Dynamics of Majority Rule in Two-State Interacting Spin Systems

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We introduce a two-state opinion dynamics model where agents evolve by majority rule. In each update, a group of agents is specified whose members then all adopt the local majority state. In the mean-field limit, where a group consists of randomly selected agents, consensus is reached in a time that scales $\ln N$, where N is the number of agents. On finite-dimensional lattices, where a group is a contiguous cluster, the consensus time fluctuates strongly between realizations and grows as a dimension-dependent power of N. The upper critical dimension appears to be larger than 4. The final opinion always equals that of the initial majority except in one dimension.

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In this Letter, we introduce an exceedingly simple opinion dynamics model—majority rule (MR)—that exhibits rich dynamical behavior. The model consists of N agents, each of which can assume the opinion (equivalently spin) states $+1$ or -1, that evolve as follows: (i) Pick a group of G spins from the system (with G an odd number). This group could be any G spins, in the mean-field limit, or is contiguous, for finite-dimensional systems. (ii) The spins in the group all adopt the state of the local majority. These two steps are repeated until the system reaches a final state of consensus. While the MR model ignores psychosociological aspects of real opinion formation [1], this simple decision-making process leads to rich collective behavior. We seek to understand two basic issues: (i) What is the time needed to reach consensus as a function of N and of the initial densities of plus and minus spins? (ii) What is the probability of reaching a given final state as a function of the initial spin densities?

To set the stage for our results, we recall the corresponding behavior in the classical two-state voter model (VM) [2], where a spin is selected at random and it adopts the opinion of a randomly chosen neighbor. For a system of N spins in d dimensions, the time to reach consensus scales as N for $d > 2$, as $N \ln N$ for $d = 2$ (the critical dimension of the VM), and as N^2 in $d = 1$ [2,3]. Because the average magnetization is conserved, the probability that the system eventually ends with all plus spins equals the initial density of plus spins in all spatial dimensions.

The MR model has the same degree of simplicity as the VM but exhibits very different behavior. Part of the reason for this difference is that MR does not conserve the average magnetization. Another distinguishing trait of MR is the many-body nature of the interaction. This feature also arises, for example, in the Sznajd model [4], where two neighboring agents that agree can influence a larger local neighborhood, or in Galam's rumor formation model [5], where an entire population is partitioned into disjoint groups that each reach their own consensus. The updating of an extended group of spins was also considered by Newman and Stein [6] in the Ising model with zero-temperature Glauber kinetics [7].

We now outline basic features of the MR model. In the mean-field limit we give an exact solution for the approach to consensus, while for finite dimensions we give numerical and qualitative results.

Mean-field limit.—Consider the simplest case where arbitrary groups of size $G = 3$ are selected and updated at each step. To determine the ultimate fate of the system, let E_n denote the "exit probability" that the system ends with all spins plus when starting with n plus spins. Now

$$\binom{3}{j}\binom{N-3}{n-j}\Big/\binom{N}{n}$$

is the probability that a group of size 3 has j plus and $3-j$ minus spins in an N-spin system that contains n plus spins. The group becomes all plus for $j = 2$, it becomes all minus for $j = 1$, while for $j = 0$ or 3 there is no evolution. Thus E_n obeys the master equation [8]

$$\binom{N}{n}E_n = 3\binom{N-3}{n-2}E_{n+1} + 3\binom{N-3}{n-1}E_{n-1} + \left[\binom{N-3}{n-3}+\binom{N-3}{n}\right]E_n, \quad (1)$$

which simplifies to

$$(n-1)(E_{n+1} - E_n) = (N-n-1)(E_n - E_{n-1}). \quad (2)$$

Writing $D_n = E_{n+1} - E_n$, Eq. (2) becomes a first-order recursion whose solution is

$$D_n = \frac{B}{\Gamma(n)\Gamma(N-n-1)}. \quad (3)$$

To compute the constant B we use the fact that $\sum_{1\le n\le N-2} D_n = E_{N-1} - E_1 = 1$, due to the boundary conditions $E_1 = 0$ and $E_{N-1} = 1$. Thus we find

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$$E_n = \sum_{j=1}^{n-1} D_j = \frac{1}{2^{N-3}} \sum_{j=1}^{n-1} \frac{\Gamma(N-2)}{\Gamma(j)\Gamma(N-j-1)}. \quad (4)$$

The probability to end with all spins minus is simply E_{N-n}. Since consensus is the ultimate fate of the system $E_n + E_{N-n} = 1$.

While the minority may win in a finite system, the probability for this event quickly vanishes as $N \to \infty$. In the continuum limit $n, N \to \infty$ with $p = n/N < 1/2$ the exit probability is exponentially small: $E_n \propto X^N$, with $X = 1/[2p^p(1-p)^{1-p}]$. Only near $n = N/2$ does the exit probability rapidly increase. Employing Stirling's approximation we may recast (4) into

$$E_n \to \mathcal{E}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} dz\, e^{-z^2/2}, \quad (5)$$

where $y = (2n - N)/\sqrt{N}$.

We now study the mean time T_n to reach consensus (either all plus or all minus) when the initial state consists of n plus and $N - n$ minus spins. (The time to reach a specified final state can also be analyzed within this framework.) Similar to the reasoning for the exit probability, the equation for T_n is [8]

$$\begin{aligned} \binom{N}{n} T_n = {} & 3\binom{N-3}{n-2}(T_{n+1} + \delta T) \\ & + 3\binom{N-3}{n-1}(T_{n-1} + \delta T) \\ & + \left[\binom{N-3}{n-3} + \binom{N-3}{n}\right](T_n + \delta T), \end{aligned} \quad (6)$$

subject to the boundary conditions $T_0 = T_N = 0$. The natural choice for the time interval between elementary steps is $\delta T = 3/N$, so that each spin is updated once per unit time, on average.

The master equation for $U_n = T_{n+1} - T_n$ simplifies to

$$(n-1)U_n = (N-n-1)U_{n-1} - \frac{(N-1)(N-2)}{n(N-n)}, \quad (7)$$

with the boundary conditions $U_0 = 1$ and $U_{N-1} = -1$. Apart from the inhomogeneous term, Eq. (7) is identical to (2). Thus, we seek a solution in a form similar to (3):

$$U_n = \frac{(N-1)(N-2)}{\Gamma(n)\Gamma(N-n-1)} V_n. \quad (8)$$

This transforms (7) into the difference equation

$$V_{n-1} = V_n + \frac{\Gamma(n-1)\Gamma(N-n-1)}{n(N-n)}. \quad (9)$$

The symmetry of Eq. (7) and ansatz (8) under the transform $n \to N-1-n$, and the antisymmetry of the boundary conditions $U_0 = 1$ and $U_{N-1} = -1$ imply that $U_n = -U_{N-1-n}$ and $V_n = -V_{N-1-n}$.

For concreteness, we take N to be odd and define $k = (N-1)/2$. Then the above boundary conditions on U_n and V_n imply that $V_k = 0$. Starting from this value and using Eq. (9) we recursively obtain, for all $j \le k$,

$$V_j = \sum_{i=1}^{k-j} \frac{\Gamma(k-i)\Gamma(k+i-1)}{(k-i+1)(k+i)}. \quad (10)$$

For $n \ge 1$, the average time $T_n = T_1 + \sum_{j=1}^{n-1} U_j$ becomes

$$T_n = 1 + 2k(2k-1) \sum_{j=1}^{n-1} \frac{V_j}{\Gamma(j)\Gamma(2k-j)} \quad (11)$$

with V_j given by (10).

For the maximal time $T_{\max} = T_k$ [with $k = (N-1)/2$], we obtain

$$T_{\max} = 1 + 2k(2k-1) \sum_{m=2}^{k} S_{k,m} \quad (12)$$

with

$$S_{k,m} = \sum_{j=1}^{m-1} \frac{\Gamma(k+j-m)\Gamma(k+m-j-1)}{\Gamma(j)\Gamma(2k-j)(k+j-m+1)(k+m-j)}.$$

A detailed asymptotic analysis [9] shows that

$$T_{\max} \to 2 \ln N \qquad N \to \infty. \quad (13)$$

For biased initial conditions, it is convenient to consider the limits $n, N \to \infty$, but with $p = n/N$ kept fixed and distinct from $1/2$. Now the leading behavior of T_n is

$$\begin{aligned} T_n & \to 2k(2k-1) \sum_{j=1}^{n-1} \frac{1}{(2k-j-1)(j+1)(2k-j)} \\ & \to \ln N \quad \text{as } N \to \infty. \end{aligned} \quad (14)$$

Thus for biased initial conditions, the consensus time also scales as $\ln N$, but with amplitude equal to 1. A detailed analysis [9] indicates that the amplitude sharply changes from 2 to 1 within the layer $N^{-1/2} \ll p - 1/2 \ll 1$ as the system is moved away from the symmetric initial condition $p = 1/2$ (Fig. 1.)

One dimension.—To implement the dynamics in one dimension, we define a group to be G consecutive spins. If there is no consensus in the selected group, the opinion of the minority-opinion agents is changed so that local consensus obtains. We parametrize the opinions by the spin states $S = \pm 1$. For the simplest case of group size $G = 3$, let S, S', S'' be the spins in the group that is being updated. Focusing on spin S, this spin flips with rate

$$W(S \to -S) = (1 + S'S'')\left[1 - S\frac{(S' + S'')}{2}\right]. \quad (15)$$

The factor $1 + S'S''$ ensures that spin S can flip only when $S' = S''$, while the quantity within the square brackets ensures consensus after spin S flips. Since $S^2 = 1$, this rate can be simplified to $W = 1 + S'S'' - S(S' + S'')$.

In one dimension, each spin S_j belongs to three groups: (S_{j-2}, S_{j-1}, S_j), (S_{j-1}, S_j, S_{j+1}), and (S_j, S_{j+1}, S_{j+2}). Therefore the total spin-flip rate is given by

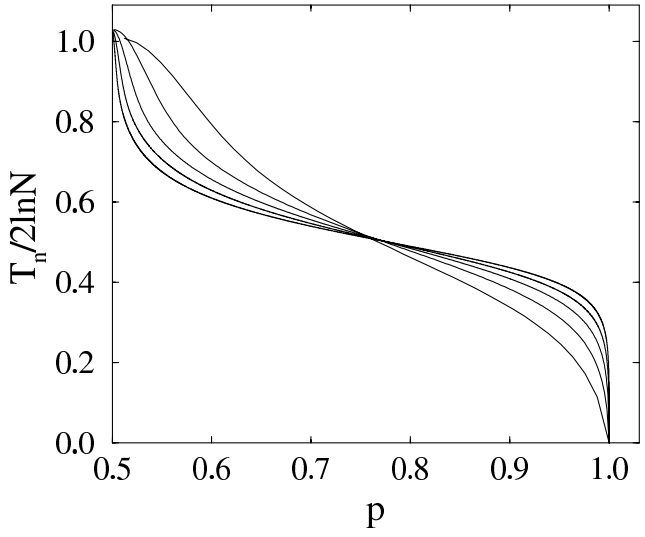

FIG. 1. Consensus time T_n versus $p = n/N$ for $N = 81$, 401, 2001, 10 001, and 50 001 (gradual steepening). The curves are symmetric about $p = 1/2$.

$$W(S_j \to -S_j) = 3 + S_{j-2}S_{j-1} + S_{j-1}S_{j+1} + S_{j+1}S_{j+2} - S_j(S_{j-2} + 2S_{j-1} + 2S_{j+1} + S_{j+2}),$$

which depends on the state of the two nearest neighbors and the two next-nearest neighbors of S_j. The equation of motion for the mean spin is [7]

$$\frac{d}{dt}\langle S_j\rangle = -2\langle S_j W(S_j \to -S_j)\rangle. \quad (16)$$

With the flip rate given above, first-order terms $\langle S_j\rangle$ are coupled to third-order terms $\langle S_{j-1}S_jS_{j+1}\rangle$. For a spatially homogeneous system this gives

$$\frac{dm_1}{dt} = 6(m_1 - m_3), \quad (17)$$

where $m_1 = \langle S_j\rangle$ and $m_3 = \langle S_{j-1}S_jS_{j+1}\rangle$. This coupling of different-order correlators makes analytical progress challenging. In contrast, the different-order correlators decouple in the VM and $dm_1/dt = 0$ [2,7]. In the mean-field limit of MR, $m_3 = m_1^3$ and the resulting equation reproduces the consensus time growing as $\ln N$ and the fact that the initial majority determines the final state.

Despite this distinction with the VM in one dimension, the dynamics of the MR can be usefully reformulated in terms of the domain walls between neighboring opposite spins. As long as walls are separated by at least two sites, each undergoes a symmetric random walk, exactly as in the VM. However, when two domain walls occupy neighboring bonds, then the local spin configuration is $\ldots - - - + - - - \ldots$ and these two walls are doomed to annihilate. Because of this close correspondence with the VM, we expect, and verified numerically, that the density of domain walls $N(t)$ decays as $t^{-1/2}$.

More quantitatively, we study the densities of plus and minus domains of length n, P_n, and Q_n, respectively [10]. The number densities of plus and minus domains are identical, $N(t) = \sum P_n = \sum Q_n$, while the fractions of positive and negative spins are given by $L_+(t) = \sum nP_n$ and $L_-(t) = \sum nQ_n$, respectively (with $L_+ + L_- = 1$). The equations of motion for these moments are

$$\frac{dN}{dt} = -3(P_1 + Q_1), \qquad \frac{dL_+}{dt} = 3(Q_1 - P_1). \quad (18)$$

Substituting $N \sim t^{-1/2}$ into (18) gives $P_1 \sim Q_1 \sim t^{-3/2}$. Therefore $L_+(t) - L_+(\infty) \sim t^{-1/2}$. Thus even though the fractions of plus and minus spins vary with time, they ultimately saturate to finite values. Correspondingly, the exit probability has a nontrivial dependence on the initial magnetization in the thermodynamic limit (Fig. 2). This should be compared with the VM result $E(p) = p$ that follows from the conserved VM magnetization.

Higher dimensions.—There are many natural ways to implement MR in dimension $d > 1$. One possibility is to update groups of three spins at the corners of elementary plaquettes on the two-dimensional triangular lattice. This gives to a spin-flip rate of a similar form to that in Eq. (15) and again leads to the equation of motion for the mean magnetization being coupled to a third-order correlator. On hypercubic lattices, a natural definition for a group is a spin plus its $2d$ nearest neighbors in the d coordinate directions (von Neumann neighborhood).

On these finite-dimensional lattices, MR differs from both the VM and the Ising model with zero-temperature Glauber kinetics (IG) [7]. On the triangular lattice, for example, an elementary plaquette of three plus spins in a sea of minus spins cannot grow in the IG model, but it can grow in the MR model. Additionally, straight interfaces are not stable in the MR model, but they are stable in the IG model. On the other hand, there is a (small) surface

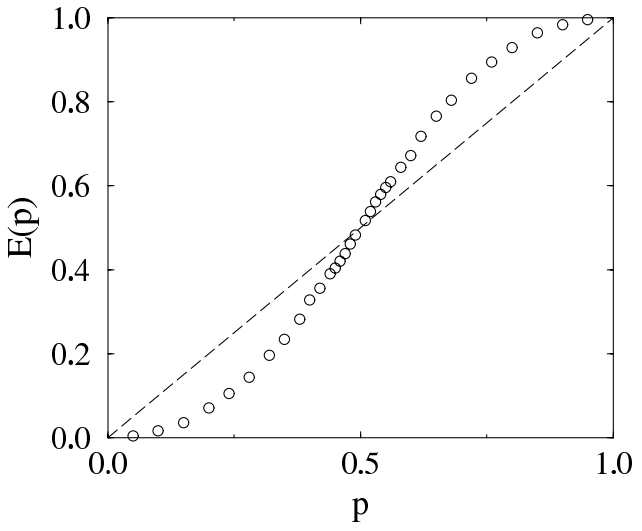

FIG. 2. Probability that a one-dimensional system of length 10^4 ends with all spins plus as a function of the initial density p of plus spins. Each data point is based on 2000 realizations. The dashed line is the corresponding VM result.

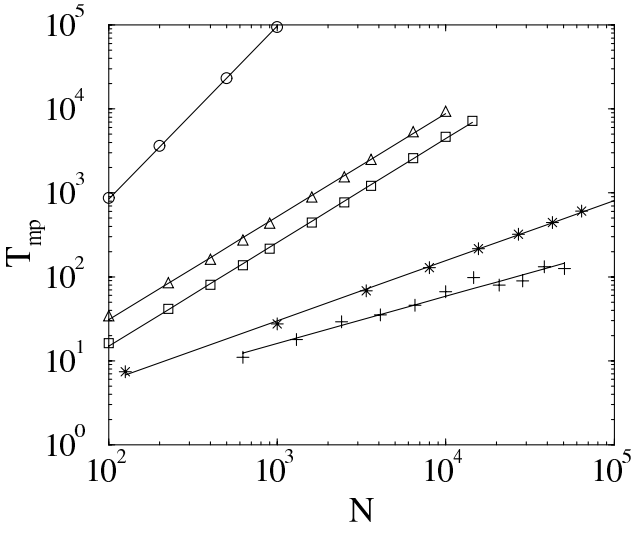

FIG. 3. Most probable consensus time T_{mp} versus N for zero initial magnetization in one dimension (○), on the triangular lattice (△), and hypercubic d-dimensional lattice with a $(2d+1)$-site neighborhood for $d=2$ (□), $d=3$ ($*$), and $d=4$ ($+$). The lines represent the best power-law fits with respective slopes 2.06, 1.23, 1.24, 0.72, and 0.56.

tension in the MR dynamics that smooths convex corners, just as in the IG model. Thus both the MR and the IG dynamics lead to relatively compact clusters, while the VM naturally gives ramified clusters.

We simulated the time T until consensus is reached in one through four dimensions when the system starts from zero magnetization. In one dimension, there is only a single characteristic time scale, the mean time $\overline{T}$, that grows as N^2. The distribution $\rho_N(\tau)$ of the scaled time $\tau = T/\overline{T}$ approaches a well-defined limiting distribution $\rho_\infty(\tau)$ for $N \to \infty$. At late stages of the evolution such that only two domain walls remain, we estimate analytically and verify numerically that this distribution has an exponential long-time tail $\rho_\infty(\tau) \sim e^{-\tau}$ as $\tau \to \infty$.

In two dimensions ($d=2$), we find evidence of at least two characteristic times, corresponding to two different routes to consensus. In most realizations, one opinion quickly becomes dominant and eventually wins. In the remaining realizations, however, the system reaches a configuration in which opinions segregate into two (or very rarely four or more) nearly straight stripes. For the IG model, the interfaces between such stripes would eventually become perfectly flat and this would be the final state [11]. However, such stripes are ultimately unstable in the MR model so that consensus is eventually reached, albeit very slowly.

In three and higher dimensions, there are apparently numerous characteristic times that all scale differently with N. A similar behavior, associated with a vast number of metastable states, was previously observed in the three-dimensional IG model [11]. The broad distribution of consensus times suggests that the most probable time T_{mp} is an appropriate characteristic scale. This quantity indeed has much more convincing power-law behavior than the mean time $\overline{T}$ for $d=2$ and 3. We obtain $T_{mp} \sim N^z$ with $z = 1.23$, 0.72, and 0.56 in $d=2$, 3, and 4, respectively, with large fluctuations occurring in the most probable consensus time in four dimensions (Fig. 3).

In summary, majority rule leads to rich dynamics. In the mean-field limit, there is ultimate consensus in the state of the initial majority. The mean consensus time $\overline{T}$ scales as $\ln N$, where N is the number of agents in the system, and the amplitude of $\overline{T}$ changes rapidly between 1 and 2 as a function of the initial magnetization. One dimension is the only case where the minority can ultimately win. Here, single-opinion domains grow as $t^{1/2}$ so that $\overline{T} \propto N^2$. Because the magnetization is not conserved, the probability to reach a given final state has a nontrivial initial state dependence. For $d > 1$, the initial majority determines the final state. The consensus time grows as a power law in N with a dimension-dependent exponent. Large sample-to-sample fluctuations in T arise whose magnitude increases with dimension. Mean-field behavior is not reproduced in four dimensions, so that the upper critical dimension is at least greater than four.

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