Choice-Driven Phase Transition in Complex Networks

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Abstract. We investigate *choice-driven* network growth. In this model, nodes are added one by one according to the following procedure: for each addition event a set of target nodes is selected, each according to linear preferential attachment, and a new node attaches to the target with the highest degree. Depending on precise details of the attachment rule, the resulting networks has three possible outcomes: (i) a non-universal power-law degree distribution; (ii) a single macroscopic hub (a node whose degree is of the order of N, the number of network nodes), while the remainder of the nodes comprises a non-universal power-law degree distribution; (iii) a degree distribution that decays as $(k \ln k)^{-2}$ at the transition between cases (i) and (ii). These properties are robust when attachment occurs to the highest-degree node from at least two targets. When attachment is made to a target whose degree is *not* the highest, the degree distribution has the double-exponential form $\exp(-\text{const.} \times e^k)$, from which the largest degree grows only as $\ln \ln N$.

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1. Introduction

Choice plays an essential role in queuing and optimization theory [1–5], in the structure of random recursive trees [6] and evolving random graphs [7–9], in explosive percolation [10–18], and in the control of avalanches in self-organized criticality [19]. We all familiar with choice in grocery checkout, customs, and security lines, where we would like to be in the line with the shortest waiting time. Picking one of N lines at random results in a maximal waiting time of the order of $\ln N$. If instead one initially selects two lines at random and then chooses the line with the smaller number of customers, the maximal waiting time drops to $O(\ln\ln N)$. Further increasing the number of initially selected lines improves the maximal waiting time only by a constant factor, thereby illustrating the "power of two choices" [1–5].

Growing networks with choice were investigated in [6], where the choice was made to the node closest to the root. Choice has also been implemented in evolving random graphs (networks with fixed number of nodes and growing number of links), where it has been shown that appropriate choice may delay [7] or speed up [8,9] the appearance of the giant component. One particular example of choice-driven link addition in evolving random graphs has recently attracted considerable attention [10–18], as it leads a percolation transition which is explosive in character.

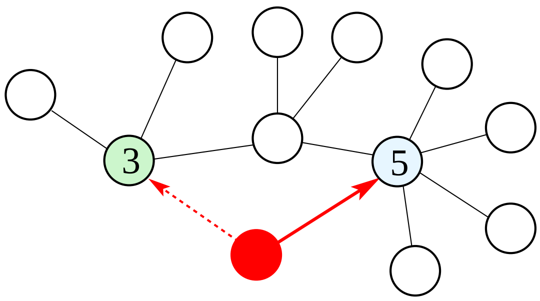
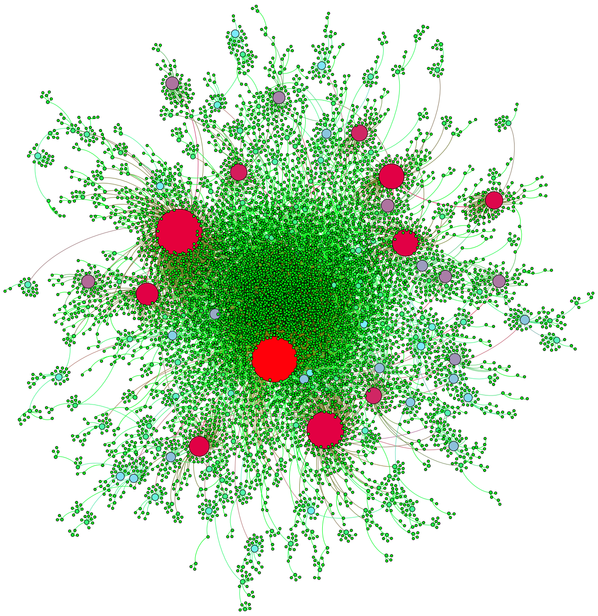


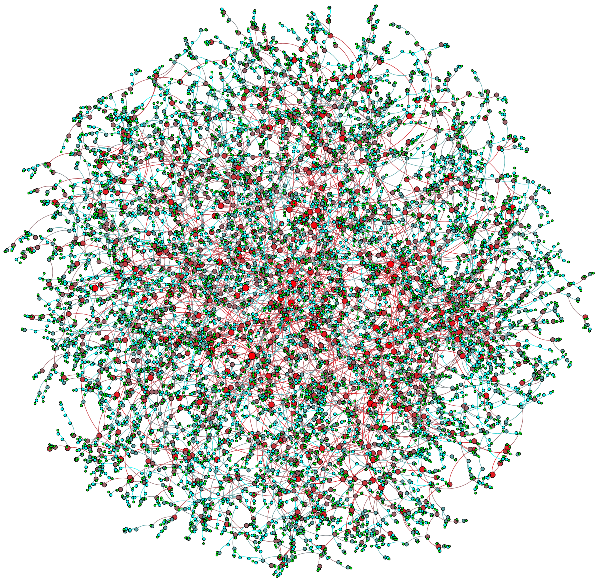
Figure 1. Illustration of network growth by greedy choice. Two nodes (shaded) in the network are selected according to preferential attachment. A new node (solid) attaches to the target with the larger degree, in this case, degree 5.

In this work, we determine how a degree-based choice affects the growth of complex networks [20]. Instead of a new node attaching to a target node according to a specified rate, we select a fixed number of targets according to this rate and the new node attaches to the target with the largest degree—"greedy" choice (Fig. 1). When the targets are selected randomly and independent of their degrees [6], it was found that the degree distribution decays exponentially with degree, but at a slower rate than in the case with no choice. When the targets are selected according to the preferential attachment mechanism, the effect of the choice is much more dramatic as we show below.

As an example, consider the situation where *two* targets are provisionally selected, each with the probability proportional to $A_k = k + \lambda$ for a target of degree k. Then our results can be summarized as follows. For $\lambda > 0$, the network has a degree distribution with an algebraic tail that possesses a non-universal exponent (i.e., dependent on λ). This exponent is smaller than in the case of no choice; thus choice broadens the degree distribution. For $\lambda = 0$ (strictly linear preferential attachment), the degree distribution has a power-law tail with the smallest exponent that is consistent with the network remaining sparse. More precisely, the fraction of nodes of degree k asymptotically decays as $(k \ln k)^{-2}$, with the logarithmic factor ensuring that the network is sparse. For $-1 < \lambda < 0$, a macrohub (a node whose degree grows linearly with the number of nodes in the network) emerges; the remainder of the degree distribution is still characterized by a non-universal algebraic tail. These properties are qualitatively robust for greedy choice with at least two alternatives, although the critical value of λ depends on the number of alternatives; in the case when p target nodes are provisionally selected, then $\lambda_c = p - 2$.



(a)



(b)

Figure 2. Example networks of 10^4 nodes that are grown by strictly linear preferential attachment for (a) greedy and (b) meek choice from two alternatives. The maximal degree is 3399 in (a) and 8 in (b). Red are high-degree nodes.

In contrast, when attachment occurs to a target whose degree is less than the largest among the target set—which we term "meek choice"—a double-exponential degree distribution arises, where $n_k \sim \exp(-\text{const.} \times e^k)$. Somewhat surprisingly, this behavior occurs even if attachment occurs to the second-largest out of a large number of targets. Thus greedy choice is the unique case and all other less greedy attachment choices lead to a double-exponential degree distribution. Two examples of small networks grown by greedy and meek choice from two alternatives are shown in Fig. 2.

2. Greedy Choice

2.1. Two Alternatives

We start by studying the degree distribution in networks where growth is driven by greedy choice between two alternatives. Let $N_k(N)$ be the number of nodes of degree k when the network contains N total nodes. Although the $N_k(N)$ are random variables, fluctuations in these quantities are small when the network is large. We thus focus on the averages $\langle N_k(N)\rangle$ in the limit of large N, where we may replace $N_k(N{+}1) - N_k(N)$ by dN_k/dN. We also drop the angle brackets henceforth.

The evolution of the degree distribution in this greedy choice model is governed by the master equations

$$\begin{aligned}\frac{dN_k}{dN} &= \frac{A_{k-1}N_{k-1}}{A}\sum_{j<k-1}\frac{A_jN_j}{A/2} - \frac{A_kN_k}{A}\sum_{j<k}\frac{A_jN_j}{A/2}\\ &+ \left[\frac{A_{k-1}N_{k-1}}{A}\right]^2 - \left[\frac{A_kN_k}{A}\right]^2 + \delta_{k,1}\,.\end{aligned} \tag{1}$$

Here A_k is the rate at which a node of degree k is selected as a potential target and $A = \sum_j A_jN_j$ the total rate. The first term on the right-hand side of Eq. (1) accounts for the increase in N_k due to the new node attaching to a node of degree $k{-}1$. Such an event occurs if the two initial targets have degrees k–1 and $j < k$–1. The complementary gain term has a similar origin, while the quadratic terms on the second line account for events where the two targets have the same degree. The master equations satisfy the sum rules $\sum_{k\geq 1} N_k = N$ and $\sum_{k\geq 1} kN_k = 2(N{-}1)$.

In the following, we focus on the class of shifted linear attachment rates given by $A_k = k + \lambda$. In this case the total rate becomes $A = \sum_j A_jN_j = (2+\lambda)N - 2$. We are interested in the $N \to \infty$ limit, so we simply write $A = \sum_j A_jN_j = (2+\lambda)N$. The fraction of nodes of fixed degree becomes size independent when $N \to \infty$, so that $N_k(N) \to Nn_k$ (see, e.g., [21, 22]). Using this fact, we recast (1) into

$$n_k = \frac{\psi_{k-1} - \psi_k}{(2+\lambda)^2/2}\sum_{j<k}\psi_j - \frac{\psi_{k-1}^2 + \psi_k^2}{(2+\lambda)^2} + \delta_{k,1}\,, \tag{2}$$

where $\psi_k \equiv (k{+}\lambda)n_k$

Let us now specialize to strictly linear preferential attachment, or $\lambda = 0$. The solutions to the first few of the recurrences (2) can be found straightforwardly and give

$$\begin{aligned} n_1 &= 2\sqrt{2} - 2 \approx 0.82843\,,\\ n_2 &= \frac{1}{2} - \sqrt{2} + \frac{1}{2}\sqrt{21 - 12\sqrt{2}} \approx 0.08945\,,\\ n_3 &= \frac{1}{9} - \frac{1}{3}\sqrt{21-12\sqrt{2}} + \frac{2}{9}\sqrt{70 - 6\sqrt{21-12\sqrt{2}} - 36\sqrt{2}} \approx 0.03179\,,\end{aligned} \tag{3}$$

etc. To obtain the asymptotic form of the degree distribution, it is convenient to analyze (2) in the continuum approximation. To lowest order, we use the asymptotic behavior

$\sum_{j<k} jn_j \to 2$ as $k \to \infty$, which follows from $\sum_{k\geq 1} kN_k = 2(N-1)$, and we also ignore the terms on the second line. These approximations simplify Eq. (2) to $(kn_k)' = -n_k$, which gives $n_k \sim k^{-2}$. However, this solution cannot be correct, as the sum $\sum_{k\geq 1} kn_k$ logarithmically diverges. The inconsistency arises because the terms that were dropped are of the same order, namely k^{-2}, as those in the approximate equation $(kn_k)' = -n_k$.

As will become plausible with hindsight, a logarithmic correction in the asymptotic degree distribution can be anticipated. We thus seek a solution of the form

$$n_k = k^{-2} u(\ell), \qquad \ell = \ln k\,. \tag{4}$$

Substituting this ansatz into (2), keeping all terms, and using the continuum approximation, gives

$$2u = \left(u - \frac{du}{d\ell}\right) \int_0^\ell dx u(x) - u^2\,, \tag{5a}$$

or, in terms of the cumulative variable $v(\ell) = \int_0^\ell dx\, u(x)$,

$$2 = \left(1 - \frac{du}{dv}\right) v - u\,, \tag{5b}$$

where we now view u as a function of v. This equation can be rewritten as $(2 - v)dv + udv + vdu = 0$, with solution $2v - \frac{1}{2}v^2 + uv = 2$. (The integration constant is set by the sum rule $\sum_{k\geq 1} kn_k = 2$, which implies $v(\infty) = 2$ and $u(\infty) = 0$.) Thus

$$u = \frac{dv}{d\ell} = \frac{2}{v}\left(1 - \frac{v}{2}\right)^2\,. \tag{5c}$$

Integrating gives

$$\frac{\ell}{2} = \ln\left(1 - \frac{v}{2}\right) + \frac{v}{2 - v}\,, \tag{5d}$$

or $2 - v \simeq 4/\ell$, as $\ell \to \infty$. Combining this result with $v(\ell) = \int_0^\ell dx\, u(x)$ ultimately leads to $u \simeq 4/\ell^2$, so that the asymptotic degree distribution is (see Fig. 3)

$$n_k \simeq \frac{4}{k^2\,(\ln k)^2}\,. \tag{6}$$

Attempting a power-law fit to the data for n_k versus k leads to an effective exponent that q appears to be slowly changing with k; this is often the symptom of a logarithmic correction, as predicted by (6).

This slow decay of the degree distribution implies the existence of an almost macroscopic hub—a node whose degree is nearly of the order of N. To estimate this maximal degree $k_{\max}$ in a network that contains N nodes, we apply the standard extremal criterion [23] that there is of the order of one node with degree $k_{\max}$ or larger,

$$\sum_{k\geq k_{\max}} n_k \sim \frac{1}{N}\,, \tag{7}$$

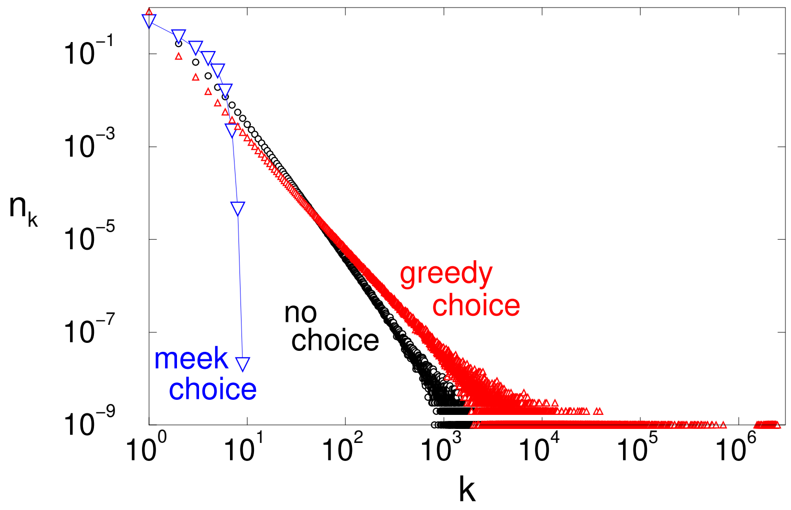


Figure 3. Influence of choice from two alternatives on the degree distributions of networks grown by strictly linear preferential attachment. The distribution without choice asymptotically decays as k^{-3}. Data are based on 10^2 realizations of 10^7 nodes.

to the degree distribution (6) to give

$$k_{\max} \sim \frac{N}{(\ln N)^2} \ , \tag{8}$$

that is, a maximal degree that is almost of the order of N.

For shifted linear preferential attachment, $A_k = k + \lambda$, the degree distribution without choice has the closed form [21]

$$n_k = (2+\lambda)\,\frac{\Gamma(3+2\lambda)}{\Gamma(1+\lambda)}\;\frac{\Gamma(k+\lambda)}{\Gamma(k+3+2\lambda)} \ , \tag{9}$$

whose asymptotic behavior is the non-universal power law $n_k \sim k^{-(3+\lambda)}$. (Note that $\lambda > -1$, so that attachment can occur to nodes of degree 1.)

A convenient way to implement shifted linear preferential attachment is by the redirection algorithm [21, 22, 24]. This algorithm consists of: (i) selecting a target node uniformly at random from the existing network; (ii) a new node either attaches to this target with probability $1 - r$ or to the parent of the target with probability r, where $r = (2 + \lambda)^{-1}$. This algorithm exactly reproduces network growth by shifted linear preferential attachment with shift λ, where the redirection probability is related to λ via $r = (2 + \lambda)^{-1}$. This algorithm is extremely simple and efficient, as the time to simulate a network of N nodes scales linearly with N.

We now determine how greedy choice affects the degree distribution when the network grows by *positive* shifted linear preferential attachment, $A_k = k + \lambda$ with $\lambda > 0$. For large k, we again drop the quadratic terms in (2), replace $\sum_{j<k}(j+\lambda)n_j$ by $\sum_{j\geq 1}(j+\lambda)n_j = 2+\lambda$, and employ the continuum approximation. It may subsequently be verified that the dropped terms are indeed subdominant when $\lambda > 0$. These steps yield

$(kn_k)' = -(2+\lambda)n_k/2$, with solution $n_k \sim k^{-(2+\lambda/2)}$. As in positive shifted preferential linear attachment without choice, the asymptotic behavior of the degree distribution is non-universal, but with a much more slowly decaying tail (Fig. 4).

For *negative* shifted linear preferential attachment, $\lambda < 0$, (corresponding to $\frac{1}{2} < r < 1$), the same analysis of the recurrence (2) as given above predicts $n_k \sim k^{-2}$, which violates the sum rule $\sum_{k\geq 1} kn_k = 2$. The source of this inconsistency is that our analysis has ignored the possibility of a transition to a new type of "condensed" network that contains a macrohub—a node whose degree is of the order of N. Let us assume that such a macrohub of degree hN exists, with h of the order of 1. To determine the degree of this macrohub, we now exploit the equivalence between shifted linear attachment and the redirection algorithm. According to redirection, whenever a random target node is selected, redirection will lead to the macrohub being chosen with probability hr. The probability of choosing this hub at least once in the two independent selection events is $1-(1-hr)^2$. This quantity gives the growth rate of the hub, so that

$$h = 1-(1-hr)^2\,. \tag{10}$$

This equation has two solutions, $h=0$, and

$$h = \frac{2r-1}{r^2}\,. \tag{11}$$

The former (trivial) solution is relevant when the redirection probability $r \leq \frac{1}{2}$, while the non-trivial solution (11) is realized when $\frac{1}{2}<r<1$.

An important feature of this macrohub is that it is unique. To justify this statement, suppose that more than one macrohub exists. Denote the degrees of the largest and second-largest hub by h_1N and h_2N, respectively. The degree of the largest hub is determined from Eq. (10), whose solution is given by (11). For the second-largest hub, the same reasoning that led to Eq. (10) now gives

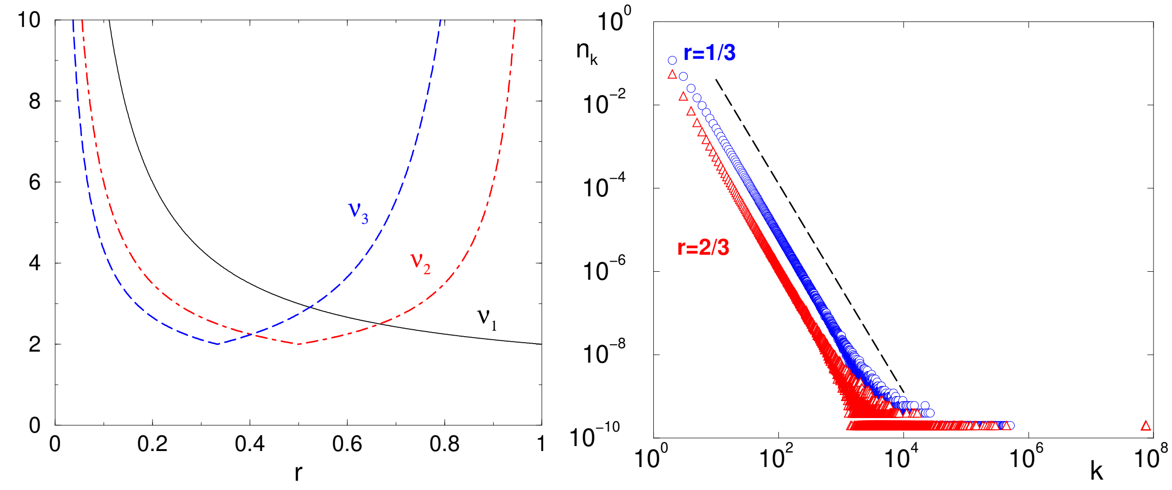
$$h_2 = (1-h_1r)^2 - (1-h_1r-h_2r)^2\,.$$

This equation has two solutions, $h_2 = 0$ and an unphysical solution $h_2 = -h_1$. Thus a second-largest hub does not exist and greedy choice generates one hub when $\frac{1}{2} < r < 1$.

To compute the degree distribution, we must now explicitly include the effect of the macrohub in the recurrence (2) when $\frac{1}{2} < r < 1$. In particular, when we replace $\sum_{j<k}(j+\lambda)n_j$ by $\sum_{j\geq 1}(j+\lambda)n_j$ as $k\to\infty$, the summation must be limited to nodes of finite degree. Thus we now write $\sum_{j\geq 1}(j+\lambda)n_j = 2+\lambda-h$, where the last term represents the contribution of the macrohub. Using the connection $\lambda = \frac{1}{r}-2$ and (11) to rewrite $2+\lambda-h$ as $r^{-2}-r^{-1}$, the recurrence (2) simplifies to

$$n_k = -2(1-r)\,\frac{d}{dk}\,(kn_k) - 2r^2(kn_k)^2\,. \tag{12}$$

The second term on the right-hand side is asymptotically negligible and the asymptotic solution is $n_k \sim k^{-[1+1/(2-2r)]}$.



(a) (b)

Figure 4. (a) The exponents $\nu_1 = 2 + \frac{1}{r}$, ν_2 from Eq. (14), and ν_3 from Eq. (23). (b) Representative degree distributions for shifted linear preferential attachment with greedy choice for redirection probabilities $r = \frac{1}{3}$ and $r = \frac{2}{3}$ for 50 realizations of a network of 10^8 nodes. The dashed line corresponds to exponent $\nu_2 = 2.5$, as given by (14) and (23). For $r = \frac{1}{3}$, the isolated data point at $k = 7.5 \times 10^7$ corresponds to macrohubs whose degree is given by Eq. (11).

To summarize, the degree distribution for greedy choice has the algebraic tail

$$n_k \sim k^{-\nu_2}\,, \tag{13}$$

where the decay exponent is given by (Fig. 4(a))

$$\nu_2(r) = \begin{cases} 1 + 1/(2r) & 0 < r < \frac{1}{2}\,, \\ 1 + 1/(2-2r) & \frac{1}{2} < r < 1. \end{cases} \tag{14}$$

and the subscript refers to greedy choice from two alternatives. Unexpectedly, $\nu_2(r)$ satisfies mirror symmetry, $\nu_2(r) = \nu_2(1-r)$. Also notice that the two forms for $\nu_2(r)$ coincide when $r = \frac{1}{2}$. This feature, together with the emergence of a macrohub for $r > \frac{1}{2}$ indicates that a structural transition occurs at $r = \frac{1}{2}$, and it is natural to anticipate the appearance of a logarithmic correction at this point, as we postulated to derive Eq. (6). For comparison, in the situation without choice, the decay exponent is $\nu_1 = 1 + \frac{1}{r}$. For the special case of strictly linear preferential attachment, $\lambda = 0$ or $r = \frac{1}{2}$, the degree distribution is

$$n_k \simeq 4 \times \begin{cases} k^{-3} & \text{no choice,} \\ (k \ln k)^{-2} & \text{binary choice.} \end{cases} \tag{15}$$

Using the above exponent ν_2 in the extremal criterion (7), the maximal degree $k_{\max}$

in a network of N nodes with greedy choice is given by:

$$k_{\max} \sim \begin{cases} N^{2r} & 0 < r < \frac{1}{2}\,, \\ N(\ln N)^{-2} & r = \frac{1}{2}\,, \\ N^{2-2r} & \frac{1}{2} < r < 1\,. \end{cases} \tag{16}$$

The latter case actually gives the second-largest degree, as the macrohub has the maximal degree whose value is hN.

To numerically implement greedy choice for shifted linear preferential attachment, we simply allow for choice in the redirection algorithm [21]. That is, we independently identify two target nodes by redirection and the new node attaches to the target with the higher degree. Figure 4(b) shows representative simulation results for the degree distribution with greedy choice when $r = \frac{1}{3}$ and $r = \frac{2}{3}$. According to Eq. (14), the exponent of the two degree distributions should be the same, as seen in our data. For $r = \frac{2}{3}$, a unique macrohub also emerges whose average degree is predicted from Eq. (11) to be hN, with $h = \frac{3}{4}$. As an illustration, simulations of 50 realizations of networks of 10^8 nodes gives $h = 0.7503 \pm 0.0012$, in excellent agreement with the theory.

2.2. More Than Two Alternatives

We may readily generalize to greedy choice with $p > 2$ options where p target nodes are selected and attachment occurs to the target with the largest degree. The influence of the number of options p can be easily determined for the emergence of a macrohub. Now the analog of (10) is

$$h = 1 - (1 - hr)^p\,, \tag{17}$$

from which a macrohub emerges when the redirection probability exceeds $r_c = \frac{1}{p}$. For $p = 3$, the explicit solution is

$$h = \frac{3r - \sqrt{4r - 3r^2}}{2r^2} \tag{18}$$

for $r > \frac{1}{3}$, while for arbitrary p

$$h \simeq \frac{2(r - r_c)}{r_c(1 - r_c)}\,, \qquad r_c = \frac{1}{p}\,. \tag{19}$$

near the transition $0 < r - r_c \ll 1$. For any p, the macrohub degree grows linearly in $r - r_c$ close to the transition.

For $p = 3$ choices, the analog of (2) for the degree distribution is

$$n_k = 3\,\frac{\psi_{k-1} - \psi_k}{(2{+}\lambda)^3}\Big(\sum_{j<k}\psi_j\Big)^2 + 3\,\frac{\psi_{k-1}^2 - \psi_k^2}{(2{+}\lambda)^3}\sum_{j<k}\psi_j + \frac{\psi_{k-1}^3 - \psi_k^3}{(2{+}\lambda)^3} + \delta_{k,1}\,, \tag{20}$$

with again $\psi_k = (k{+}\lambda)n_k$. The first term accounts for events where a unique maximal-degree node exists from among three choices, while the second and third terms account for events with a two-fold and three-fold degeneracy in the maximal-degree node, respectively.

When $-1 < \lambda < 1$, or equivalently $0 < r < \frac{1}{3}$, the terms in the first line of (20) are dominant and the equation reduces to $(kn_k)' = -\frac{1}{3}(2+\lambda)n_k$ for $k \to \infty$. We thereby obtain $n_k \sim k^{-[1+1/(3r)]}$. In the marginal case of $r = \frac{1}{3}$, we again expect a logarithmic correction of the form given in (4). With this ansatz, the terms in the first and second lines of (20) are now of the same order, while the terms in the third line are negligible. The governing equation for $u(v)$ is

$$9 = \left(1 - \frac{du}{dv}\right)v^2 - 2uv\,, \tag{21}$$

which gives $u = (3-v)^2(6+v)/(3v^2)$. Combining this with $u = \frac{dv}{d\ell}$ and specializing to the limit of large ℓ, we find

$$n_k \simeq \frac{3}{k^2\,(\ln k)^2}\,. \tag{22}$$

When $\lambda > 1$ (equivalently $\frac{1}{3} < r < 1$), the first term on the right-hand side of (20) is dominant. However, we should again exclude the macrohub from the sum $\Sigma_k = \sum_{j<k}(j+\lambda)n_j$. Hence $\Sigma_k \to 2 + \lambda - h$ and (20) reduces to

$$n_k = -3r[1-hr]^2 \frac{d}{dk}\,(kn_k)\,.$$

Thus for the greedy three-choice model, the degree distribution scales as $n_k \sim k^{-\nu_3}$, with

$$\nu_3(r) = \begin{cases} 1 + 1/(3r) & 0 < r < \frac{1}{3}\,, \\ 1 + 1/(3r[1-hr]^2) & \frac{1}{3} < r < 1. \end{cases} \tag{23}$$

For arbitrary $p \geq 2$, the generalization of (23) is

$$\nu_p(r) = \begin{cases} 1 + 1/(pr) & 0 < r < \frac{1}{p}\,, \\ 1 + 1/(pr[1-hr]^{p-1}) & \frac{1}{p} < r < 1. \end{cases} \tag{24}$$

with $h = h(r)$ implicitly determined by (17). In the marginal case of $r = \frac{1}{p}$, the generalization of (22) is

$$n_k \simeq \frac{p(2p-2)!}{(p-2)!}\,\frac{1}{k^2\,(\ln k)^2}\,, \tag{25}$$

and the maximal degree $k_{\max}$ in a network of N nodes is

$$k_{\max} \sim \begin{cases} N^{pr} & 0 < r < \frac{1}{p}\,, \\ N(\ln N)^{-2} & r = \frac{1}{p}\,, \\ N^{pr[1-hr]^{p-1}} & \frac{1}{p} < r < 1\,. \end{cases} \tag{26}$$

As in optimization and queuing theory, the possibility of choosing between more than two options leads only to quantitative changes compared to the more fundamental case of two options.

2.3. Networks With Loops

Thus far, we studied the situation where every new node attaches to one already existing node, leading to tree networks. However, we can also treat networks with loops. Here we outline how to deal with the situation where loops are created when each new node attaches to m already existing nodes, with each attachment event created by the same choice-driven algorithm as in the previous section. Limiting ourselves to shifted linear attachment and focusing on greedy choice from two alternatives, the recursion for n_k is given by (compare with Eq. (2))

$$n_k = m\,\frac{\psi_{k-1}-\psi_k}{(2m+\lambda)^2/2}\sum_{j<k}\psi_j - m\,\frac{\psi_{k-1}^2+\psi_k^2}{(2m+\lambda)^2} + \delta_{k,m}\,. \tag{27}$$

This recurrence can be analyzed using the same methods as in the case of trees. For instance when $\lambda > 0$, we replace $\sum_{j<k}\psi_j$ by $\sum_{j\geq m}\psi_j = 2m+\lambda$ when $k \gg 1$, and then employ the continuum approximation to recast (27) into the differential equation $(kn_k)' = -\big(1+\frac{\lambda}{2m}\big)n_k$. This equation again has an algebraic solution of the form (13), with decay exponent $\nu_2 = 2+\lambda/(2m)$.

A macrohub of degree hN again emerges when $\lambda < 0$, with h determined by the relation

$$h = m\left[1-\left(1-\frac{h}{2m+\lambda}\right)^2\right]\,, \tag{28}$$

which generalizes (10). Thus

$$h = -\frac{\lambda(2m+\lambda)}{m}\,. \tag{29}$$

Note that the range of the shift parameter is now $\lambda > -m$, since the minimal degree is m and we must ensure that the attachment to nodes of degree m is non-negative. The degree distribution associated with the remaining nodes still has an algebraic tail. To summarize, the decay exponent is given by

$$\nu_2 = \begin{cases} 2+\lambda/(2m) & \lambda > 0,\\ (4m+3\lambda)/(2m+2\lambda) & 0 > \lambda > -m. \end{cases} \tag{30}$$

For the special case of strictly linear preferential attachment $\lambda = 0$, the tail of the degree distribution is

$$n_k \simeq \begin{cases} 2m(m+1)\times k^{-3} & \text{no choice,}\\ 4m\times (k\ln k)^{-2} & \text{binary choice.} \end{cases} \tag{31}$$

3. Meek Choice

The complementary situation of meek choice, where a set of target nodes is first selected and a new node attaches to a target with less than the largest degree leads to very different phenomenology. The simplest case is that of first selecting two nodes according to linear preferential attachment and the new nodes attaches to the smaller-degree target; this specific example was also recently investigated in [25].

We determine the degree distribution in this meek choice model by following the same approach as in greedy choice. The analog of (2) for the degree distribution is

$$n_k = \tfrac{1}{2}\big[\psi_{k-1} - \psi_k\big] \sum_{j \geq k} \psi_j + \tfrac{1}{4}\big[\psi_{k-1}^2 + \psi_k^2\big] + \delta_{k,1} \,. \tag{32}$$

Using identity $\sum_{j\geq k} j n_j = 2 - \sum_{j<k} j n_j$ recasts (32) as a recurrence. In the case of strictly linear preferential attachment, $\lambda = 0$, the solutions for small degrees are:

$$\begin{aligned}
n_1 &= 4 - 2\sqrt{3} \approx 0.53589\,,\\
n_2 &= -\tfrac{1}{2} + \sqrt{3} - \tfrac{1}{2}\sqrt{25 - 12\sqrt{3}} \approx 0.20548\,,\\
n_3 &= -\tfrac{1}{9} + \tfrac{1}{3}\sqrt{25 - 12\sqrt{3}} - \tfrac{2}{9}\sqrt{79 - 6\sqrt{25 - 12\sqrt{3}} - 36\sqrt{3}} \approx 0.11099\,,
\end{aligned} \tag{33}$$

etc. Notice that while the first few n_k are larger than those for greedy choice in Eqs. (3), the asymptotic degree distribution decays precipitously with k (Fig. 3). For example, in simulations of 50 realizations of networks grown to 10^8 nodes, the largest observed degree is only 9!

We now exploit this rapid decay to determine the asymptotic behavior of the degree distribution. For large k, an increase in n_k can occur only if the two target nodes have degree $k-1$. Thus we posit that the dominant term in (32) is $\frac{1}{4}(k-1)^2 n_{k-1}^2$. Keeping only this term, the asymptotic behavior of the logarithm of the degree distribution is given by

$$\ln n_k \sim -C \times 2^k\,, \tag{34}$$

up to some amplitude C that cannot be determined within this simplified analysis. One can then verify that the remaining terms in (32) are subdominant. From this asymptotic degree distribution, we estimate the maximal degree in a network of N nodes to be $k_{\max} \simeq \log_2 \log_2 N$, as recently proven in Ref. [25].

When p distinct initial target nodes are selected by preferential attachment, there are p possibilities for the attachment event: to the highest-degree node, to the second-highest degree node, all the way to the lowest-degree node. While the combinatorics become unwieldy for the general case of identifying the target node with the m^{th}-largest degree out of p choices, the dominant contribution to n_k for large k arises when m targets have degree $k-1$ and the remaining $p-m$ targets have degrees less than $k-1$. Following the same reasoning as in the case of attaching to the smallest-degree node out of two choices, the dominant term in the generalization of (32) is proportional to $(k-1)^m n_{k-1}^m$. This leads to $n_k \sim \exp(-\text{const.} \times m^k)$. Thus for all but greedy choice, the degree distribution decays precipitously with degree.

From this asymptotic degree distribution, the maximal degree grows with N as

$$k_{\max} \sim \begin{cases} N^{\omega_p} & \text{greedy choice} \\ \log_2 \log_2 N & 2^{\text{nd}} \text{ highest degree} \\ \log_3 \log_3 N & 3^{\text{rd}} \text{ highest degree} \\ \cdots & \\ \log_p \log_p N & \text{smallest degree} \end{cases} \tag{35}$$

for $p \geq 2$. The exponent ω_p that appears in (35) depends on the number of alternatives p and on details of the attachment rate. For strictly linear preferential attachment, $\omega_p = p(1-h)/(2-h)$, where the degree h of the macrohub is the positive solution of the equation $h = 1 - (1 - h/2)^p$. The other ultra-slow growth laws in (35) are robust with respect to the details of the attachment rule. These latter behaviors do not depend on the details of the selection rule as long as the choice is less than greedy.

4. Summary

Incorporating choice in preferential attachment network growth leads to rich phenomenology in which the effect of preferential attachment can be strongly amplified or entirely eliminated. We have explored a general class of models in which a set of target nodes in the network are first selected according to preferential attachment and then a new node joins the network by attaching to one of these target nodes according to a specified criterion. In greedy choice, attachment is made to the target with the largest degree. We also investigated attaching to a node in the target set whose degree is not the largest. For a target set of p nodes, there are $p - 1$ possible such choices—to the 2^{nd}-largest degree node, the 3^{rd}-largest, ..., to the smallest-degree node. We term this class of models as meek choice.

Past work on the power of choice on the random recursive tree [6] found that greedy choice broadens the degree distribution, but only in a quantitative way. We have shown that greedy choice plays a much more significant role for networks that grow by preferential attachment. We focused on shifted linear preferential attachment, but our methods apply to other models with asymptotically linear preferential attachment. The details depend on the model, but the general outcome is robust. In the sub-critical phase, the degree distribution has a power law tail that is considerably broader than in the case of no choice. In the super-critical phase, a macrohub emerges, while the remainder of the degree distribution is still algebraic. At the boundary between these two phases, the degree distribution decays as $(k \ln k)^{-2}$. This form for the degree distribution is consistent with a finite average degree in the network because of the presence of the logarithmic factor.

The influence of meek choice is perhaps even more dramatic, as it effectively counteracts preferential attachment. When p target nodes are initially selected, meek choice means that the new node attaches to a target whose degree is less than the highest in the target set. For the case where a new node attaches to the $m^{\rm th}$-largest degree out of a target set of p nodes that are each selected by linear preferential attachment, meek choice leads to a double-exponential degree distribution of the form $\exp(-\text{const.} \times e^k)$, and a maximal degree that is of the order of $\log_m \log_m N$. It is surprising that this sharp decay should hold for attachment to the target with the $2^{\rm nd}$-highest degree out of $p \gg 1$ targets. In this case, the degree distribution will initially resemble that of greedy choice and the crossover to a precipitous decay will occur at an extremely large degree value.

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