Depletion-Controlled Starvation of a Diffusing Forager

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We study the starvation of a lattice random walker in which each site initially contains one food unit and the walker can travel $\mathcal{S}$ steps without food before starving. When the walker encounters food, the food is completely eaten, and the walker can again travel $\mathcal{S}$ steps without food before starving. When the walker hits an empty site, the time until the walker starves decreases by 1. In spatial dimension $d = 1$, the average lifetime of the walker $\langle\tau\rangle \propto \mathcal{S}$, while for $d > 2$, $\langle\tau\rangle \simeq \exp(\mathcal{S}^\omega)$, with $\omega \to 1$ as $d \to \infty$. In the marginal case of $d = 2$, $\langle\tau\rangle \propto \mathcal{S}^z$, with $z \approx 2$. Long-lived walks explore a highly ramified region so they always remains close to sources of food and the distribution of distinct sites visited does not obey single-parameter scaling.

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Searching for a randomly located resource is an essential task of all living organisms [1–9]. Examples include searching for nourishment, an abode, or a particular individual. Stochastically driven search processes also underlie diffusion-controlled reactions [10] and a variety of physiological processes [11]. In all these examples, the time for a successful search is the typical metric that the organism is trying to optimize. A related aspect of stochastic search is the tradeoff between continued exploitation of a familiar resource or the exploration of new domains for potentially more fruitful resources [12].

Motivated by the rich phenomenology of stochastic search, we introduce a minimal search model that is based on the as yet unexplored feature of *depletion*. In our model, a random walker gradually depletes the resource contained in a medium—here a d-dimensional lattice with a unit of food initially at each site—as it moves. The walker has an intrinsic starvation time $\mathcal{S}$, defined as the number of steps it can take without encountering food before starving to death. If the walker encounters a food-containing site, the walker instantaneously and completely consumes the food and can again travel $\mathcal{S}$ additional steps without eating before starving. Each time the walker encounters an empty site, it comes one time unit closer to starvation.

We focus on two key observables of starving random walks: the average lifetime $\langle\tau\rangle$ and the average number $\langle\mathcal{N}\rangle$ of distinct sites visited when starvation occurs. In dimension $d = 1$, $\langle\tau\rangle \propto \mathcal{S}$ and we determine the distribution of $\mathcal{N}$ at starvation, $P(\mathcal{N})$. For $d > 2$, the transience of the random walk leads to τ scaling as $\exp(\mathcal{S}^\omega)$, with $\omega \to 1$ as $d \to \infty$. When successive visits to new sites are uncorrelated, corresponding to $d = \infty$, we find $\tau \sim e^{k\mathcal{S}}$, with k exactly calculable. In $d = 2$, numerical simulations suggest that $\tau \simeq \mathcal{S}^z$, with $z \approx 2$. We develop a mean-field approximation for $d = 2$ that gives a rigorous lower bound for τ and suggests that $P(\mathcal{N})$ does not obey single-parameter scaling, as seen in our simulations.

Our mortality mechanism differs from previous models in which a random walker can die or be absorbed at a fixed rate, independent of its location [13–16]. Here, the lifetime distribution of a random walker is not given *a priori* but is generated by the random-walk trajectory, which renders the problem highly non-trivial. It is worth mentioning the related problem of the "excited" random walk, in which the hopping of the walker depends on whether it has just encountered food or an empty site [17–22]. While the excited random walk has surprising behavior, we will show that even when the motion of the walker is not explicitly affected by the environment unusual properties arise.

One dimension. As the walker moves, an interval devoid of food—a desert—is gradually carved out, and the survival of the walker is controlled by the interplay between wandering within the desert and reaching food at the edge of this desert (Fig. 1). For a starving random walker to survive for times beyond its intrinsic lifetime $\mathcal{S}$, excursions of more than $\mathcal{S}$ steps without food cannot occur in its past history. Thus a long-lived walk must spend less time wandering in the interior of a desert than unrestricted walks, and the mean number of distinct sites visited should be larger than that for unrestricted random walks for the same number of steps N; the latter asymptotically scales as $\sqrt{8N/\pi}$ [23, 24].

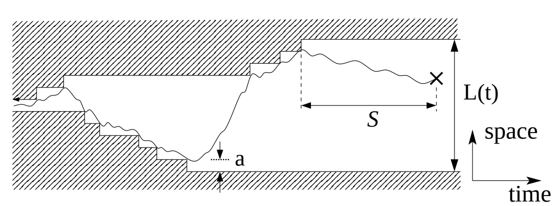

FIG. 1: A $d = 1$ starving random walker clears out an interval where food (shaded) has been eaten. The walker starves ($\times$) when it travels $\mathcal{S}$ steps without encountering food.

We first determine $P(\mathcal{N})$, the probability that a random walker has visited $\mathcal{N}$ distinct sites when it starves. This probability can be expressed as

$$P(\mathcal{N}) = Q_2\, Q_3\, Q_4 \ldots Q_{\mathcal{N}}(1 - Q_{\mathcal{N}+1})\,, \tag{1}$$

where $Q_j = \int_0^{\mathcal{S}} dt\, F_j(t)$, and $F_j(t)$ is the probability that the walker reaches either end of an interval of length ja (with a the lattice spacing) within $\mathcal{S}$ steps when starting a distance a from one end. Each Q_j accounts for the interval growing from length $j-1 \to j$ because the walker reaches either endpoint within $\mathcal{S}$ steps, while the factor $Q_{\mathcal{N}+1}$ accounts for the last excursion in which the walker starves.

In the long-time limit, we use the continuum expression for $F_j(t)$ [25]:

$$F_j(t) = \frac{4\pi D}{(ja)^2} \sum_{n\geq 0} (2n+1)\, e_n(j/\sqrt{t})\, \sin\lambda_{j,n}\,, \tag{2}$$

where $D \equiv a^2/2$ is the diffusivity of the corresponding continuous process, $\lambda_{j,n} \equiv (2n+1)\pi/j$, and $e_n(j/\sqrt{t}) \equiv \exp\left[-(2n+1)^2\pi^2 Dt/(ja)^2\right]$. Consequently,

$$Q_j = \int_0^{\mathcal{S}} dt\, F_j(t) = 1 - \frac{4}{\pi} \sum_{n\geq 0} \frac{e_n(j/\sqrt{\mathcal{S}})\, \sin\lambda_{j,n}}{2n+1}\,. \tag{3}$$

This expression for Q_j, together with Eq. (1), provides a formal solution for the distribution of the number of distinct sites visited at the starvation time.

To obtain the explicit result, we start by taking the logarithm of $U_{\mathcal{N}} \equiv \prod_{2\leq j\leq \mathcal{N}} Q_j$:

$$\ln U_{\mathcal{N}} = \sum_{2\leq j\leq \mathcal{N}} \ln\left[1 - \frac{4}{\pi} \sum_{n\geq 0} \frac{e_n(j/\sqrt{\mathcal{S}})\, \sin\lambda_{j,n}}{2n+1}\right]. \tag{4}$$

We now convert the outer sum to an integral, introduce the variable $z \equiv j/\sqrt{\mathcal{S}}$, and, for $\mathcal{S} \gg 1$, replace $\sin\lambda_{j,n}/(2n+1)$ by π/j. These steps give

$$\ln U_{\mathcal{N}} \simeq \sqrt{\mathcal{S}} \int_0^{\mathcal{N}/\sqrt{\mathcal{S}}} dz\, \ln\left[1 - \frac{4}{z\sqrt{\mathcal{S}}} \sum_{n\geq 0} e_n(z)\right]. \tag{5}$$

Expanding the logarithm, which applies for large $\mathcal{S}$, gives

$$\begin{aligned} \ln U_{\mathcal{N}} &\simeq -4 \int_0^{\mathcal{N}/\sqrt{\mathcal{S}}} \frac{dz}{z} \sum_{n\geq 0} e_n(z), \\ &= -2 \sum_{n\geq 0} \mathrm{E}_1\left[(2n+1)^2/\theta^2\right], \end{aligned} \tag{6}$$

where $\theta = a\mathcal{N}/(\pi\sqrt{D\mathcal{S}})$ is the scaled number of distinct sites visited and E_1 is the exponential integral, which is defined as $E_1(x) \equiv \int_1^\infty dt\, e^{-xt}/t$.

Similarly, the factor $1 - Q_{\mathcal{N}+1}$ in (1) has the limiting behavior for $\mathcal{S} \gg 1$

$$1 - Q_{\mathcal{N}+1} \simeq \sum_{n\geq 0} \frac{4a}{\pi\theta\sqrt{D\mathcal{S}}}\; e^{-(2n+1)^2/\theta^2}\,, \tag{7}$$

so that the distribution of θ is

$$P(\theta) = \frac{4}{\theta} \sum_{n\geq 0} \exp\left\{-\frac{(2n+1)^2}{\theta^2} - 2\sum_{n\geq 0} \mathrm{E}_1\left[\frac{(2n+1)^2}{\theta^2}\right]\right\}. \tag{8}$$

From this result, the average number of visited sites at the starvation time is

$$\langle \mathcal{N} \rangle \simeq \frac{\pi\sqrt{D\mathcal{S}}}{a} \int_0^\infty \theta\, P(\theta)\, d\theta \;\approx\; A\sqrt{\mathcal{S}}\,, \tag{9}$$

with $A \approx 2.90222$. As mentioned above, this exceeds the number of distinct visited for the unrestricted random walk after $N = \mathcal{S}$ steps, where the amplitude is $\sqrt{8/\pi} \approx 1.5957\ldots$.

The average lifetime $\langle\tau\rangle$ of starving random walks is formally given by

$$\langle\tau\rangle = \sum_{j\geq 1} \Big(\langle\tau_1\rangle + \langle\tau_2\rangle + \cdots + \langle\tau_j\rangle + \mathcal{S}\Big) P(j)\,, \tag{10}$$

where $\langle\tau_j\rangle$ is the average time for the random walk to hit either end of the interval in the j^{th} excursion, conditioned on the walker hitting either end before it starves, while the factor $\mathcal{S}$ accounts for the final excursion that causes the walker to starve. By definition

$$\langle\tau_j\rangle = \frac{\int_0^{\mathcal{S}} dt\, t\, F_j(t)}{\int_0^{\mathcal{S}} dt\, F_j(t)}\,, \tag{11}$$

The numerator, defined as N_j, reduces to

$$N_j = \frac{4\pi D}{(ja)^2} \sum_{n\geq 0} (2n+1) \sin\lambda_j(n) \int_0^{\mathcal{S}} dt\; t\; e^{-\beta t}\,, \tag{12}$$

with $\beta \equiv \lambda_{j,n}^2/(ja)^2$.

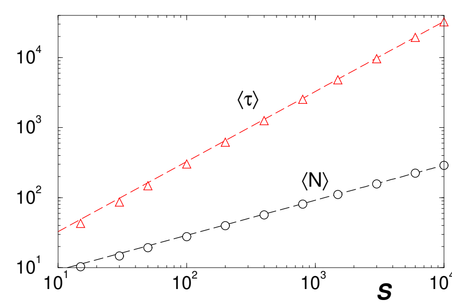

FIG. 2: Average number of distinct sites visited $\langle N\rangle$ ($\circ$) and average lifetime $\langle\tau\rangle$ ($\triangle$) for 10^6 realizations of starving random walks in one dimension at the starvation time versus $\mathcal{S}$. The dashed lines are the respective asymptotic predictions of $\langle\mathcal{N}\rangle \sim 2.90222\sqrt{\mathcal{S}}$ and $\langle\tau\rangle \sim 3.26786\; \mathcal{S}$.

For large j, we again approximate the sine function by its argument and perform the temporal integral to give

$$N_j \simeq \frac{4a^2}{\pi^2 D} \sum_{n\geq 0} \frac{1}{(2n+1)^2} \left[1 - e^{-\beta\mathcal{S}}(1 + \beta\mathcal{S})\right]. \tag{13}$$

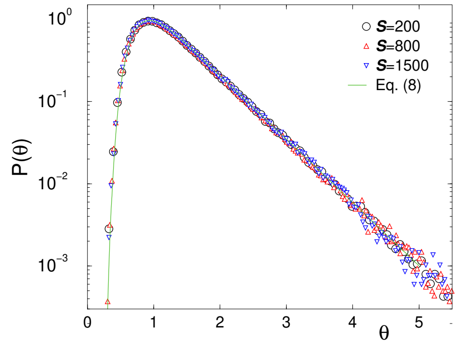

FIG. 3: The scaled distribution of number of distinct sites visited for three representative values of $\mathcal{S}$. The curve is the theoretical prediction from Eq. (8) and the data are based on 10^6 walks for each value of $\mathcal{S}$.

Using this in Eq. (11) leads to $\langle\tau\rangle \simeq 3.26786\,\mathcal{S}$. These results for the number of distinct sites visited and the lifetime agree with numerical simulations shown in Fig. 2.

Large Dimensions. When $d > 2$, a random walk is transient, so that new sites are visited at a non-zero rate [23–25]. Thus a walker is unlikely to first create a local desert and then wander strictly within this desert until it starves, so that its survival time should be much longer than in $d = 1$ at the same value of $\mathcal{S}$.

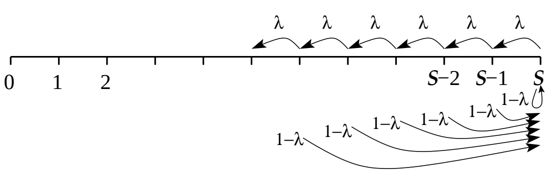

FIG. 4: Evolution of a high-dimensional random walk in starvation space. A particle at position n in this space can survive n additional steps without encountering food.

For $d = \infty$, the probability of hitting a previously visited or a previously unvisited site at each step follows a Poisson process with respective rates that we define as λ and $1-\lambda$. Schematically, the time until the walker starves undergoes one-dimensional hopping in "starvation space"—an interval of length $\mathcal{S}$—and starvation occurs when 0 is reached (Fig. 4). A particle at site n in starvation space can wander in physical space another n steps without encountering food before it starves. When the walker hits a previously visited site in physical space, the time to starvation decreases by one time unit, corresponding to a hop to the left with rate λ in starvation space. Conversely, when the random walker encounters a new food-containing site, it can wander an additional $\mathcal{S}$ steps until starvation occurs, corresponding to a long-range rightward hop to site $\mathcal{S}$ in starvation space with rate $1-\lambda$.

Using this equivalence to hopping in starvation space, we now compute t_n, the average time until the walker starves when starting from site n. These starvation times satisfy the recursions [25]

$$\begin{aligned} t_n &= 1 + \lambda t_{n-1} + (1-\lambda)t_{\mathcal{S}} \qquad 2 \le n \le \mathcal{S}\,,\\ t_1 &= 1 + (1-\lambda)t_{\mathcal{S}}\,, \end{aligned} \tag{14}$$

from which

$$t_{\mathcal{S}} = \langle\tau\rangle = \frac{1}{\lambda^{\mathcal{S}}}\left(\frac{1-\lambda^{\mathcal{S}}}{1-\lambda}\right). \tag{15}$$

The high-dimensional limit corresponds to $\lambda \to 0$, for which the average starvation time grows exponentially with $\mathcal{S}$, in contrast to the linear dependence found in $d = 1$.

We may also obtain the distribution of the number of distinct sites visited by the walker in physical space at the starvation time. For this quantity, we need the probability R_n that the walk reaches $\mathcal{S}$ without first hitting 0, when starting from site n in starvation space. Each such return corresponds to the random walker visiting a new site in physical space without starving. These return probabilities satisfy the recursions

$$\begin{aligned} R_n &= (1-\lambda) + \lambda R_{n-1} \qquad 2 \le N \le \mathcal{S}\,,\\ R_1 &= (1-\lambda)\,, \end{aligned} \tag{16}$$

with solution $R_n = (1-\lambda)^n$.

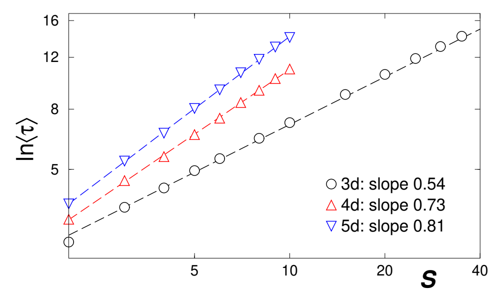

FIG. 5: Logarithm of the average starvation time $\langle\tau\rangle$ versus $\mathcal{S}$ for starving random walks in $d = 3$, 4, and 5. Data are based on 10^6 for each value of $\mathcal{S}$.

For a walker that starts at site $\mathcal{S}$ in starvation space, the probability that $\mathcal{N}$ distinct sites are visited before the walker starves is given by

$$P(\mathcal{N}) = (R_{\mathcal{S}})^{\mathcal{N}}(1-R_{\mathcal{S}})\,, \tag{17}$$

from which the average number of distinct sites visited before the walker starves is

$$\langle\mathcal{N}\rangle = \frac{R_{\mathcal{S}}}{1-R_{\mathcal{S}}} = \lambda^{-\mathcal{S}}(1-\lambda^{-\mathcal{S}})\,, \tag{18}$$

which is the same as $\langle\tau\rangle$ from Eq. (15), except for the factor of $1-\lambda$.

Numerical simulations for $d > 2$ give $\ln\langle\tau\rangle \sim \mathcal{S}^{\omega}$ with $\omega = 0.54$, 0.73, and 0.81 in $d = 3, 4, 5$ (Fig. 5). Even though a random walk is transient for $d > 2$, so that there is a finite rate of visiting new sites, temporal correlations between successive visits to new sites lead to $\langle\tau\rangle$ not conforming to the mean-field result given in Eq. (15).

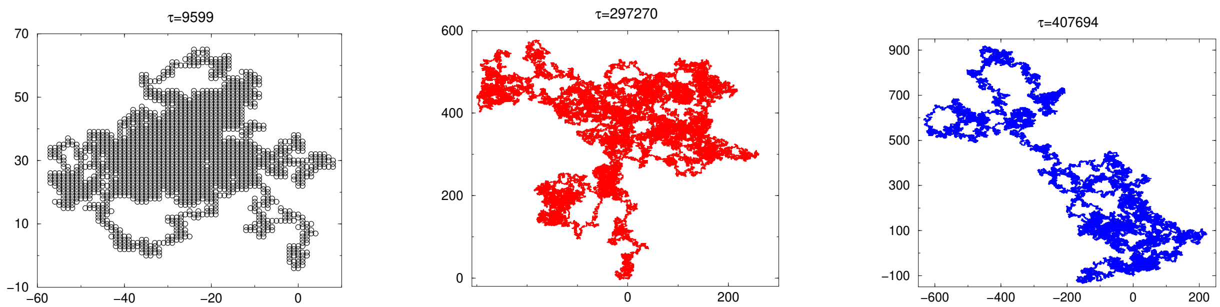

FIG. 6: Example two-dimensional trajectories for $\mathcal{S} = 500$. The lifetime τ of each walk is indicated.

Two Dimensions. The two-dimensional system is enigmatic because it is at the critical dimension between recurrence and transience, and it is more relevant to model the movement of an animal that wanders in an ecosystem to find food. Sample random-walk trajectories are shown in Fig. 6 for intrinsic lifetime $\mathcal{S} = 500$. The trajectories of short-lived walks are compact, while those of long-lived walks are quite stringy so that the walker remains close to food-containing sites.

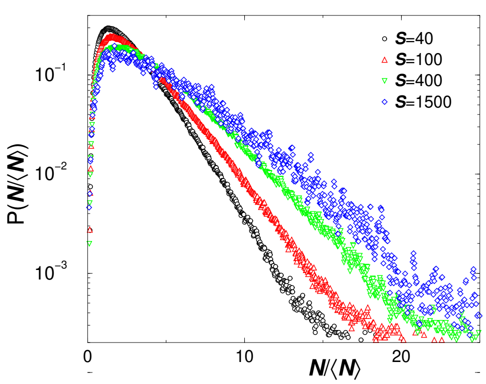

FIG. 7: The scaled distribution of number of distinct sites visited for 4 representative value of $\mathcal{S}$. The data have been averaged over a 15-point range and only every fifth data point is displayed. All data are based on 10^6 for each value of $\mathcal{S}$.

An unexpected feature in two dimensions is that the underlying distribution of $\mathcal{N}$ does not satisfy single-parameter scaling (Fig. 7). We have developed a mean-field description for the evolution of starving random walks in two dimensions that relies on the assumption that the desert remains circular at all times [26]. While this assumption is unrealistic, this approach provides rigorous lower bounds for both the average number of distinct sites visited at starvation, $\langle\mathcal{N}\rangle \propto \mathcal{S}/\ln\mathcal{S}$, and for the average lifetime, $\langle\tau\rangle \propto \mathcal{S}^{3/2}/(\ln\mathcal{S})^3$. This theory also predicts that $P(\mathcal{N})$ does not satisfy single-parameter scaling, as observed in our simulations.

To summarize, starving random walks represents a minimalist description of the consumption of a depleting resource by a stochastic searcher. The motion of the walker is limited by the number of steps $\mathcal{S}$ that it can take without encountering food before starving. The spatial dimensionality plays a crucial role in the dynamics, as the lifetime of a starving random walker grows faster than algebraically in $\mathcal{S}$ for $d > 2$ and algebraically with $\mathcal{S}$ for $d \leq 2$. We also obtained comprehensive results for the starvation dynamics in $d = 1$. The two-dimensional case is quite challenging, as the distribution of the distinct sites visited does not seem to obey scaling, and the region visited by the random walker is spatially complex.

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