Journal of Statistical Mechanics:
Theory and Experiment

PAPER: CLASSICAL STATISTICAL MECHANICS, EQUILIBRIUM AND NON-EQUILIBRIUM

# Reputation-driven voting dynamics

To cite this article: D Bhat and S Redner *J. Stat. Mech.* (2019) 063208

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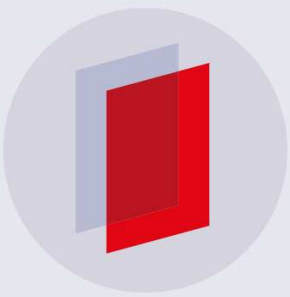

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Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Classical statistical mechanics, equilibrium and non-equilibrium

# Reputation-driven voting dynamics

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Received 22 February 2019
Accepted for publication 8 April 2019
Published 21 June 2019

Online at stacks.iop.org/JSTAT/2019/063208
https://doi.org/10.1088/1742-5468/ab190c

**Abstract.** We introduce the reputational voter model (RVM) to account for the time-varying abilities of individuals to influence their neighbors. To understand of the RVM, we first discuss the fitness voter model (FVM), in which each voter has a fixed and distinct fitness. In a voting event where voter $i$ is fitter than voter $j$, only $j$ changes opinion. We show that the dynamics of the FVM and the voter model are identical. We next discuss the adaptive voter model (AVM), in which the influencing voter in a voting event increases its fitness by a fixed amount. The dynamics of the AVM is non-stationary and slowly crosses over to that of FVM because of the gradual broadening of the fitness distribution of the population. Finally, we treat the RVM, in which the voter $i$ is endowed with a reputational rank $r_i$ that ranges from 1 (highest rank) to $N$ (lowest), where $N$ is the population size. In a voting event in which voter $i$ outranks $j$, only the opinion of $j$ changes. Concomitantly, the rank of $i$ increases, while that of $j$ does not change. The rank distribution remains uniform on the integers $1, 2, 3, \ldots, N$, leading to stationary dynamics. For equal number of voters in the two voting states with these two subpopulations having the same average rand, the time to reach consensus in the mean-field limit scales as $\exp(\sqrt{N})$. This long consensus time arises because the average rank of the minority population is typically higher than that of the majority. Thus whenever consensus is approached, this highly ranked minority tends to drive the population away from consensus.

J. Stat. Mech. (2019) 063208

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1742-5468/19/063208+19$33.00

**Keywords:** agent-based models, fluctuation phenomena, stochastic processes

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## 1. Introduction

The way people form an opinion about a given issue, such as making a political decision of choosing a product is a complex social phenomenon. An individual's opinion can be influenced by economic factors, advertising, mass media, as well the opinions of others. When opinion changes occur only through interactions between individuals, a natural model for this dynamics is the voter model (VM) [1–10]. In the VM, each individual, or voter, can assume one of two states, denoted as $+$ and $-$, with one voter at each node of an arbitrary network. A voter is selected at random and it adopts the state of a randomly chosen neighboring voter. This update is repeated at a fixed rate until a population of $N$ voters necessarily reaches consensus. Each voter is influenced only by its neighbors and has no self confidence in its own opinion.

https://doi.org/10.1088/1742-5468/ab190c


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The paradigmatic nature of the VM has sparked much research in probability theory [1–4] and statistical physics [5–7, 9–11]. Because of its flexibility and utility, the VM has been applied to diverse problems, such as population genetics [12], ecology [13, 14], and epidemics [15], and voting behavior in elections [16]. However, consensus is not the typical outcome for many decision-making processes. This fact has motivated a variety of extensions of the VM to include realistic elements of opinion formation that can forestall consensus. Examples include: stochastic noise [17–19], the influence of multiple neighbors [20], self confidence [21], heterogeneity [22], partisanship [23, 24], and multiple opinion states [25–27].

An important extension of the VM that is relevant to this work arises when either the underlying network or the decision-making rule of each voter changes with time [28–34]. The latter scenario represents an attempt to account for the feature that the influence of individuals may be time dependent—some individuals may become more influential and others less so as the opinions of the population evolve. A natural way to account for this feature is to assign each individual a fitness that can change with time. In a single update, the higher-fitness voter imposes its opinion on its neighbor and correspondingly, the fitness of the influencer increases by a fixed amount, while the fitness of the influenced voter does not change. This adaptive voter model (AVM), introduced in [34], leads to a consensus time on the complete graph that appears to scale as $N^{\alpha}$, with $\alpha \approx 1.45$, a slower approach to consensus compared to the classic VM. We will argue, however, this model exhibits a very slow crossover that masks the asymptotic approach to consensus.

This AVM also provides the motivation for our *reputational voter model* (RVM) to help understand the role of individual reputation changes on the consensus dynamics. In the RVM, each voter is endowed with a unique integer-valued reputation that ranges from 1, for the voter with the best reputation, to $N$, for the voter with the worst reputation, in addition to its voting state. In an update, two voters in different opinion states are selected at random and the voter with the higher reputation imposes its voting state on the voter with the lower reputation. After this interaction, only the reputation of the influencer voter rises, in analogy with the AVM. As we will show, the effect of these reputational changes significantly hinder the approach to consensus. When the population initially contains equal numbers in the two voting states and the average rank of these two subpopulations are the same, the time to reach consensus scales as $\exp(\sqrt{N})$. This slow approach to consensus arises because close to consensus the average rank of the minority population is typically higher than that of the majority. This imbalance tends to drive the population away from consensus and thereby leads to a long consensus time.

In section 2, we define the models under study: (i) the fitness voter model (FVM), where each voter is assigned a unique and unchanging fitness value, (ii) the adaptive voter model (AVM) [34], and (iii) the reputational voter model (RVM). In section 3, we will show that the FVM has the same dynamics as the classic VM. In section 4, we will argue that the consensus time scaling as $N^{\alpha}$, with $\alpha \approx 1.45$ in the AVM [34], is a finite-time artifact and that the dynamics of the AVM eventually crosses over to that of the FVM. In section 5, we introduce the RVM and discuss the role of the time-dependent individual reputations on the opinion dynamics. In section 6, we give some concluding remarks.



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## 2. Models

We begin by defining a set of voter models that culminate with the RVM, which is the focus of this work. All our models are defined on the complete graph; this structure is assumed throughout.

### 2.1. Classic voter model (VM)

We define the classic VM in a form that is convenient for our subsequent extensions. In the VM, voters are situated on a complete graph of $N$ nodes, with one voter per node. Each voter is initially assigned to one of two opinion states, $+$ or $-$. The number of voters in the $+$ and $-$ states are denoted by $N_+$ and $N_-$. The opinion update is the following:

(a) Pick two random voters in opposite opinion states.

(b) One of these two voters changes its opinion.

(c) Repeat steps (a) and (b) until consensus is necessarily reached.

Figuratively, each agent has no self-confidence and merely adopts the state of one of its neighbors. After each update, the time is incremented by an exponential random variable with mean value $\delta t \equiv N/(N_+N_-)$.

There are two basic observables in the VM: the consensus time and the exit probability. The consensus time, $T_N(m)$, is the average time for a population of $N$ voters to reach unanimity when the initial magnetization, which is the difference in the density of $+$ and $-$ voters, equals $m$. For the complete graph, the consensus time is (see e.g. [8])

$$T_N(m) = -N\Big\{(1+m)\ln\big[\tfrac{1}{2}(1+m)\big] + (1-m)\ln\big[\tfrac{1}{2}(1-m)\big]\Big\}. \tag{1a}$$

We are often interested in the zero-magnetization initial condition, in which case, we write the consensus time as $T_N$. The main feature of the consensus time on the complete graph is that it grows linearly with $N$.

The exit probability $E(m)$ is defined as the probability that a population of $N$ voters with initial magnetization $m$ reaches $+$ consensus. The form of the exit probability is especially simple because the average magnetization is conserved:

$$E(m) = \frac{1}{2}(1+m). \tag{1b}$$

In voter models where the magnetization is not conserved, the exit probability is a nonlinear function of $m$.

### 2.2. Fitness voter model (FVM)

In the FVM, each voter is assigned an opinion state as well as a unique and fixed fitness that is drawn from a uniform distribution in the range $[0,F_0]$. A voter with a larger fitness value is regarded as more fit. The opinion update is now:



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(a) Pick two random voters in opposite opinion states.

(b) The less fit voter changes its opinion.

(c) Repeat steps (a) and (b) until consensus is necessarily reached.

The time increment for each update is again an exponential random variable with mean value $\delta t$. The crucial feature of the FVM is the *unique* fitness of each voter; the actual fitness values are immaterial. We will show below that the dynamics of the FVM is the same as the VM.

### 2.3. Adaptive voter model (AVM)

In our version of the AVM, each voter is assigned a unique fitness that is drawn from the uniform distribution $[0,F_0]$. The fitness of each voter also changes as a result of opinion updates. The opinion update is given by:

(a) Pick two random voters in opposite opinion states, with fitnesses $f_i$ and $f_j$.

(b) The less fit voter changes its opinion.

(c) For the fitter voter $i$, $f_i \to f_i + \delta f$.

(d) Repeat steps (a)–(c) until consensus is necessarily reached.

After each update, the time is incremented by an exponential random variable with mean value $\delta t$. As we shall see, the initial fitness range $F_0$, the fitness increment $\delta f$ in each voting event, and $N$ play important roles in determining the long-time dynamics.

### 2.4. Reputational voter model (RVM)

In the RVM, each voter is assigned a unique and integer-valued reputation, or rank, between 1 and $N$, with 1 corresponding to the best-ranked voter and $N$ to the worst-ranked. The opinion update is given by:

(a) Pick two random voters in opposite opinion states, with ranks $r_i$ and $r_j$.

(b) The lower-ranked voter changes its opinion.

(c) The higher-ranked voter $i$ gains rank, $r_i \to r_i - 1$.

(d) The rank of the voter with rank adjacent to $i$ is adjusted to eliminate ties (figure 1).

(e) Repeat steps (a)–(d) until consensus is necessarily reached.

As we will see, when the population is close to consensus, minority-species voters are typically well ranked and more likely to influence the majority rather than be influenced. This effective bias drives the population back to equal densities of + and − voters, and leads to a large consensus time.



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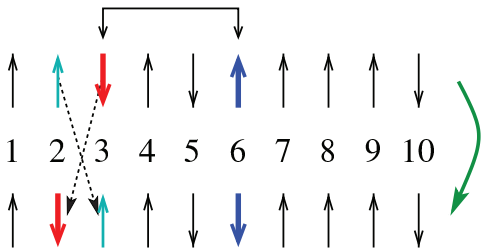


**Figure 1.** Update event in the RVM. Voters are arranged in rank order. The voter with rank 3 changes the opinion of the voter with rank 6. After the voting event, the ranks of the influencer and an adjacently ranked voter are shuffled to avoid ties.

## 3. Dynamics of the fitness voter model (FVM)

The main feature of the FVM is that its dynamics is identical to that of the VM. This equivalence will be important to understand the dynamics of the AVM, that will be treated in the next section. First consider the dependence of the exit probability $E(m)$ on the initial magnetization $m$. By construction, the fittest voter in the population can never change its opinion. Consequently, the final consensus state coincides with the initial voting state of this fittest voter. The probability that the fittest voter is in the + state equals $\frac{1}{2}(1+m)$. Thus $E(m) = \frac{1}{2}(1+m)$, as in the VM.

Let us now treat the consensus time. For the VM on the complete graph, the initial magnetization uniquely specifies the system. From this initial state, there are many trajectories that eventually take the system to consensus. To compute fundamental quantities like the exit probability and the consensus time, we need to average over all stochastic trajectories of the voting dynamics. For the FVM on the complete graph, the initial state is specified by both the magnetization and the fitness of each voter. The computation of the exit probability and the consensus time requires averaging over all stochastic trajectories and over all fitness values.

Thus let us compare the fate of a single pair of voters $ij$ in the state $+-$ in the VM and in the FVM. For the VM, this pair changes to either $++$ or $--$ equiprobably. In the FVM, if the fitness of voter $i$, $f_i > f_j$, then this pair changes from the state $+-$ to $++$. However, if $f_i < f_j$, then this pair changes from $+-$ to $--$. Since it is equally likely that $f_i > f_j$ or $f_i < f_j$, then in averaging over all stochastic trajectories *and* over all fitness assignments, the $ij$ pair in the FVM equally likely changes to $++$ or to $--$. Thus the dynamics of the VM, averaged over all stochastic trajectories, is the same as that of the FVM, when averaged over all stochastic trajectories and over all initial fitness assignments. A detailed microscopic derivation of this equivalence is given appendix A.

## 4. Dynamics of the adaptive voter model (AVM)

### 4.1. Consensus time and exit probability

In [34], it was reported that the consensus time scales as $T_N \sim N^{\alpha}$, with $\alpha \approx 1.45$. Instead, we will argue that this exponent estimate is a finite-time effect. To support



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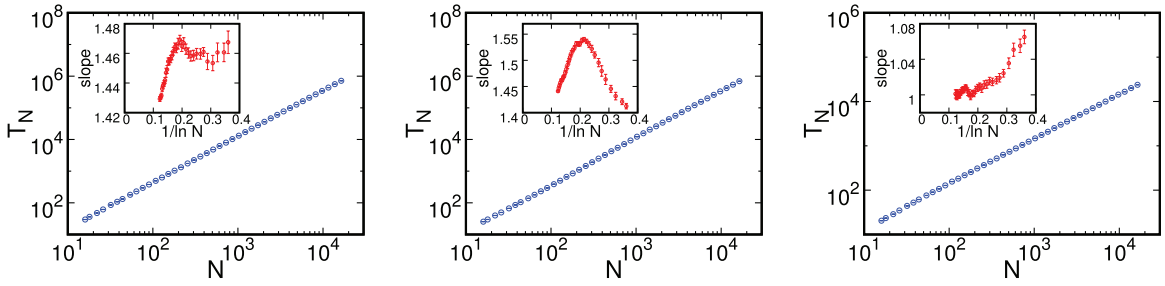


**Figure 2.** Average consensus time $T_N$ versus $N$ for the AVM on the complete graph of $N$ sites with: (a) $F_0 = 1$, (b) $F_0 = N$, and (c) $F_0 = N^2$, with $\delta f = 1$. The insets show local 10-point slopes as a function of $1/\ln N$. The error bars are the standard deviation in a linear least-squares fit.

this assertion, we show simulation data for the dependence of $T_N$ versus $N$ in figure 2 for representative parameter values: (a) initial width of the fitness distribution $F_0 = 1$ (b) $F_0 = N$ and (c) $F_0 = N^2$, and $\delta f$, the change in individual fitness in a voting event, fixed to be 1. The data in the figure are based on $10^4$ realizations for $N$ up to $2^{14} = 16\,384$. On a double logarithmic scale, the data of $T_N$ versus $N$ appears relatively straight, which suggests that a linear fit is warranted. However, there is a small but consistent downward curvature in the data, a feature that becomes apparent by studying local slopes of $T_N$ versus $N$ based on $k$ successive data points (insets to figure 2). The choice of $k$ is important: for too-small $k$ values, successive local slopes fluctuate strongly and cannot be reliably extrapolated, while for $k$ too large, the systematic trend in the local slope is averaged away. We find that for $k = 10$, there is a good compromise between minimizing statistical fluctuations and uncovering systematic local trends.

In figures 2(a) and (b), corresponding to $F_0 = 1$ and $F_0 = N$ respectively, the local slope is non-monotonic in $N$. The source of this crossover behavior appears to be the broadening of the fitness distribution as a function of time. This leads to rank-changing events becoming progressively less frequent. When rank changes stop occurring, the dynamics should be the same as the FVM, for which $T_N \sim N$. However, consensus interrupts this gradual crossover. Conversely, for $F_0 = N^2$, the initial fitness distribution is sufficiently broad that rank-changing events never occur. The dynamics thus coincides with that of the FVM, for which $T_N \sim N$. For this case, the simulation data for the local slope appears to extrapolate to a value that is close to the expected value of 1 (figure 2(c)).

Simulation results for the exit probability is shown in figure 3 for: (a) $F_0 = 1$, (b) $F_0 = N$, and (c) $F_0 = N^2$, with $\delta f = 1$ in all cases. In (a) and (b), the exit probability $E(m)$ is a non-linear function of $m$, which means that the magnetization is not conserved. The non-linearity indicates that there is an effective bias in the dynamics that tends to drive a population with non-zero magnetization back to the zero-magnetization state and thus forestalls consensus. Note the curious feature, for which we have no explanation, is that $E(m)$ is non-monotonic in $m$ for small $N$. When $F_0 = N^2$, the exit probability is linear in $m$. As discussed above, rank-changing events no longer occur for $F_0 = N^2$, so that the dynamics should be the same as the VM.

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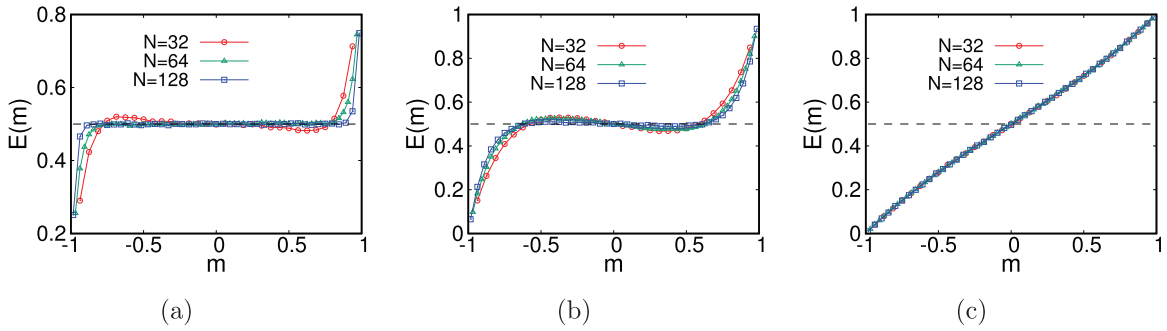


**Figure 3.** Exit probability as a function of initial magnetization $m$ for the AVM on the complete graph of $N$ sites with: (a) $F_0 = 1$, (b) $F_0 = N$, and (c) $F_0 = N^2$, for $\delta f = 1$ in (a)–(c). These data are obtained by averaging over $10^5$ trajectories.

To summarize, in spite of the simplicity of the AVM update rule, its basic properties are surprisingly complex. When the initial fitness distribution is sufficiently broad or equivalently, the fitness increment $\delta f$ in a single voting effect is sufficiently small, rank-changing events do not occur, so that the dynamics is the same as the FVM, which, in turn, is the same as that of the classic VM. The dynamics of the AVM has a paradoxical character in the time range where rank-changing events do occur. Figure 3 shows that the average magnetization is not conserved because $E(m)$ strongly deviates from the form $E(m) = \frac{1}{2}(1+m)$ that arises in the magnetization-conserving VM. The non-linear dependence of $E(m)$ in this figure indicates the presence of an underlying bias that tends to drive the system to zero magnetization whenever $m \neq 0$. In other examples of voter-like models with non-conserved magnetization [35, 36], a similar non-linearity for $E(m)$ was observed. As a result of the effective bias that drives the system to zero magnetization, the consensus times in these models were found to grow faster than a power law in $N$ [35, 36]. The observation of an apparent power-law dependence of $T_N$ on $N$ found above and in [34] is possibly a manifestation of the gradually diminishing effective bias. The main message from our analysis is that the exponent $\alpha$ in $T_N \sim N^\alpha$ is strongly $N$-dependent and less than the value 1.45 reported in [34].

### 4.2. Dynamical non-stationarity

By directly adapting the theory given in [37, 38] for the fitness distribution in a model of social competition, the distribution of individual fitnesses in the AVM approaches a uniform distribution in $[0, F(t)]$, with $F(t) = F_0 + \delta f\, t/2$. Consequently, as the fitness distribution broadens, changes in fitness rank become more rare. When rank changes can no longer occur, the subsequent dynamics approaches that of the FVM.

To understand this transition, we estimate the time dependence of fitness-rank changes. Consider two voters $i$ and $j$ of adjacent ranks, with $f_i(0) > f_j(0)$; that is, voter $i$ is initially fitter than voter $j$. Their fitnesses $f_i$ and $f_j$ at a later time $t$ are

$$\begin{aligned} f_i(t) &= f_i(0) + v_i t \pm \sqrt{Dt}, \\ f_j(t) &= f_j(0) + v_j t \pm \sqrt{Dt}. \end{aligned} \tag{2}$$

Here $v_i$ is the systematic change in fitness because a higher-ranked voter typically is more influential than a lower-ranked voter. The 'speed' $v_i$ at which the $i^{\text{th}}$ voter gains



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fitness is proportional to the fraction of voters with lower fitness. For a uniform fitness distribution, $v_i = f_i\,\delta f/F$. Thus the speed of the best-ranked voter is $\delta f$ and that of the worst-ranked voter is 0. The term $\pm\sqrt{Dt}$ denotes the change in fitness due to stochastic effects, which give rise to rank-changing events.

In the absence of stochasticity, no rank-changing events occur. To assess the role of stochasticity on rank changes, we assume a negative stochastic term for $f_i$ and a positive stochastic term for $f_j$ and find the condition under which the ranks of these two voters can switch [39, 40]. That is, suppose that at some time $t$, $f_i(t) < f_j(t)$. From equation (2), this criterion gives

$$f_i(0) - f_j(0) + (v_i - v_j)t < \sqrt{4Dt}. \tag{3a}$$

Now $v_i - v_j = \delta f/N$, while the diffusion coefficient associated with the stochasticity is proportional to $(\delta f)^2$. Thus equation (3*a*) becomes

$$\frac{F_0}{N} + \frac{\delta f}{N}t < \delta f\,\sqrt{4t}. \tag{3b}$$

Dividing through by $\delta f$, defining $a = 1/N$, and $b = F_0/(N\delta f)$, the solution to (3*b*) is

$$t = \frac{1}{2a^2}\left[(1-2ab) \pm \sqrt{1-4ab}\right]. \tag{4}$$

There are no solutions for $4ab > 1$, which translates to $F_0/\delta f > N^2$. That is, for a given $N$, if either the initial fitness range is sufficiently large or the fitness change in a single voting event is sufficiently small, no rank changes occur. In this limit, the dynamics of the AVM reduces to the FVM, which, in turn, is the same as the VM. For $4ab < 1$, the physically relevant situation is $4ab \ll 1$. Now there are two solutions (see figure 4):

$$t_* \approx \left(\frac{F_0}{N\delta f}\right)^2, \qquad\qquad t^* \approx N^2. \tag{5}$$

Between these two times, rank-changing events occur. We may estimate the time dependence of the number of rank changes as follows. The typical fitness difference of neighboring-ranked voters at time $t$ is $\Delta \equiv F(t)/N$. In a single voting event, the typical number of rank changes is $\mathrm{d}r \approx \delta f/\Delta = N\delta f/F(t)$, as long as $\delta f > \Delta$. Thus we estimate the number of rank changes per unit time as

$$N_r(t) \simeq \frac{N\delta f/F(t)}{\delta t} = \frac{N\delta f/\delta t}{F_0 + \delta f\, t/2}. \tag{6a}$$

We can make this estimate more precise by computing the number of rank changes averaged over the uniform distribution of fitnesses. Consider the case where $F_0 = 1$ and $\delta f = 1$. For the first voting event between two voters with fitnesses $f_i$ and $f_j < f_i$, their fitnesses after the voting event will be $f_i + 1$ and $f_j$ respectively. The number of rank changes due to these changes is $\mathrm{d}r = N(1 - f_i)$. Averaging this expression over the uniform distribution of fitnesses subject to the constraint $f_i > f_j$, gives $\mathrm{d}r = N/3$. Then using $\delta t = 4/N$ as the time increment for this first voting event, the initial number of rank changes per unit time is $N^2/12$. Using this for $N_r(t = 0)$ in (6*a*), the number of rank changes per unit time at any later time is



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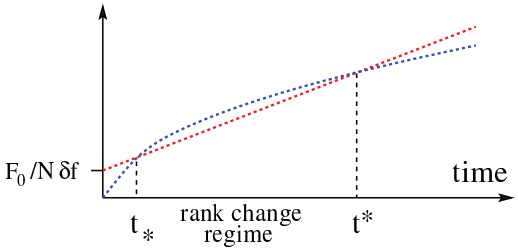


**Figure 4.** Schematic of the left-hand and right-hand sides of equation (3*b*) (red and blue respectively). For this example, rank changes can occur only in the intermediate time regime between $t_*$ and $t^*$.

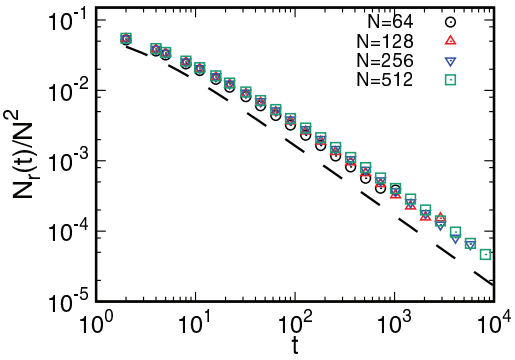


**Figure 5.** Time dependence of the number of rank changes per unit time, $N_r(t)$, scaled by $N^2$ in the AVM for $F_0 = 1$ and $\delta f = 1$. Data for $t > T_N$ are dominated by noise and are not shown. The data are generated by averaging over $10^5$ realizations. The dashed line is the prediction in equation (6*b*).

$$N_r(t) = \frac{N^2 \delta f/12}{F_0 + \delta f\, t/2}. \tag{6b}$$

This prediction is consistent with the simulations shown in figure 5.

The simple reasoning given above shows that the dynamics of the AVM is non stationary. At early times, rank-changing events occur frequently (as long as $\delta f$ is not pathologically small) and these rank changes are responsible for the slow approach to consensus. However, at sufficiently long times, rank changing events stop occurring and the dynamics crosses over to that of the FVM. Thus over a substantial time range the dynamics of the AVM is governed by crossover effects.

### 4.3. Magnetization zero crossings

The non-stationarity of the AVM also manifests itself in the times between successive zero crossings of the magnetization. For a system that starts at zero initial magnetization, there are typically multiple instances when the magnetization returns to zero before consensus is reached. We define $\tau_n$ as the average time between the $(n-1)$st and $n$th zero crossing, with the 0th crossing occurring at $t = 0$. Each $\tau_n$ is averaged over those trajectories that have not yet reached consensus by the $n$th crossing. A basic feature of these magnetization zero crossings for the AVM is that $\tau_n$ varies non-monotonically



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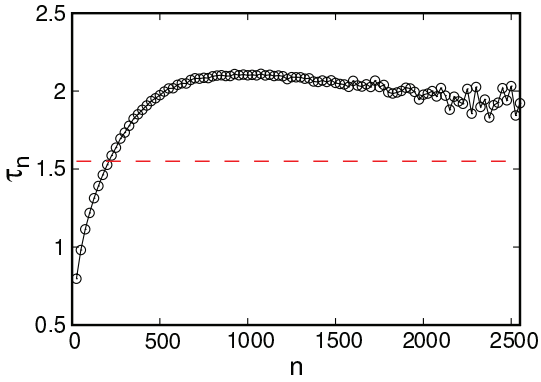


**Figure 6.** Dependence of $\tau_n$, the $n$th zero-crossing time on $n$ for $10^6$ realizations with $N = 256$. The data are smoothed by averaging over 15 successive points. The parameters are $F_0 = 1$ and $\delta f = 1$. The dashed line is the exact zero-crossing time for the VM (appendix B). The average number of zero crossings is 900.3.

with $n$ (figure 6). In this plot, the number of 'surviving' trajectories decreases as $n$ increases (roughly a fraction $10^{-3}$ of all realizations survive until $n = 2500$), and the behavior of $\tau_n$ becomes progressively noisier. In contrast, the dynamics of the classic VM is stationary and successive zero-crossing times are all the same; the derivation of the crossing time for the VM is given in appendix B.

We can qualitatively understand the non-monotonicity of the AVM zero-crossing times in terms of the time dependence of the rank changes of the voters. As derived in equation (6*b*), rank changes are frequent at early times and become progressively less common. These rank changes give rise to an effective bias $v(m)$ towards zero magnetization (see also the next section). At early times, these frequent rank changes imply a strong bias to zero magnetization; this leads to zero-crossing times that are smaller than in the VM. At later times, we can assess the role of the bias on magnetization trajectories in terms of the Péclet number [41], $Pe \equiv |v(m)m|/D(m)$, where $D(m)$ is the diffusion coefficient associated with the trajectories. As time increases and the bias becomes weaker, only those trajectories that approach close to $m = \pm 1$ experience a Péclet number $Pe > 1$ and get driven back towards zero magnetization. These large-deviation trajectories lead to a zero-crossing time that is larger than that of the VM. Finally, at late times ($t \gg N^2$), rank changes become sufficiently rare that the dynamics approaches that of the VM and the zero-crossing times also approach that of the VM. This asymptotic limit will be reached only when the number of zero crossings $n$ is of the order of $N^2$ when rank changes no longer occur.

## 5. Dynamics of the reputational voter model (RVM)

### 5.1. Effective potential

In RVM, each voting event leads to a fixed number of rank changes (figure 1). This implies that the dynamics is stationary, which simplifies the analysis of this model. We will argue that the dynamics of the magnetization is equivalent to that of a random



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walk that is confined to an effective potential well, leading to an anomalously long consensus time compared to the VM and the AVM.

In a single voting event, the magnetization $m$ changes by $\delta m \equiv \pm 2/N$ and the average time for such a voting event is $\delta t = N/(N_+ N_-)$, where $N_\pm$ are the number of voters in the $+$ and $-$ states, respectively. We define $w(m \to m')$ as the probability that the magnetization changes from $m$ to $m'$ in a single voting event and $P(m,t)\delta m$ as the probability that the system has a magnetization between $m$ and $m + \delta m$. The Chapman–Kolmogorov equation for the time dependence of $P(m,t)$ is

$$P(m, t+\delta t) = w(m - \delta m \to m) P(m - \delta m, t) + w(m + \delta m \to m) P(m + \delta m, t). \tag{7a}$$

Expanding this equation to second order in a Taylor series gives the Fokker–Planck equation

$$\frac{\partial}{\partial t} P(m,t) = -\frac{\partial}{\partial m}[v(m)P] + \frac{\partial^2}{\partial m^2}[D(m)P], \tag{7b}$$

where the drift velocity and diffusion coefficient are given by

$$\begin{aligned} v(m) &= 2[2w(m \to m + \delta m) - 1]/(N\delta t) = [2w(m \to m + \delta m) - 1](1 - m^2)/2, \\ D(m) &= 2/(N^2 \delta t) = (1 - m^2)/2N, \end{aligned}$$

and where the second equalities follow by expressing the time step $\delta t = N/(N_+ N_-)$ in terms of the magnetization, $\delta t = 4/[N(1 - m^2)]$.

In figure 7, we plot the ratio $v(m)/D(m)$ versus $m$. For this data, we take the initial magnetization to be zero, and define the initial average ranks of voters in the $+$ and $-$ states to be equal. The quantity $w(m \to m + \delta m)$ is measured as the probability that the magnetization of the system increases from $m$ to $m + \delta m$. The important feature is the non-zero drift velocity that drives the population *away* from consensus and ultimately leads to a long consensus time. Empirically, we also find that the curves of $v/D$ for different $N$ all collapse onto a single universal curve when the data is scales by $\sqrt{N}$ (inset to figure 7). The resulting scaled curve has a sigmoidal shape that is turned on its side. We find, therefore, that this curve is well fit by the archetypal sigmoidal function $f(m) = -0.65 \tanh^{-1} m$, where the amplitude 0.65 gives the minimum deviation between the data for $v/m$ and the fit.

### 5.2. Consensus time and exit probability

Because the drift velocity drives the system away from consensus, we anticipate that the consensus time will grow faster than a power law in $N$, as shown in figure 8. For this data, the initial magnetization is set to $m = 0$ and the voter ranks are chosen so that the average ranks of $+$ and $-$ voters are, on average, equal. The data in this figure indicate that $T_N$ grows faster than a power law in $N$. There is also an extremely slow crossover to the asymptotic behavior (inset to figure 8(a)) and it is not possible to determine the functional form of $T_N$ based on simulation data up to $N = 1024$.

To give a more principled and reliable estimate for the $N$ dependence of $T_N$, we write the backward Kolmogorov equation for the consensus time [8, 42]

$$T_N(m) = w(m \to m + \delta m) T_N(m + \delta m) + w(m \to m - \delta m) T_N(m - \delta m) + \delta t. \tag{8a}$$



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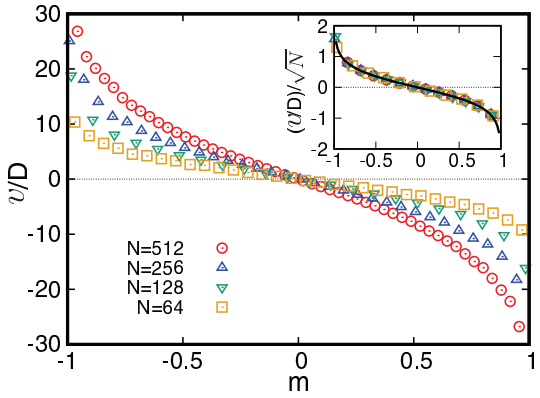
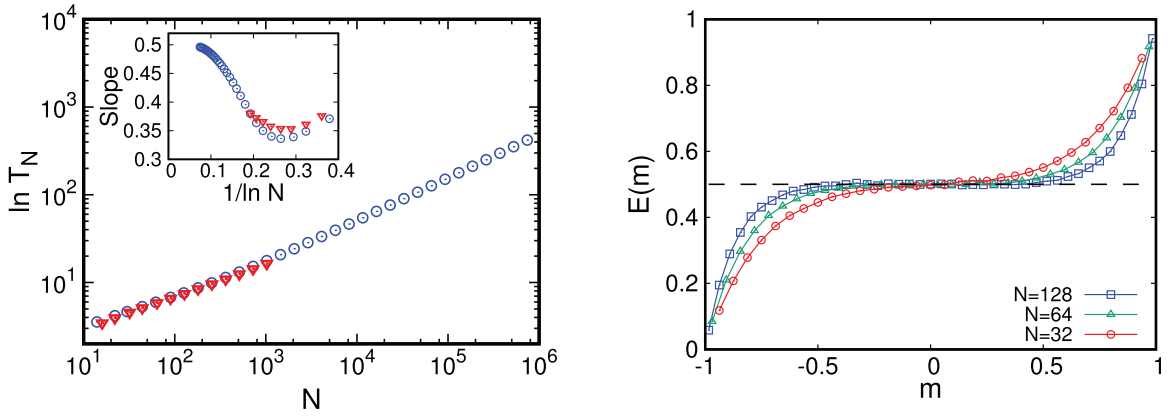



**Figure 7.** Dependence of $v(m)/D(m)$ for the RVM on $m$ for different $N$. The inset shows the data collapse when $v/D$ is divided by $\sqrt{N}$. The solid curve is the empirical fit $f(m) = -0.65 \tanh^{-1} m$ (see text). The data represent averages over $10^4$ realizations.




(a) (b)

**Figure 8.** (a) Dependence of $\ln T_N$ versus $N$ on a double logarithmic scale based on: (i) $10^4$ realizations of the RVM (red triangles), and (ii) numerical integration of equation (9) (blue circles). The inset shows the local slopes of these two datasets as a function of $1/\ln N$. (b) Exit probability of the RVM as a function of initial magnetization $m$ for different $N$. These data are based on $10^5$ realizations.

In the continuum limit this recursion becomes [8, 42]

$$\frac{v(m)}{D(m)}\frac{\partial T_N(m)}{\partial m} + \frac{\partial^2 T_N(m)}{\partial m^2} = -\frac{1}{D(m)}. \tag{8b}$$

For arbitrary functional forms of $v(m)$ and $D(m)$, the formal solution of (8$b$) is [42]

$$T_N(m) = \int_m^1 \mathrm{e}^{-A(m')} \left[ \int_0^{m'} \frac{\mathrm{e}^{A(m'')}}{D(m'')} \mathrm{d}m'' \right] \mathrm{d}m', \tag{9}$$

where $A(m) = \int_0^m [v(m')/D(m')]\mathrm{d}m'$. While it is generally not possible to solve (9) analytically, we can numerically integrate this equation. Here, we use our empirical observation that $v(m)/D(m) = \sqrt{N} f(m)$, with $f(m) = -0.65 \tanh^{-1} m$ (inset to figure 7).



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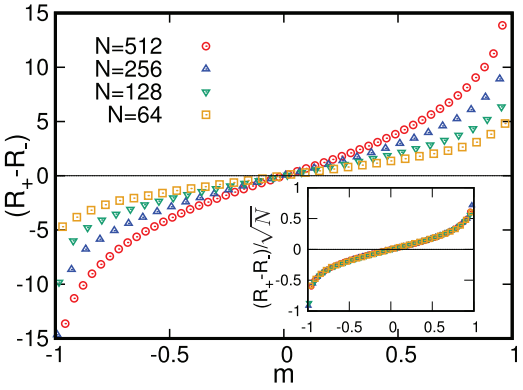


**Figure 9.** Difference between the average rank of + and − voters, $R_+ - R_-$, as a function of $m$ for different $N$. The inset shows the data collapse when $R_+ - R_-$ is scaled by $\sqrt{N}$. The data represent averages over $10^4$ realizations.

The outcome of this numerical integration for $N$ up to $10^6$ is also shown in figure 8(a). The simulation data and the integration data are nearly the same, and the local slopes of these two datasets show similar behaviors. However, since we can obtain integration data up to $N = 10^6$, we can now see the asymptotic trend in the local slope, which indicates that the local slope eventually converges to $\frac{1}{2}$ (inset to this figure). Thus we argue that the consensus time for the RVM has the dependence $T_N \sim \exp(\sqrt{N})$.

Due to the non-zero drift velocity in the RVM, the magnetization is not conserved, a feature that again manifests itself in the non-linear dependence of the exit probability $E(m)$ on initial magnetization (figure 8(b)). We again define the initial state so that the ranks of + and − voters are equal, on average. As $N$ increases, $E(m)$ gradually approaches a step function; this is a consequence of $v(m)/D(m)$ being an increasing function of $N$. The step-like form of $E(m)$ is also consistent with the consensus time growing faster than any power law in $N$ as shown in figure 8(a).

## 6. Summary and discussion

We studied a set of voter-like models on the complete graph, corresponding to the mean-field limit, in which each voter has a characteristic fitness that is a measure of its influence on others. Our motivation for investigating these models is that, in real life, some individuals are influential and other less so; moreover, the influence of each individual can change with time as opinions evolve. While our models are highly stylized, perhaps they provide a useful first step to understand the role of individual persuasiveness on how opinions change in a population.

For the fitness voter model (FVM), where the fitness of each voter is distinct and fixed, a simple, but striking result is that its voting dynamics turns out to be identical to the classic VM. Our main focus was on voter models in which the fitness of each voter, as well as its opinion, can change in an elemental update event. We found that the coupled dynamics of the fitness and voting state of each voter leads to rich dynamics and also to very slow and subtle crossover effects. This type of coupled dynamics between voting state and fitness also shares some conceptual commonality with voter



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models in which the connections between voters can change as their opinions in each update [28–34].

We investigated two examples in which changing individual voter fitness controls the consensus dynamics. In the adaptive voter model (AVM), the fitness of the influencer voter increases by a fixed amount while the fitness of the influenced voter is unchanged in a single voting event. This same model was recently investigated in [34], where it was reported that the consensus time $T_N \sim N^\alpha$, with $\alpha \approx 1.45$. We argued instead that the dynamics of the AVM is more subtle than this simple power law. In particular, the dynamics has a non-stationary character, in which the fitness distribution of the population broadens in time. Eventually, the width of this distribution broadens to the point where fitness updates no longer change the relative ranks of individual voters. When this occurs, the opinion dynamics slowly crosses over to that of the FVM, which in turn is the same as the classic VM. This crossover is interrupted by consensus, and the dependence of $T_N$ on $N$ appears to be a power law, but with an exponent that is smaller than 1.45 (figure 2).

We also introduced the reputational voter model (RVM), which has the advantage that its dynamics is stationary. In the RVM, each voter is assigned a unique integer-valued rank that ranges from 1, for the best-ranked voter, to $N$, for the worst-ranked voter. In an update, the influencer voter moves up by 1 in rank while the rank of the influenced voter is unchanged. A salient feature of this dynamics is that the voters in the minority opinion tend to be higher ranked than those with the majority opinion.

Consider a single update event, in which a voter with a $-$ opinion is converted to $+$. For this to occur, the reputation of this $-$ voter must be lower than the $+$ voter. After this update, the average rank of the $+$ voters becomes a bit worse: the influencer voter moves up by one rank, but the influenced voter, whose rank is typically much lower, now joins the list of $+$ voters. Concomitantly, the $-$ voters have lost one voter whose average rank is low, so that the average rank of this group improves. We have not been able to go beyond this heuristic observation to compute the magnitude of the rank difference as a function of the magnetization. Nevertheless, the trend from figure 9 is clear: for nonzero $m$, the minority voters are better ranked and for fixed $m$ this rank difference appears to grow as $\sqrt{N}$ (inset to figure 9). This rank difference is the mechanism underlying the drift velocity that drives the system away from consensus. The primary consequence of this bias is that $T_N$ grows faster than a power law in $N$ and the numerical evidence suggests that $T_N \sim \exp(\sqrt{N})$ (figure 8(a)).

There are multiple ways in which fitness, or rank changes of voters can be implemented; we only treated the case where the influencer voter becomes 'stronger', while the influenced voter is not affected. It is also natural to consider the cases where: (i) the influencer voter becomes stronger and the influenced voter becomes weaker, and (ii) the influencer voter is unaffected and only the influenced voter becomes weaker. In case (i), simulations indicate that the dynamical behavior is similar to the situation where only the influencer voter becomes stronger. In case (ii), however, the dynamics appears to be in the same universality class as the VM. Namely, the consensus time $T_N \sim N$ and the exit probability $E(m) = (1 + m)/2$. The latter behavior arises because the highest-ranked voter does not change its opinion throughout the dynamics, a situation that also arises in the FVM.

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## Acknowledgments

We gratefully acknowledge financial support from NSF grant DMR-1608211.

## Appendix A. Equivalence between the FVM and the VM

For notational simplicity, let $L$ and $M = N - L$ denote the number of voters with $+$ and $-$ opinions, respectively. We define the probability for $L$ to increase by 1 in a time step $\delta t$ as $\phi(L \to L+1)$ and $\phi(L \to L-1) = 1 - \phi(L \to L+1)$ for the probability for $L$ to decrease by 1. We also define $f$ and $g$ to represent the fitness of voters with $+$ and $-$ opinions, respectively. We define a voter configuration as

$$C(\mathbf{f}, \mathbf{g}, L+1) \equiv \{(f_1, f_2, \ldots f_{L+1})(g_1, g_2, \ldots g_{M-1})\} \tag{A.1}$$

in which there are $L+1$ voters with $+$ opinion (left set in $C$) and $M-1$ voters with $-$ opinion (right set). In each set, we order the individual fitnesses so that $f_i < f_{i+1}$ and $g_i < g_{i+1}$. When the number of $+$ voters increases from $L$ to $L+1$, the system moves from one of $\mathcal{N}(L) = \binom{N}{L}$ configurations to one of $\mathcal{N}(L+1) = \binom{N}{L+1}$ configurations.

We focus on one such event in which $L$ increases and thereby system ends in the configuration $C(\mathbf{f}, \mathbf{g}, L+1)$ specified in equation (A.1). In this event, one out the $L+1$ voters which are currently in $+$ opinion set of $C(\mathbf{f}, \mathbf{g}, L+1)$, must have left $-$ opinion set of the previous configuration. Let $f_i$ $(1 \leqslant i \leqslant L+1)$ be fitness of this relocated voter and denote $C_i(\mathbf{f}', \mathbf{g}', L)$ as the previous configuration from which system reached $C(\mathbf{f}, \mathbf{g}, L+1)$. It is important to note: (a) the difference between $C(\mathbf{f}, \mathbf{g}, L+1)$ and $C_i(\mathbf{f}', \mathbf{g}', L)$ is only the opinion of the voter with fitness $f_i$; (b) in both configurations, the number of $+$ opinion voters whose fitness is larger than $f_i$ is $L+1-i$. Using these two facts, the probability for the system to move from $C_i(\mathbf{f}', \mathbf{g}', L)$ to $C(\mathbf{f}, \mathbf{g}, L+1)$ is given by

$$h_i = \frac{1}{\mathcal{N}(L)} \times \frac{1}{M} \times \frac{L+1-i}{L}. \tag{A.2}$$

The first factor is the probability for the system to be in one out of $\mathcal{N}(L)$ possible configurations. The second factor is the probability that the voter with fitness $f_i$ is picked from $M = N - L$ elements in the group of $-$ voters. The third factor is the probability to pick a voter in the $+$ set in $C_i(\mathbf{f}', \mathbf{g}', L)$ whose fitness larger than $f_i$.

This probability $h_i$ should be summed over all $1 \leqslant i \leqslant L+1$ to enumerate all possibilities that lead to $C(\mathbf{f}, \mathbf{g}, L+1)$. Thus the probability to reach $C(\mathbf{f}, \mathbf{g}, L+1)$ from all eligible configurations by an event in which $L$ increases is

$$\sum_{i=1}^{L+1} h_i = \frac{1}{\mathcal{N}(L)} \frac{L+1}{2M}. \tag{A.3}$$

This probability is independent of the fitness of the voters in $C(\mathbf{f}, \mathbf{g}, L+1)$. Therefore, in an event in which $L$ increases by 1, any one out of $\mathcal{N}(L+1)$ configurations can be reached with the probability given in equation (A.3).

In equation (A.2), we assumed that all configurations with $L$ voters in the $+$ voting state have the same probability $1/\mathcal{N}(L)$. This assumption is justified because, at time



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$t = 0$, all $\mathcal{N}(L = N/2)$ configurations are chosen with equal probability. Because of equation (A.3), at any later time all $\mathcal{N}(L)$ configurations with fixed $L$ are visited by the system an equal number of times, on average.

In equation (A.3) we found the probability to reach one out of $\mathcal{N}(L+1)$ configurations by an event in which $L$ increases. To obtain $\phi(L \to L+1)$ we must sum over all possibilities that result in all $\mathcal{N}(L+1)$ different configurations. Because each of these probabilities are same (equation (A.3)), we have

$$\phi(L \to L+1) = \sum_{i=1}^{L} h_i \; \mathcal{N}(L+1) = \frac{1}{\mathcal{N}(L)} \frac{L+1}{2M} \mathcal{N}(L+1) = \frac{1}{2}. \tag{A.4}$$

That is, the probabilities for the number of + opinion voters to increase and decrease in a single time step are equal, i.e. $\phi(L \to L+1) = \phi(L \to L-1) = 1/2$. These are the same as the transition probabilities in the VM, which establishes the equivalence between the FVM and the classic VM.

## Appendix B. Zero crossings statistics of the magnetization in the VM

For the VM, we want to determine: (a) the conditional time $T_-(m)$ for the population to start at magnetization $m = 0$ and return to $m = 0$ without reaching consensus, and (b) the conditional time $T_+(m)$ to start at $m = 0$ and reach consensus without return. These conditional exit times satisfy the backward Kolmogorov equations

$$D(m) \frac{\mathrm{d}^2 \big[E_\pm(m) T_\pm(m)\big]}{\mathrm{d}m^2} = -E_\pm(m), \tag{B.1}$$

subject to the boundary conditions: $E_\pm(0)T_\pm(0) = E_\pm(1)T_\pm(1) = 0$. Here $E_\pm(m)$ are the exit probabilities to $m = 0$ and $m = 1$ and are given by $E_+(m) = m$ and $E_-(m) = 1 - m$.

The solutions to (B.1) are

$$\begin{aligned} E_+(m)T_+(m) &= N\big[(1+m)\ln(1+m) - (1-m)\ln(1-m) - 2m\ln 2\big], \\ E_-(m)T_-(m) &= 2N\big[2m\ln 2 - (1+m)\ln(1+m)\big]. \end{aligned} \tag{B.2}$$

To obtain the escape and return times, we consider the initial condition $m = 2/N$, which is the outcome after a single voting event (where a $+-$ pair changes to $++$) and include the time increment to go from $m = 0$ to $m = 2/N$. Then from (B.2), the escape time is

$$\begin{aligned} \tau_e &= \frac{4}{N} + T_+\left(\frac{2}{N}\right) \\ &= \frac{4}{N} + \frac{N^2}{2}\left[\left(1+\frac{2}{N}\right)\ln\left(1+\frac{2}{N}\right) - \left(1-\frac{2}{N}\right)\ln\left(1-\frac{2}{N}\right) - \frac{4}{N}\ln 2\right] \\ &\approx 2N(1-\ln 2) + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned} \tag{B.3a}$$

https://doi.org/10.1088/1742-5468/ab190c



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Similarly, the return time, equivalent to the zero-crossing time, is

$$\begin{aligned}\tau_0 &= \frac{4}{N} + T_-\left(\frac{2}{N}\right)\\ &= \frac{4}{N} + \frac{2N}{1-\frac{2}{N}}\left[\frac{4}{N}\ln 2 - \left(1+\frac{2}{N}\right)\ln\left(1+\frac{2}{N}\right)\right]\\ &\approx 4(2\ln 2 - 1) + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned} \tag{B.3$b$}$$

Note that the average escape time $\tau_e$ is $\mathcal{O}(N)$, while the average return time $\tau_0$ is $\mathcal{O}(1)$.

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