \to 3+3+3$ can also occur freely. The rate equations (8) and (9) are thus modified to

$$\dot{A} \simeq 3B - 3R - r \tag{19}$$

$$\dot{B} \simeq R - r - B. \tag{20}$$

To determine the asymptotic behaviour, it proves convenient to transform these equations to

$$\dot{A} - \dot{B} \simeq -4(R - B) \tag{21}$$

$$\dot{A} + 3\dot{B} \simeq -4r. \tag{22}$$

We now ignore $\dot{B}$ on the left-hand sides. Combining the estimate $r \sim BA^2$ with equation (22) gives $B \sim -\dot{A}/A^2$. This implies $B \gg -\dot{A}$, so that equation (21) now gives $B \simeq R$, or $\dot{A} \sim -A^5/\ln(1/A)$. We therefore conclude that when $J_2 = J_1$ the density of elementary excitations are

$$A(t) \sim \left(\frac{\ln t}{t}\right)^{1/4} \qquad \text{and} \qquad B(t) \sim \left(\frac{1}{t^3 \ln t}\right)^{1/4}. \tag{23}$$

Thus in the marginal case of $J_2 = J_1$, the Ising–Glauber chain still coarsens, but at a much slower rate than in the case of strong antiferromagnetism, $J_2 > J_1$.

3. Third- and more distant-neighbour interaction

Our general approach can be adapted to longer range antiferromagnetic interactions. We first outline basic relaxational features for an antiferromagnetic third-neighbour interaction; the behaviour for arbitrary range antiferromagnetic interaction follows inductively. The Hamiltonian now is $\mathcal{H} = -\sum_i (J_1 s_i s_{i+1} - J_3 s_i s_{i+3})$ and for sufficiently strong antiferromagnetic interaction $J_3 > J_1/3$ the ground state consists of alternating domains of length 3, $\cdots + + + - - - + + + \cdots$. Starting from the ferromagnetic up state, the evolution to the ground state again proceeds by a two-stage process when $J_3 > J_1$. While the ground state occurs when $J_3 > J_1/3$, we shall employ the stronger inequality $J_3 > J_1$

in the following to guarantee that this ground state is accessible via single spin-flip Glauber dynamics.

In the initial nucleation stage, it is energetically favourable for a spin within an large up domain to flip if this spin is three or more lattice spacings from any domain boundary. Thus domains of length $\geqslant 7$ are unstable to such nucleation events. It is also energetically favorable for this isolated down spin domain to grow to size 3, as long as the expanding domain wall remains at least three lattice spacings from adjacent domain walls. At the end of the nucleation stage, therefore, all domains have length $\leqslant 6$. Further, domains of length 1 or 2 must be surrounded by domains of length 3; otherwise the central domain would expand until its size reached 3.

A basic observation is that the true elementary excitations are domains of length 2 and of length 4, as these are the only objects which diffuse freely within a stable sea of 3's. All other defects are resonant states of these two elementary excitations. To determine the nature of the coarsening, first consider the resonances of 4-domains. For example, a 5-domain is formed when two 4's meet and interact, so that one 4 shrinks to length 3, while the other grows to length 5. There is no energy cost associated with this process, so that a 5-domain can be viewed as a [4, 4] resonance. Similarly, a 6-domain is a [5, 4], or equivalently a [4, 4, 4] resonance. Finally, a 7-domain may be produced by the conversion of $6 + 4 \to 7 + 3$; thus the 7 is a [4, 4, 4, 4] resonance. At the centre of the 7-domain, a single spin can flip, thereby nucleating a 1-domain and two surrounding stable 3-domains. The resultant stoichiometry of this process is therefore, $4 + 4 + 4 + 4 \to 1$.

Conversely, consider the resonances and interactions associated with 2-domains. When two 2's within a stable sea of 3's meet, a 1-domain is formed, as indicated by

$$\begin{array}{c} ++\,+--+\,+---++\,+ \\ \qquad\quad\Downarrow \\ ++\,+---\,+---++\,+\,. \end{array}$$

Since there is no energy cost associated with this process, a 1-domain is simply a [2, 2] resonance. When three 2's meet, there are several possible zero-energy-cost outcomes. If the interior spin of the outer 2-domain flips, then the result is $2 + 2 + 2 \to 1 + 3 + 2$. Since the 1-domain is a [2, 2] resonance, this process can be considered as the first step in separating the three initial domains. However, if one of the spins in the middle domain flips, then the outcome is $2 + 2 + 2 \to 3 + 1 + 2$. Once a 3, 1, 2 state is reached, it is energetically favorable for the central isolated spin to flip thus giving $3 + 1 + 2 \to 6$, i.e. a [4, 4, 4] resonance. This last step is accompanied by the energy loss $-4J_1$. The outcome of more than three 2's meeting can be obtained by grouping the 2's into triplets and analysing the outcome of each triplet in series. Finally, when a 2 and a 4 meet, it is energetically favourable for $2 \to 3$ and $4 \to 3$. This can be viewed as the two-species annihilation $2 + 4 \to 0$, since the two 3's formed in the reaction belong to the stable ground state.

From these basic processes, the governing reactions for this system are

$$\begin{aligned} &4 + 4 + 4 + 4 \to 1 \to 2 + 2 \\ &2 + 4 \to 0 \\ &2 + 2 + 2 \to 6 \to 4 + 4 + 4. \end{aligned} \tag{24}$$

For these reactions, the associated rate equations for the density of 4's (A) and 2's (B) are

$$\dot{A} = -4A^4 + 3B^3 - AB \tag{25}$$

$$\dot{B} = 2A^4 - 3B^3 - AB. \tag{26}$$

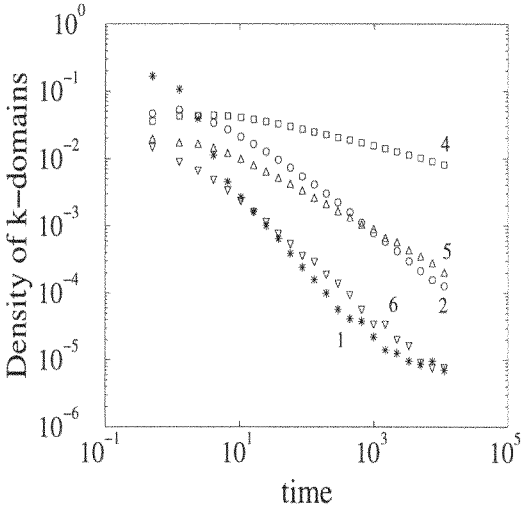

Figure 2. Density of domains of length 1 (∗), 2 (○), 4 (□), 5 (△), and 6 (▽). Simulation results are based on 500 realizations of a chain of 20000 sites with $J_3/J_1 > 1$.

The structure of these equations is similar to the second-neighbour interaction case. Following the same reasoning as used previously, we find $A(t) \sim t^{-1/3}$ and $B(t) \sim t^{-1}$. Similarly, we may adapt the rate equation above to describe the system in one dimension by following the approach used to write equations (8) and (9). This leads to

$$\dot{A} \simeq -4A^4 - A^2B \tag{27}$$

$$\dot{B} \simeq 2A^4 - A^2B \tag{28}$$

with the asymptotic behaviour $A(t) \sim t^{-1/3}$ and $B(t) \sim t^{-2/3}$.

As in the case of first- and second-neighbour interactions, simulations for the densities of the various elementary excitations indicate that they decay at different temporal rates (figure 2). Over the last two decades of data, linear least-squares fits give 0.25 and 0.80 for the exponent associated with the density of 4's and 2's, respectively. Once again, the sense of the curvature in these two data sets is consistent with the respective asymptotic exponents of $\frac{1}{3}$ and $\frac{2}{3}$. However, even more so than in figure 1, the linear fit is not indicative of asymptotic behaviour. We merely point out that our picture of elementary excitations allows one to express all densities in terms of the density of 4's. This predicts that the density of 1's scales as A^4, the densities of 2's and 5's scale as A^2, and the density of 6's scales as A^3. This is only marginally consistent with the data, a feature which we attribute to slow approach to asymptotic behaviour.

Finally, we may apply a similar geometrical picture of elementary excitations to treat the general case of the Ising–Glauber chain with competing first- and nth-neighbour interactions when $J_n > J_1$. In this case, the basic excitations are domains of $n+1$ spins (A) and domains of $n-1$ spins (B), each of which diffuses freely within a stable ground state sea of alternating ferromagnetic strings of length n. Other types of excitations are resonances of these elementary excitations. The basic kinetic mechanisms that govern the coarsening of the spin system are the $(n+1)$-particle annihilation of contiguous groups of A excitations via $(n+1)A \to 1$, and the two-species annihilation $A + B \to 0$. An analysis of the corresponding rate equations again indicates that these two processes are of the same order of magnitude. Consequently, the ground state is approached as $t^{-1/n}$ and the densities of the basic excitations decay according to $A(t) \sim t^{-1/n}$ and $B(t) \sim t^{-(n-1)/n}$.

4. Conclusions

We investigated the coarsening kinetics of the Ising chain with single spin-flip dynamics when a distant nth-neighbour antiferromagnetic interaction $-J_n$ competes with the nearest-neighbour ferromagnetic interaction J_1. For $J_n > J_1$, this competition leads to slower zero-temperature coarsening compared with the case of nearest-neighbour ferromagnetic interactions only. Other types of non-universal relaxation phenomena have been reported for alternating interactions and other modifications of the pure Ising chain [13]. The case of a competing interaction, however, is amenable to an intuitively appealing description in terms of elementary excitations which makes clear the mechanism for the new kinetic behaviour. It is intriguing that the nature of the elementary excitations and the spectrum of resonances are not obviously connected with the microscopic interaction of the spin system.

For general nth-neighbour antiferromagnetic interaction, the elementary excitations are ferromagnetic strings of $n+1$ and $n-1$ spins. The former interact and disappear through $(n+1)$-particle single-species annihilation, while $n+1$ and $n-1$ mutually annihilate when they meet. These two processes are of the same order of magnitude so that the rate of the overall coarsening process can be viewed as being limited by $(n+1)$-particle annihilation. This leads to a coarsening which proceeds as $t^{-1/n}$. The marginal case $J_n = J_1$ admits additional microscopic processes which leads to even slower coarsening.

It is interesting that Glauber dynamics provides only a tenuous connection with equilibrium properties of the system. That is, the dynamical change of behaviour at $J_n = J_1$ is disconnected from the corresponding equilibrium behaviour, where ferromagnetism occurs for $J_n < J_1/n$, a ground state of alternating domains of n ferromagnetic spins occurs for $J_n > J_1/n$, and an infinitely degenerate ground state consisting of alternating domains of $\geqslant n$ spins occurs for $J_n = J_1/n$.

Finally, it is worth noting that the presence of two kinds of elementary excitations implies the existence of two characteristic length scales (the reciprocals of their corresponding densities). This is in contrast to conventional coarsening phenomena which typically exhibit a *single* length scale. However, violation of scaling has been reported in several coarsening systems. In particular, *two* length scales arise in vector order parameter systems [14], in the cyclic Lotka–Volterra model [15], in single-species annihilation with combined diffusive and convective transport [16], and in three-dimensional Ising systems with competing interactions [17].

Acknowledgments

We thank D Das and the referee for helpful comments on the manuscript. We also gratefully acknowledge NSF grant DMR-9632059 and ARO grant DAAH04-96-1-0114 for financial support.

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