Can a Lamb Reach a Haven Before Being Eaten by Diffusing Lions?

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Abstract. We study the survival of a single diffusing lamb on the positive half line in the presence of N diffusing lions to the right of the lamb and a haven at $x = 0$. If the lamb reaches this haven before meeting any lion, the lamb survives. We investigate the survival probability of the lamb, $S_N(x, L)$, as a function of N and the respective initial positions of the lamb and the lions, x and L. We determine $S_N(x, L)$ analytically for the special cases of $N = 1$ and $N \to \infty$. For large but finite N, we determine the unusual asymptotic form whose leading behavior is $S_N(z) \sim N^{-z^2}$, with $z = x/L$. Simulations of the capture process very slowly converge to this asymptotic prediction as N reaches 10^{500}.

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1. Introduction

We investigate the one-dimensional diffusive capture process in which a marked particle — a "lamb" — diffuses on the positive half line $x > 0$ in the presence of N independently diffusing predators — "lions" — that are all initially at $L > x$. If the lamb meets any lion, the lamb is killed. Additionally, the origin is a *haven* for the lamb. If the lamb reaches the haven before meeting any of the lions, then the lamb survives. We are interested in the survival probability of the lamb as a function of the starting positions of the two species, as well as on the number of lions.

This model is a natural counterpoint to the well-studied capture process of a single diffusing lamb in the presence of N independent, diffusing lions on the infinite line [1, 2, 3, 4]. In the most interesting situation where the lions are all on one side of the lamb, the survival probability $\mathcal{S}_N(t)$ of the lamb asymptotically decays as a power-law in time, $\mathcal{S}_N(t) \sim t^{-\beta_N}$, with the survival exponent β_N exhibiting a non-trivial dependence on the number of lions N and also on the diffusivities of each animal. For simplicity, the case where the diffusivities of all animals are the same (and set to one) is normally considered. The initial positions of the lamb and the lions are irrelevant in this asymptotic behavior.

For this diffusive capture on the infinite line, the exponent β_N is known exactly only for $N = 1$ and $N = 2$: $\beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{3}{4}$ [1, 2, 3, 4, 5, 6]. The latter result shows that even though the two lions are independent, their effect on the capture process is not, since $t^{-\beta_2} > \left(t^{-\beta_1}\right)^2$. For the case $N = 3$, a mapping to an equivalent electrostatic problem leads to the accurate estimate $\beta_3 = 0.91342 \pm 0.00008$ [7]. For $N > 3$, the value of β_N has been estimated with moderate accuracy only for a few values of N [1]; however, it is known that $\beta_N > 1$ for $N > 3$, so that the average lifetime of the lamb is finite [8]. Because β_N grows more slowly than linearly with N, each additional lion has a progressively weaker influence on the capture process. As $N \to \infty$, both asymptotic and rigorous arguments give $\beta_N \to \frac{1}{4} \ln N$ [2, 3, 4]. Parenthetically, the capture process with lions sited on both sides of the lamb is much more efficient than in the one-sided system. For N lions with approximately equal numbers of them on either side of the lamb, the lamb survival probability asymptotically decays as $t^{-\gamma_N}$, with γ_N growing linearly with N for large N.

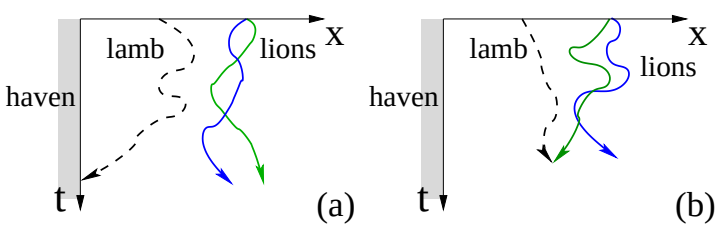

Figure 1. Space-time trajectories of a lamb and two lions with a haven at $x = 0$ when: (a) the lamb survives and (b) the lamb is eaten.

In this work, we incorporate the new feature of a *haven* at $x = 0$ and ask whether

the lamb can reach the haven before meeting any of the lions. If the haven is reached, we say that the lamb survives (Fig. 1). Our goal is to determine how the ultimate survival probability $S_N(x, L)$ depends on the initial positions of the lamb and all the lions, x and $L > x$, respectively, as well as on the number of lions. As we shall see, the survival probability depends on $z \equiv \frac{x}{L}$ rather than on x and L separately and thus we write the ultimate survival probability as $S_N(z)$. Our main result is that $S_N(z)$ has an unusual form whose leading behavior is $S_N(z) \sim N^{-z^2}$, but this behavior does not become apparent until N becomes of the order of 10^{500}.

We begin by solving the simplest and exactly-soluble case of one lion in Sect. 2. We also outline the formal solution to the problem for any number of lions. In Sect. 3 we treat the extreme case where the number of lions is infinite, so that the lion that is closest to the lamb moves ballistically. We then investigate arbitrary N in Sect. 4. When N is large, we can replace the N lions by a single "closest lion" that moves deterministically. We develop approximation schemes to estimate $S_N(z)$ in this large-N limit. We also present numerical results for the survival probability in Sect. 5. A straightforward simulation of the random-walk motion of the particles is prohibitively slow when N is large, and we present two alternative approaches that are considerably more efficient and allow us to probe the survival probability in the regime where N is extremely large — of the order of 10^{500}. Finally, in Sect. 6, we summarize and also discuss some natural and intriguing extensions of the model.

2. Exact Analysis

2.1. One Lion

As a preliminary, we can readily solve the case of one lamb at $x = x_1$ and one lion at $L = x_2 > x_1$ for the general situation where the diffusivities of the two species are distinct — D_1 for the lamb and D_2 for the lion. We compute the survival probability that the lamb reaches the haven at $x = 0$ before being eaten by the lion, $S(x_1, x_2)$, by mapping the coordinates of the lamb and the lion on the line to diffusion in a two-dimensional wedge, from which the survival probability follows easily.

It is convenient to transform from the coordinates (x_1, x_2) to $y_1 = x_1/\sqrt{D_1}$ and $y_2 = x_2/\sqrt{D_2}$. In the y_1-y_2 plane, the motions of the lamb and lion on the half line can be viewed as the isotropic diffusion of a fictitious composite particle with unit diffusivity [6, 9]. If y_1 reaches zero while the condition $y_1 < y_2$ is always satisfied, the lamb survives (Fig. 2). Conversely, if $y_1\sqrt{D_1} = y_2\sqrt{D_2}$ at some time (corresponding to $x_1 = x_2$) while y_1 always remains positive, then the lamb has been eaten by the lion before the haven is reached.

In the y_1-y_2 plane, the initial position of the composite particle is

$$\left(\frac{x_1}{\sqrt{D_1}}, \frac{x_2}{\sqrt{D_2}}\right) ,$$

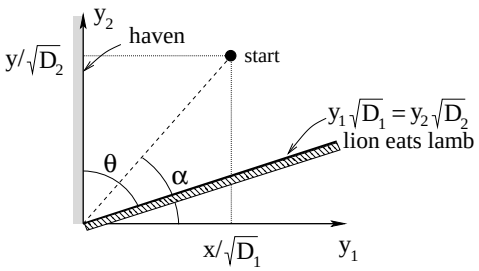

Figure 2. Mapping the diffusion of a lamb and a lion on the half line $x > 0$ to isotropic diffusion in a wedge of opening angle θ.

corresponding to the polar angle

$$\alpha = \tan^{-1}\left(\frac{x_2/\sqrt{D_2}}{x_1/\sqrt{D_1}}\right) \ . \tag{1}$$

The allowed region for the composite particle is a wedge of opening angle

$$\theta = \tan^{-1}\sqrt{D_2/D_1}\,. \tag{2}$$

We want the probability $S(x_1, x_2)$ that the composite particle first hits the line $y_1 = 0$ (corresponding to the lamb reaching the haven) without hitting the line $y_1\sqrt{D_1} = y_2\sqrt{D_2}$. This probability satisfies the Laplace equation [9]

$$D_1 \frac{\partial^2 S}{\partial {x_1}^2} + D_2 \frac{\partial^2 S}{\partial {x_2}^2} = 0$$

for $x_2 \geq x_1$, with boundary conditions $S(x_1{=}0, x_2) = 1$ and $S(x_1, x_2{=}x_1) = 0$. Clearly the solution is a function that linearly interpolates between 0 and 1 in the angular direction, so that the ultimate survival probability is [9, 10]

$$S(x_1, x_2) = \frac{\alpha - (\pi/2 - \theta)}{\theta} \ . \tag{3}$$

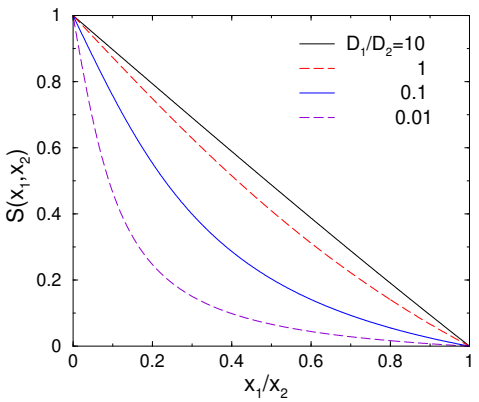

Figure 3. Ultimate survival probability $S(x_1, x_2)$ as a function of x_1/x_2 for various values of the lamb and lion diffusivities, D_1 and D_2, respectively.

As is obvious from Fig. 3, the closer that the lamb starts to the haven the more likely it is to survive. Moreover, as can be inferred from Fig. 2, the best strategy for the

lamb to survive for a given initial condition is to diffuse quickly. As the diffusivity of the lamb D_1 increases, the wedge angle in Fig. 2 approaches $\frac{\pi}{2}$ while the starting position of the fictitious particle in the plane moves close to the y_2 axis, i.e., closer to the haven. Finally notice that in the limit $D_2 \to 0$ (stationary lion), the survival probability decays linearly with x_1/x_2.

As a byproduct of the wedge mapping, we can immediately determine the probability that the lamb is still diffusing — that is, the lamb has not yet reached the haven and has not yet been eaten by the lion. This situation corresponds to the fictitious particle having not yet reached either of the sides of an infinite wedge defined by $x_1 = 0$ and $x_1 = x_2$. In the isotropic y_1-y_2 coordinates, this wedge has opening angle θ (Fig. 2), and the survival probability asymptotically decays as $t^{-\pi/2\theta}$. In particular, when $D_1 = D_2$, then $\theta = \frac{\pi}{4}$ (see Eq. (2)), and the survival probability decays as t^{-2}.

2.2. Formal Solution for General N

The reasoning given above can be readily generalized to map the problem of a diffusing lamb in the presence of N diffusing lions to a single diffusing fictitious particle in $N+1$ dimensions, with boundary conditions that reflect the lamb reaching the haven or being eaten by a lion. For simplicity, we set the diffusivities of the lamb and the lions to one. We first discuss the case of two lions; the generalization to any number of lions is immediate.

Suppose that the lamb is initially at $x_1 > 0$ and that the two lions are initially at $x_2 = x_3 > x_1$. The lamb survives if it reaches $x = 0$ without meeting either of the lions on the way to $x = 0$. We now map the diffusion of the three interacting particles on the positive half line to the isotropic diffusion of a composite particle at (x_1, x_2, x_3) in three dimensions, with constraints that correspond to the interactions in the lamb-lion system. By this mapping, the allowed region for the composite particle is defined by $x_1 > 0$, corresponding to the lamb not yet reaching the refuge, as well as by $x_1 < x_2$ and $x_1 < x_3$, corresponding to the lamb not yet eaten by either of the lions. This defines a wedge-shaped region that are delineated by three planar sides that is known as a Weyl chamber [11].

The survival of the lamb corresponds to the composite particle first hitting the plane $x_1 = 0$ of the Weyl chamber without hitting either of the planes $x_1 = x_2$ and $x_1 = x_3$. By the equivalence between first-passage and electrostatics [9], this survival probability of the lamb coincides with the electrostatic potential $\Phi(x_1, x_2, x_3)$ at the initial point of the composite particle, with the boundary conditions $\Phi = 1$ on the plane $x_1 = 0$, and $\Phi = 0$ on the planes $x_1 = x_2$ and $x_1 = x_3$. This same mapping works for any number of lions and constitutes the formal solution. Unfortunately, the analytical solution to this potential problem does not seem tractable for more than two lions.

3. Infinite Number of Lions

When the number of lions is infinite and each undergoes a nearest-neighbor random walk, the position of the lion that is closest to the lamb inexorably moves one lattice spacing to the left in each time step. In this case, we determine the ultimate survival probability $S_\infty(x, L)$ by writing the backward Kolmogorov equation [9] for $S_\infty(x, L)$ and then applying scaling. To write this equation, we consider the evolution of the system over a small time interval $[0, \Delta t]$ during which the lamb moves to $x + \eta(0)\Delta t$ and the boundary moves to $L - v\Delta t$, where v is the boundary velocity. That is, the position of the lamb $x(t)$ evolves by the Langevin equation $dx/dt = \eta(t)$, where $\eta(t)$ is Gaussian white noise with zero mean, $\langle \eta(t) \rangle = 0$, and correlation $\langle \eta(t)\eta(t') \rangle = 2\,D\,\delta(t - t')$. We now view the new positions of the lamb and the boundary after the time interval Δt as the initial conditions for the subsequent evolution. This $S(x, L) = \langle S(x + \eta(0)\Delta t, L - v\Delta t) \rangle$, where the average is over the initial noise $\eta(0)$. Expanding the right-hand side of this recursion to lowest non-vanishing order in each variable and using the properties of delta-correlated noise, we obtain the backward equation

$$D\frac{\partial^2 S}{\partial x^2} - v\frac{\partial S}{\partial L} = 0 \tag{4}$$

for $0 \le x \le L$, with the boundary conditions $S(0, L) = 1$ and $S(L, L) = 0$. To solve this equation we make the scaling ansatz $S(x, L) = f(y)$ (with $y = x/\sqrt{L}$) to give the ordinary differential equation for f:

$$f'' + \frac{v}{2D}\, y\, f' = 0\,, \tag{5}$$

subject to the boundary conditions $f(0) = 1$, $f(\infty) = 0$; here the prime denotes differentiation with respect to y.

Integrating and applying the boundary conditions gives

$$f(y) = \mathrm{erfc}\left(y\sqrt{\frac{v}{4D}}\right) = \mathrm{erfc}\left(z\sqrt{\frac{vL}{4D}}\right)\,, \tag{6}$$

with $z = x/L$. The primary feature of this result is that the lamb survival probability is non-zero only within a thin boundary layer where the starting position satisfies $x \ll \sqrt{4DL/v}$. Outside this layer the lamb is almost surely eaten by one of the lions.

4. Asymptotics for Large N

The capture process also simplifies when the number of lions N is finite but large, because the position of the closest lion becomes progressively more deterministic as N increases, even though each individual lion undergoes independent Brownian motion. Thus we only need to consider the ultimate survival of the lamb in the presence of a single effective predator [12] — the closest lion — that moves systematically towards the lamb (Fig. 4). We now exploit this physical picture to give a heuristic argument for the ultimate survival probability of the lamb.

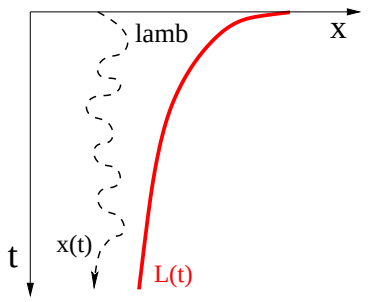

Figure 4. Schematic space-time representation of the diffusion of the lamb at x and the position $L(t)$ of the closest lion when the number of lions $N \gg 1$.

When all the lions start at $L > x$, the average number of lions at x is

$$n(x,t) = \frac{N}{\sqrt{4\pi Dt}}\, e^{-(x-L)^2/4Dt}\,.$$

We estimate the location of the closest lion, $L(t)$, by demanding that $n(L(t),t) = 1$. This criterion gives [4]

$$L(t) = L - \sqrt{At}\,, \tag{7}$$

where

$$A = 4D\ln M\left(1 - \frac{1}{2}\frac{\ln\ln M}{\ln M} + \ldots\right) \tag{8}$$

and $M = N/\sqrt{4\pi}$. Thus to lowest order, $A \approx 4D\ln N$. At a critical time $t^* = L^2/A$ the closest lion has reached the haven at $x = 0$ and the capture process is necessarily finished — either the lamb has been killed or it has reached the haven. Notice that although N must be large for the closest lion to move deterministically, N cannot be too large. As discussed in the previous section, if each lion undergoes a nearest-neighbor random walk, the closest lion moves deterministically to the left with speed $v = 1$ when N is sufficiently large. For Eq. (7) to be valid, we therefore require (to lowest order) that $\sqrt{4Dt\,\ln N} < vt$, or $N < \exp(v^2t/4D)$. Using $v = 1$ and $D = \frac{1}{2}$ for a nearest-neighbor random walk, the last lion moves deterministically as $\sqrt{4Dt\,\ln N}$ only when $t > 2\ln N$. For $t < 2\ln N$, the last lion moves with constant unit speed toward the lamb.

We now crudely estimate the ultimate survival probability of the lamb as the total probability flux that reaches $x = 0$ up to time t^* in the semi-infinite system *without* any additional constraints. This integrated flux represents an upper bound for the survival probability for large N because this estimate includes lamb trajectories that could intersect the trajectory of the last lion and then reach the haven. For a diffusing particle that starts at x, the flux to an absorbing boundary at the origin at time t is: [9]

$$j(0,t) = \frac{x}{\sqrt{4\pi Dt^3}}\, e^{-x^2/4Dt}\,.$$

Consequently, the probability S_N for the lamb to get trapped at the origin up to time t^* (corresponding to the lamb reaching the haven and surviving) satisfies the bound

$$S_N < \int_0^{t^*} \frac{x}{\sqrt{4\pi Dt^3}}\, e^{-x^2/4Dt}\, dt \;= \mathrm{erfc}(z\sqrt{\ln N}). \tag{9}$$

Here we have used the substitution $= x/\sqrt{4Dt}$ to transform to a Gaussian integral, as well as the lowest-order approximation $t^* = L^2/(4D\ln N)$ and $z = \frac{x}{L}$.

From the asymptotic form $\mathrm{erfc}(x) \sim e^{-x^2}/(\sqrt{\pi}\,x^2)$, we thus obtain an upper bound for the ultimate survival probability that has the unusual functional form

$$S_N < \frac{1}{\sqrt{\pi}}\; N^{-z^2}\,[\ln(N^{z^2})]^{-1/2}\,. \tag{10}$$

Consistent with basic intuition, S_N is a decreasing function of N and also decreases as $z \to 1$ with N fixed. It should be emphasized that Eq. (10) applies in the limit of $z\sqrt{\ln N} \gg 1$, which is extremely hard to achieve by direct simulation. For example, if the lamb starts halfway between the haven and the lions ($z = \frac{1}{2}$), then for $N = 10^4$, the argument of the complementary error function is $z\sqrt{\ln N} \approx 1.52$; for $N = 10^{16}$, $z\sqrt{\ln N} \approx 3.03$. Conversely to reach $z\sqrt{\ln N} = 10$ requires $N = e^{400} \approx 10^{174}$. Notice also that Eq. (9) matches the survival probability given by Eq. (6) for a ballistically-moving boundary when N reaches a critical value for which the completion time $t^* = L^2/A$ also equals L/v.

More rigorously, we should also incorporate the absorbing boundary condition at $L(t)$, corresponding to the lamb getting eaten by the closest lion. This problem of a fixed absorbing boundary at $x = 0$ and a moving absorbing boundary at $x = L - \sqrt{At}$ does not seem readily soluble, however. Instead, we investigate a related model in which the boundary motion mimics that of the closest lion, but is engineered to be soluble. As we shall show, the ultimate survival probability for this alternative problem has a qualitatively similar dependence on system parameters as Eq. (10). Consider the toy model in which the closest lion coordinate is $L_{\rm toy}(t) = \sqrt{L^2 - Bt}$ (compared to $L(t) = L - \sqrt{At}$, with $A = 4D\ln N$, for $N \gg 1$ diffusing lions). These two boundaries satisfy the inequality $L_{\rm toy}(t) > L(t)$ and both reach the origin at the same time when $B = A$. Thus the toy model remains an upper bound for the true survival probability.

It is again convenient to treat the evolution of the system in the two-dimensional space (x, L). Let $S(x, L)$ be the probability that the lamb successfully reaches the haven, where x and L denote the initial positions of the lamb and the boundary respectively. Following the same approach as in Sec. 3, we write the backward equation for $S(x, L)$. In a small time interval $[0, \Delta t]$ the lamb moves to $x + \eta(0)\Delta t$, where $\eta(t)$ is Gaussian white noise with zero mean, and the boundary moves to $L - (B/2L)\Delta t$, where $B/2L$ is the boundary speed. The survival probability now satisfies $S(x, L) = \langle S(x + \eta(0)\Delta t, L - (B/2L)\Delta t)\rangle$, and expanding the right-hand side to lowest order gives the backward equation

$$D\frac{d^2S}{dx^2} - \frac{B}{2L}\frac{dS}{dL} = 0 \tag{11}$$

for $0 \le x \le L$, with the boundary conditions $S(0, L) = 1$ and $S(L, L) = 0$. To solve (11) we make the scaling ansatz $S(x, L) = f(y)$, with $y = \gamma z$, where $\gamma = \sqrt{B/(2D)}$ and $z = \frac{x}{L}$, and find that the scaling function satisfies

$$f'' + y\,f' = 0\,, \tag{12}$$

for $0 \leq y \leq 1$, with the boundary conditions $f(0) = 1$ and $f(\gamma) = 0$; here the prime denotes differentiation with respect to y. Integrating once gives $f \propto e^{-y^2/2}$, and integrating again gives

$$f(z) = 1 - \mathrm{erf}(\gamma z/\sqrt{2})/\mathrm{erf}(\gamma/\sqrt{2})\,, \tag{13}$$

where the constants are determined by the boundary conditions. Substituting in $\gamma = \sqrt{B/(2D)}$ and $B = 4D\ln N$ gives the asymptotic behavior

$$f(z) \approx [N^{-z^2} - N^{-1}] \sim N^{-z^2}\ . \tag{14}$$

This upper bound has the same asymptotic behavior as (10) and suggests that the heuristic approach should be quite accurate.

5. Simulations

We now present simulation results for the lamb-lion-haven system. While a direct simulation is simple to code, it becomes prohibitively slow when N is large. We have therefore developed two complimentary approaches to determine the survival probability in the large-N limit.

5.1. Probability Propagation

Probability propagation is well-suited for probing the case of $N \gg 1$, where we replace the position of the closest lion by a deterministic absorbing boundary, $L(t)$, that moves according to Eq. (7). Here, the constant A can be chosen as the mean or most probable position of the closest lion or any other reasonable positional metric. We choose to set $A = 4D\ln N$, which is the leading behavior in Eq. (7). The omission of higher-order corrections, which slightly decrease A, lead to a more slowly-moving boundary and a correspondingly slightly larger survival probability. Thus probability propagation should provide a lower bound to the true survival probability.

Let $P(x,t)$ be the probability that the lamb is at x at time t. At each time step, the probability in the interior region $2 < x < \lfloor L(t)\rfloor - 1$ propagates according to $P(x,t+1) = \frac{1}{2}P(x-1,t) + \frac{1}{2}P(x+1,t)$; here $\lfloor L(t)\rfloor$ is the largest integer less than $L(t)$. At the edge sites $P(1,t+1) = \frac{1}{2}P(2,t)$ and $P(\lfloor L(t)\rfloor, t+1) = \frac{1}{2}P(\lfloor L(t)\rfloor - 1, t)$. Probability elements that reach either $x = 0$ or $\lfloor L(t)\rfloor + 1$ do not propagate further and remain in place. Probability propagation continues until $L(t)$ reaches $x = 0$. The total probability at $x = 0$ at this termination time gives the survival probability of the lamb.

We used probability propagation to obtain $S_N(z)$ for N up to 10^{500}. We used quadruple precision variables to ensure accuracy of the probability values throughout the propagation. The initial value of L was chosen to be the smallest such that finite-size effects were imperceptible — this ranged from $L = 1000$ for small N to $L = 30,000$ for the largest N values.

5.2. Event-Driven Simulation

A naive simulation is simply to move every lion and the lamb by ± 1 at each time step, an approach which is prohibitively slow for large N. However, there is no need to simulate every single random-walk step, particularly if the lamb is far from both the haven and nearest lion. This motivates using an event-driven simulation, in which we propagate all particles over a time that corresponds to a finite fraction of the time needed for a reaction to actually occur — either the lamb reaching the haven or getting eaten by the closest lion.

Let y be the minimum of the distances between the lamb and the nearest lion, and between the lamb and the haven. We could move every particle according to a binomial distribution of $y/2-1$ steps because there is no possibility that the lamb meets any of the lions or reaches the haven during this update. However, this approach is unnecessarily stringent because each particle moves a typical distance that is only of the order of $\sqrt{y}$. Thus we increment the number of steps by m, where

$$m = \begin{cases} y^2/2Y & \text{if } y \geq Y, \\ y/2 & \text{if } y < Y, \end{cases} \tag{15}$$

and move every particle according to a binomial distribution of m steps. Note that these update rules match at the crossover separation $y = Y$. After each such update, we check if the lamb has reached or crossed over the position of the haven or that of any lion, in which case the simulation is finished.

For $y < Y$, $m = \frac{y}{2}$ and the lamb cannot reach either the haven or any lion during the update; this part of the simulation is exact. For $y \geq Y$, there is a non-zero probability that the lamb trajectory could cross the haven or a lion trajectory and then cross back during the update. However, by choosing Y appropriately, the probability of error due to such crossing trajectories can be made vanishingly small. We found that $Y = 15$ gave an excellent compromise between accuracy and efficiency. We also checked that simulations results with the update rule (15) are essentially identical with exact results that arise by choosing $Y = \infty$ in the update rule (15).

5.3. Results

In Fig. 5(a) we show the dependence of the ultimate survival probability versus scaled initial position $z = \frac{x}{L}$ for N up to $256,000$, with 10^5 realizations for each data point, from event-driven simulations. For $N \gg 1$, these survival probabilities gradually converge, as N increases, to a limiting curve that corresponds to the system where the last lion moves ballistically. The lions are all initially at $L = 100$ and we verified that the survival probability depends only on the ratio x/L, without any explicit finite-L dependence. This independence on L emerges when $L \geq 100$ and thus we focus on the smallest system ($L = 100$) where finite-size effects are negligible.

We also examined the dependence of S_N on N for fixed z to test the asymptotic power-law behavior $S \sim N^{-z^2}$ of Eq. (10). Our analytical prediction matches the

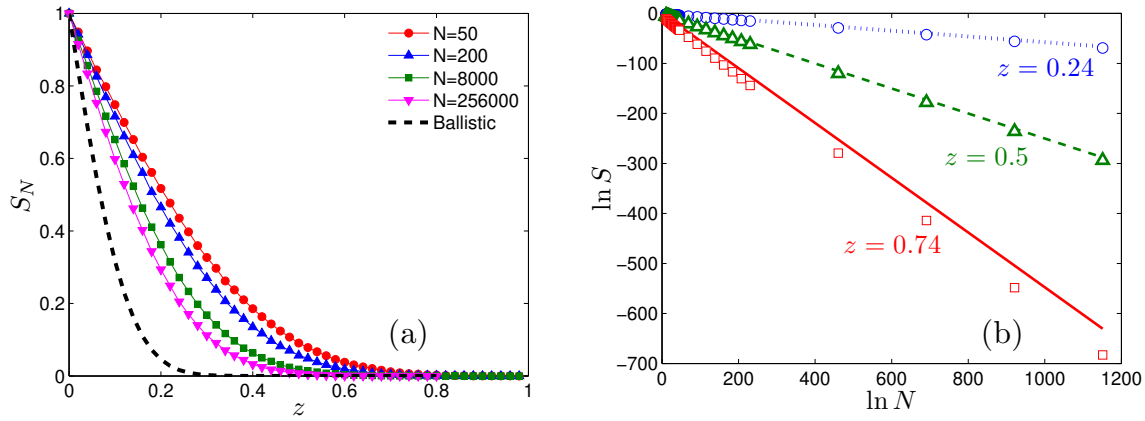

Figure 5. (a) Dependence of the survival probability on z for representative values of N from event driven simulations. Dashed curve is analytic solution for ballistic lion motion from Eq. (6) where $v = 1$, $D = 1/2$, and $L = 100$. (b) Survival probability versus number of lions N for three representative z values. Curves give the analytic prediction $S = N^{-z^2}$ and the symbols represent probability propagation results.

simulation quite well for $z \lesssim 0.5$ (Fig. 5)(b). However, a small but slowly growing discrepancy arises as z is increased beyond 0.5. The source of this discrepancy is that the heuristic derivation of Sec. 4 ignores the existence the absorbing boundary caused by the last lion. When z approaches 1, the lamb starts sufficiently close to the last lion that the assumption of ignoring the boundary caused by the last lion is no longer valid.

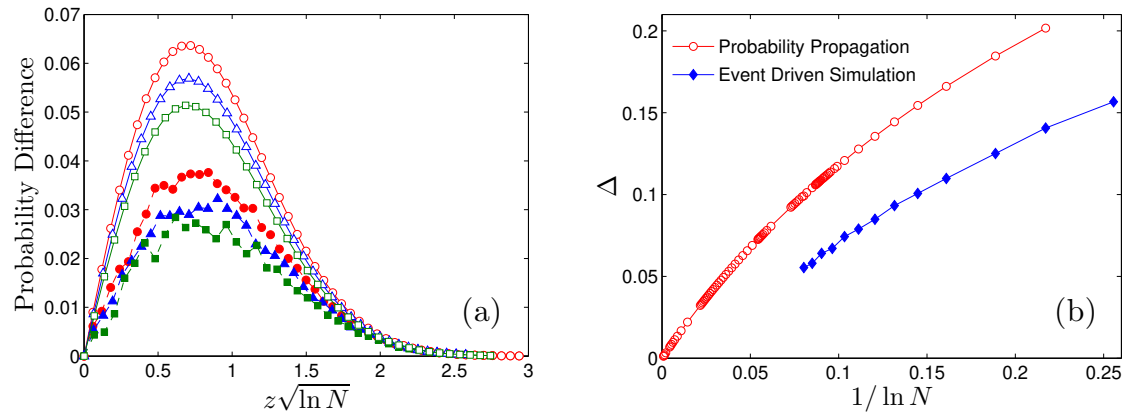

Figure 6. (a) Difference between simulated survival probability and $S_N(z) = \mathrm{erfc}(z\sqrt{\ln N})$ as a function of z for various N. Open symbols correspond to probability propagation while filled symbols correspond to event-driven simulations. Circles correspond to $N = 8,000$, triangles to $N = 32,000$, and squares to $N = 128,000$. (b) Relative area difference Δ versus $1/\ln N$ for probability propagation (○) and event-driven (⧫) simulations.

Finally, we compare our two simulation approaches with each other and with our our heuristic prediction $S_N \lesssim \mathrm{erfc}(z\sqrt{\ln N})$ from Eq. (9). By construction, the event-

driven simulation is more accurate because it explicitly follows the stochastic motion of the lamb and the lions. Our heuristic prediction (9) provides an upper bound for large N, but this regime is not feasible to simulate with the event-driven algorithm. Conversely, the probability propagation simulation can be implemented for arbitrarily large N but suffers from systematic error because it assumes the closest lion position to be deterministic.

Figure 6(a) illustrates the convergence of the simulation results to Eq. (9) where the difference between the simulated value of $S_N(z)$ and $\mathrm{erfc}(z\sqrt{\ln N})$ is plotted as a function of z for representative N values. We quantify this difference by $\Delta \equiv (A_\mathrm{a}-A_\mathrm{s})/A_\mathrm{a}$, where $A_\mathrm{a} = \int S_N(z)dz$, with $S_N(z) = \mathrm{erfc}(z\sqrt{\ln N})$, is the area beneath the analytic survival curve and similarly for the area beneath the simulated curve. Figure 6(b) shows that $\Delta \to 0$ as $N \to \infty$ for the probability propagation algorithm. A similar, but not identical convergences arises in the event-driven simulation, but the method cannot reach the large-N regime. These results provide strong evidence that the survival probability is indeed given by $S \to \mathrm{erfc}(z\sqrt{\ln N})$ as $N \to \infty$.

6. Outlook

The presence of a haven adds an intriguing element to the classic capture process of a single lamb in the presence of N diffusing lions. Now the basic question is whether the lamb can reach safety at the haven before it is eaten by one of the lions. We investigated the dependence of the ultimate survival probability of the lamb, $S_N(x, L)$, on the number of lions N and also on the initial positions of the lamb (x) and the lions (all at L, for simplicity). By a rough heuristic argument, we found that S_N has the asymptotic behavior $S_N \lesssim \mathrm{erfc}(z\sqrt{\ln N})$, and this function has the unusual leading behavior $S_N \sim N^{-z^2}$, where $z = x/L$. It is remarkable that a simplistic approach gives such an unusual and rich result. However, the approach to this asymptotic regime is extremely slow and it is necessary to simulate a system that corresponds to N of the order of 10^{500} lions before the asymptotic behavior becomes apparent.

It is natural to ask about the properties of the ultimate survival probability in higher dimensions. For diffusive capture in an unbounded system, the case of one dimension is the most interesting. However, the presence of a haven now makes the higher-dimensional problem nontrivial. For example, in two dimensions, a natural setting would be a diffusing prey, N diffusing predators, and a circular haven of radius R centered at the origin. Because of the recurrence of diffusion in two dimensions, the prey will eventually reach the haven if there are no predators, but the mean time to reach the haven is infinite. What happens when predators exist? How does the survival probability depend on the number of predators and on the initial positions of the prey and predators? How long does it take for the capture process to end? Another interesting two-dimensional geometry is a semi-infinite planar haven. Finally, in three dimensions, the transience of diffusion could lead to very different properties for the survival probability than in two dimensions.

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