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Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

On the role of zealotry in the voter model

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Received 21 June 2007
Accepted 2 August 2007
Published 23 August 2007

Online at stacks.iop.org/JSTAT/2007/P08029
doi:10.1088/1742-5468/2007/08/P08029

Abstract. We study the voter model with a finite density of zealots—voters that never change opinion. For equal numbers of zealots of each species, the distribution of magnetization (opinions) is Gaussian in the mean-field limit, as well as in one and two dimensions, with a width that is proportional to $1/\sqrt{Z}$, where Z is the number of zealots, independent of the total number of voters. Thus just a few zealots can prevent consensus or even the formation of a robust majority.

Keywords: interacting agent models, scaling in socio-economic systems, probability theory

ArXiv ePrint: 0706.2892

J. Stat. Mech. (2007) P08029

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1. Introduction

The voter model [1] is one of the simplest examples of cooperative behavior that has been used as a paradigm for the dynamics of opinions in socially interacting populations. In the voter model, each node of a graph is occupied by a voter that has two opinion states, denoted as + and −. Opinions evolve by: (i) picking a random voter; (ii) the selected voter adopts the state of a randomly chosen neighbor; (iii) repeat these steps *ad infinitum* or until a finite system necessarily reaches consensus. Naively, one can view each voter has having no self confidence and thus takes on the state of one of its neighbors. This evolution resembles that of the Ising model with zero-temperature Glauber kinetics [2], but with one important difference: in the Ising model, each spin obeys the state of the local majority; in the voter model, a voter chooses a state with a probability that is proportional to the number of neighbors in that state.

There are three basic properties of the voter model that characterize its evolution. The first is the exit probability, namely, the probability that a finite system eventually reaches consensus where all voters are in the + state, $E_+(\rho_0)$, as a function of the initial density ρ_0 of + voters. Because the mean magnetization, defined as the difference in the fraction of + and − voters (averaged over all realizations and histories), is conserved on any degree-regular graph, and because the only possible final states of a finite system are consensus, $E_+(\rho_0) = \rho_0$ [1].

A second basic property is the mean time T_N to reach consensus in a finite system of N voters. For regular lattices in d dimensions, it is known that T_N scales as N^2 in $d = 1$, as $N \ln N$ in $d = 2$, and as N in $d > 2$ [1, 3]. In contrast, T_N generally scales sublinearly

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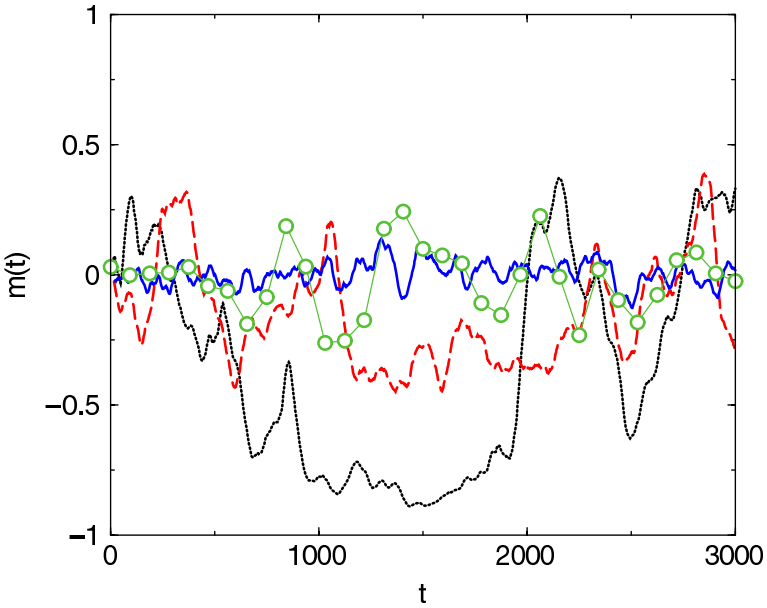

Figure 1. Time dependence of the magnetization for single realizations of 1000 voters on the complete graph for $Z = 2$ (black), 16 (red), and 128 zealots (blue). Data were smoothed over a 1% range. Also shown are US presidential election results (circles) from 1876 to 2004 (corresponding to $t = 0$ and 3000 respectively) where the magnetization is defined as the difference in the vote fraction of the top two candidates.

with N on heterogeneous graphs with broad degree distributions [4]. Defining μ_k as the kth moment of the degree distribution, then $T_N \sim N\mu_1^2/\mu_2$, which grows slower than linearly in N for a sufficiently broad degree distribution. Finally, the 2-point correlation function $G_2(r)$, defined as the probability that 2 voters a distance r apart are in the same state, asymptotically decays as r^{2-d} on a regular lattice when the spatial dimension $d > 2$ [3, 5]. This decay is the same as that of the electrostatic potential of a point charge, a correspondence that has proven useful in analyzing the voter model.

In this work, we investigate an extension of the voter model in which a small fraction of the population are zealots—individuals that never change opinion. The effect of a single zealot [6] or a small number of zealots [7] on primarily static properties of the voter model has been studied previously, and considerable insight has been gained by exploiting the previously mentioned electrostatic correspondence. The role of zealots has also been investigated in a majority rule opinion dynamics model [8], where again equal densities of zealots of each type prevent consensus from being achieved. One motivation for our work is the obvious fact that consensus is not the asymptotic outcome of repeated elections in democratic societies. One such example is the set of US presidential elections [9], where the percentage of votes for the winner has ranged from highs of 61.05% (Johnson over Goldwater 1964) and 60.80% (Roosevelt over Landon 1932) to lows of 47.80% (Harrison minority winner over Cleveland 1888) and 47.92% (Hayes over Tilden 1876). In this compilation, we exclude the votes of marginal candidates when there was substantial voting to candidates outside the top two (figure 1).

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This example, as well as election results from many democratic countries, show in an obvious way that consensus will never be achieved in large voting populations. This fact motivates us to investigate an opinion dynamics model in which consensus is stymied by the presence of zealots. Because of the competing influences of the zealots and the tendency toward consensus by the voter dynamics, the magnetization fluctuates with time in a manner that can be made to qualitatively mimic, for example, the US presidential election results (figure 1). Upon averaging over a long time period, these time-dependent fluctuations lead to a stationary magnetization distribution whose properties are the main focus of this work.

The basic question that we wish to address in the voter model with a subpopulation of zealots is: what is the nature of the global opinion as a function of the density of zealots? One of our main results is that equal but very small numbers of zealots of both types leads to a steady state with a narrow Gaussian magnetization distribution centered at zero. Here the magnetization is simply the difference in the fraction of voters of each species. Thus a small fraction of zealots is surprisingly effective in maintaining a steady state with only small fluctuations about this state.

It should also be mentioned that there are a variety of simple and prototypical opinion dynamics models, in which lack of consensus is a basic outcome, including the multiple-state Axelrod model [10], the bounded compromise model of Weisbuch *et al* [11] and its variants [12]. For these models, the consensus preventing feature typically is the absence of interaction whenever two agents become sufficiently incompatible. As a function of basic model parameters, the fraction of incompatible agents can grow, leading to cultural fragmentation and an attendant steady or static opinion state.

In the next section, we define the model. Then in sections 3 and 4, we solve the model in the mean-field limit and on a one-dimensional periodic ring. We then investigate the behavior on the square lattice by numerical simulations in section 5 and find behavior that is quantitatively close to that in the mean-field limit. Finally, we conclude and point out some additional interesting features of the role of zealotry on the voter model in section 6.

2. The model

The population consists of N voters, with a fixed number of zealots that never change opinion, while the remaining voters are susceptible to opinion change. Each voter can be in one of two opinion states, $+1$ or -1 that we term 'democrat' and 'republican', respectively. Thus the system consists of Z_+ democrat and Z_- republican zealots, as well as N_+ democrat and N_- republican susceptibles. Each type of voter evolves as follows:

(1) *susceptible democrats* can become republicans;
(2) *susceptible republicans* can become democrats;
(3) *zealot democrats* are always democrats;
(4) *zealot republicans* are always republicans.

Each agent, whether a zealot or a susceptible, has the same persuasion strength that we set to 1. That is, after a susceptible voter selects a neighbor, the voter is persuaded to adopt the state of this neighbor with probability 1. Because the total population comprises of agents in one of four possible states, we have $N = N_+ + N_- + Z_+ + Z_-$. Since the number

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of zealots is fixed, the total number of susceptible individuals $S = N - Z_+ - Z_- = N_+ + N_-$ is also conserved. The dynamics is a direct generalization of voter model and consists of the following steps:

(1) pick a random voter, if this voter is a zealot nothing happens;
(2) if the selected voter is a susceptible, then pick a random neighbor and adopt its state; note that if the selected voter and the neighbor are in the same state, nothing happens in the update;
(3) repeat steps 1 and 2 *ad infinitum* or until consensus is reached.

We will investigate this model on the two geometries of the complete graph, a natural realization of the mean-field limit, and regular lattices. For the complete graph, all other voters in the system are nearest neighbor to any voter. Thus the complete graph has no spatial structure, a feature that allows for a simple solution. In contrast, when the voters live on the sites of a regular lattice, a voter can be directly influenced only by its the nearest neighbors.

3. Dynamics on the complete graph

On the complete graph, the state of the population may be characterized by the probability $P(N_+, N_-, t)$ of finding $N_\pm$ susceptible voters at time t. Since $N_- = S - N_+$, we merely need to consider the master equation for $P(N_+, t)$, which reads

$$\frac{\partial P(N_+,t)}{\partial t} = \sum_{\delta=\pm 1} P(N_+ + \delta, t) W(N_+ + \delta \to N_+) - \sum_{\delta=\pm 1} P(N_+, t) W(N_+ \to N_+ + \delta). \quad (1)$$

The first term accounts for processes in which the number of susceptible democrats after the event equals N_+, while the second term accounts for the complementary loss processes where $N_+ \to N_+ \pm 1$. Here W represents the rate at which transitions occur and is given by

$$\begin{aligned} \delta t\, W(N_+ \to N_+ + 1) &= \frac{N_-(N_+ + Z_+)}{N(N-1)} \\ \delta t\, W(N_+ \to N_+ - 1) &= \frac{N_+(N_- + Z_-)}{N(N-1)}. \end{aligned} \quad (2)$$

The first line is the probability of choosing first a republican susceptible and then a democrat (susceptible or zealot), for which a susceptible republican converts to a susceptible democrat in the voter model interaction. We choose $\delta t = N^{-1}$, so that, on average, each agent is selected once at each time step.

While it is usually not possible to solve an equation of the form (1), analytical progress can be achieved by considering a continuum $N \to \infty$ limit of the master equation and performing a Taylor expansion [13]. For this purpose, we introduce the rescaled variables $n \equiv N_+/N$, $z_\pm = Z_\pm/N$, and also $s \equiv 1 - z_+ - z_-$ so that $s - n \equiv N_-/N$. In the continuum limit, the reaction rates now become

$$\begin{aligned} W(n \to n + N^{-1}) &= N\,(s-n)(n+z_+) \\ W(n \to n - N^{-1}) &= N\,n(s-n+z_-). \end{aligned} \quad (3)$$

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Expanding (1) to the second order in the variable n, we find the following Fokker–Planck equation [12], [14]–[17]:

$$\frac{\partial P(n,t)}{\partial t} = -\frac{\partial}{\partial n}\left[\alpha(n)P(n,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial n^2}\left[\beta(n)P(n,t)\right], \tag{4}$$

where (see e.g., chapter VII of [13])

$$\alpha(n) = \sum_{\delta n=\pm 1/N} \delta n\, W(n \to n+\delta n) = [z_+ s - n(1-s)];$$

$$\beta(n) = \sum_{\delta n=\pm 1/N} (\delta n)^2\, W(n \to n+\delta n) = [(n+z_+)(s-n) + n(s+z_- - n)]/N.$$

The first term on the right-hand side of equation (4) leads to the deterministic mean-field rate equation $\dot{n}(t) = \alpha$, with solution

$$n(t) = \frac{z_+ s}{1-s} + \left[n(0) - \frac{z_+ s}{1-s}\right] \mathrm{e}^{-(1-s)t}. \tag{5}$$

Thus an initial density of susceptible democrats in an infinite system exponentially relaxes to the steady-state value $n^* = z_+ s/(1-s)$. Correspondingly, the magnetization $m = (N_+ + Z_+ - N_- - Z_-)/N$ attains the steady-state value $(z_+ - z_-)/(z_+ + z_-)$. When the number of agents is finite, however, finite-size fluctuations arise from the diffusive second term on the right-hand side of equation (4). This term leads to a steady-state probability distribution with a finite width that is centered at n^*. In what follows, we examine these fluctuations around the mean-field steady state when N and $Z_\pm$ are both finite.

3.1. Stationary magnetization distribution

According to the Fokker–Planck equation (4), the stationary distribution $P(n)$ obeys

$$\alpha(n)P(n) - \frac{1}{2}\frac{\partial}{\partial n}\left[\beta(n)P(n,t)\right] = 0, \tag{6}$$

whose formal solution is

$$P(n) = \mathcal{Z}\,\frac{\exp\left(2\int_0^n \mathrm{d}n'\,(\alpha(n')/\beta(n'))\right)}{\beta(n)}. \tag{7}$$

Since the density n of agents in the state $+1$ ranges from 0 to s, the normalization constant $\mathcal{Z}$ is obtained by requiring $\int_0^s \mathrm{d}n\, P(n) = 1$. This condition gives

$$\mathcal{Z} = \left[\int_0^s \frac{\exp\left(2\int_0^n \mathrm{d}n'\,(\alpha(n')/\beta(n'))\right)}{\beta(n)}\,\mathrm{d}n\right]^{-1}.$$

We are particularly interested in the distribution of the magnetization $P(m)$ in the continuum limit, which directly follows from (7) through a simple change of variables. We first consider the system with the same number of zealots of each type, and then the asymmetric system with unequal numbers of zealots of each type.

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3.2. Symmetric case: $Z_+ = Z_- = Z$

When the number of zealots of each species is equal, we write $Z_+ = Z_- \equiv Z$. The rate equation (5) then gives an equal steady-state density of democrats and republicans, $n^* = n_+ = n_- = s/2$, corresponding to zero average magnetization, $m^* = 0$. We now compute the stationary distribution of magnetization by accounting for finite-size fluctuations. When $Z_+ = Z_- = Z$, $P(n)$ obeys equation (7) with

$$\alpha(n) = z(1 - 2z - 2n),$$

$$\beta(n) = [(2n + z)(1 - 2z) - 2n^2]/N.$$

Notice that $\alpha = (Nz/2)(\mathrm{d}\beta/\mathrm{d}n)$, a feature that allows us to solve for the steady-state magnetization distribution easily.

To perform the integral in equation (7), it is helpful to transform from n to the magnetization $m = (2n - s)/s$ which lies in $[-1, 1]$. We therefore find $\exp(2\int_0^n \mathrm{d}n'(\alpha(n')/\beta(n'))) = (1 + (2n(s-n)/zs))^{Nz}$. According to equation (7), this leads to the following stationary distribution of susceptible democrats:

$$P(n) = \frac{(zs + 2n(s-n))^{Nz-1}}{\int_0^s \mathrm{d}n\,(zs + 2n(s-n))^{Nz-1}}. \tag{8}$$

Using the fact that $2n(s-n) = s^2(1-m^2)/2$, we readily obtain the stationary magnetization distribution:

$$P(m) = \frac{(s^{-1} - m^2)^{Z-1}}{\int_{-1}^1 \mathrm{d}m\,(s^{-1} - m^2)^{Z-1}}. \tag{9}$$

In the limit of large Z, we may then approximate the distribution by the Gaussian $P(m) \propto \mathrm{e}^{-m^2/2\sigma^2}$, with $\sigma^2 = 1/[2s(Z-1)]$.

When zealots are present in equal numbers, the magnetization distribution quickly approaches a symmetric Gaussian, with a width that is inversely proportional to the square root of the *number* of zealots and not the density. Thus as the system size is increased, the density of zealots needed to keep the magnetization within a fixed range goes to zero. In the limiting case where there is one zealot of each type, the magnetization is uniformly distributed in $[-1, 1]$ (figure 2). Finally notice that the distribution quickly approaches the asymptotic scaling form when $Z \gtrsim 8$ (inset to figure 2).

3.3. Asymmetric case: $Z_+ \neq Z_-$

When the density of zealots of each type are unequal, we now have

$$\alpha(n) = (z_+ + n)s - n, \tag{10}$$

$$\beta(n) = [(2n + z_+)(s-n) + nz_-]/N \tag{11}$$

in equation (6). To compute $P(n)$ (and equivalently $P(m)$), it is now convenient to introduce the quantities $\delta \equiv z_+ - z_-$ and $r \equiv \sqrt{\delta^2 + 4s}$. Noticing that one can write $\alpha/\beta = [N(s-1)(\mathrm{d}\beta/\mathrm{d}n) + \delta(1+s)/4]/\beta$, one can easily compute the integral in equation (7) and thereby obtain $P(n)$. Transforming from the density to the magnetization

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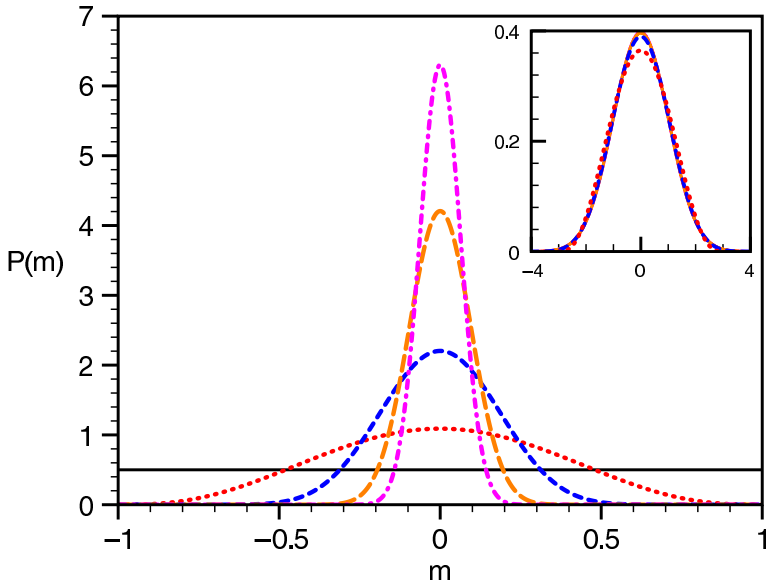

Figure 2. Steady-state magnetization distributions for 1000 voters on the complete graph for $Z = 2, 8, 32, 128$, and 512 zealots (progressively steepening curves). The inset shows the scaled form of these distributions for $Z \geq 8$; the case $Z = 8$ slightly deviates from the rest of the distributions that become visually coincident.

by $n = (m+1)s/2$, we obtain the following expression for the stationary magnetization distribution (figure 3):

$$\mathcal{Z}P(m) = [1 - m(\delta + ms)]^{(Z_+ + Z_- - 2)/2} \left[1 + \frac{r}{ms - (r-\delta)/2}\right]^{(\delta/2r)\,(2N - Z_+ - Z_-)}. \qquad (12)$$

As in the symmetric case, $\mathcal{Z}$ is a normalization constant obtained by requiring that $\int_{-1}^{1} \mathrm{d}m\, P(m) = 1$. Notice that $P(m)$ is comprised of two terms. The first term gives a Gaussian contribution (in the limit of large N) and is the analogue of equation (9). The second term is a non-trivial contribution due to the asymmetry that is responsible for the skewness of $P(m)$ which remains peaked around $m^* = (z_+ - z_-)/(z_+ + z_-)$. Close to this peak value, there is little asymmetry (i.e., $\delta \ll 1$). Additionally, for a large number of zealots we may approximate the distribution (12) by the Gaussian $P(m) \approx \mathrm{e}^{-(m-m^*)^2/2\sigma^2}\,[1 + \mathcal{O}((m-m^*)\delta))]$, with $\sigma^2 = [s(Z_+ + Z_- - 2)]^{-1}$.

4. One dimension

We now turn to the one-dimensional system, where the behavior of the classical voter model is quite different from that in the mean-field limit. When zealots are present, however, we generically obtain a Gaussian magnetization distribution, as in the mean-field case. We now derive the magnetization distribution—first for two zealots—and then for an arbitrary number of zealots.

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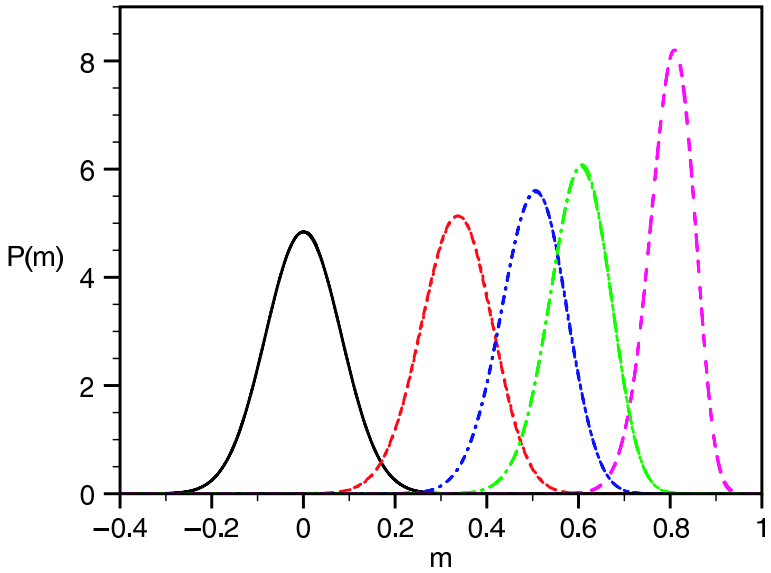

Figure 3. Steady-state magnetization distributions on a complete graph of 1000 sites with unequal numbers of zealots. Shown left to right are the cases of $(Z_+, Z_-) = (90, 90)$, $(120, 60)$, $(135, 45)$, $(144, 36)$, $(162, 18)$. The results of voter model simulations and the solution to the master equations are coincident. The mean magnetization of the system equals the magnetization of the zealots: $m = (z_+ - z_-)/(z_+ + z_-)$.

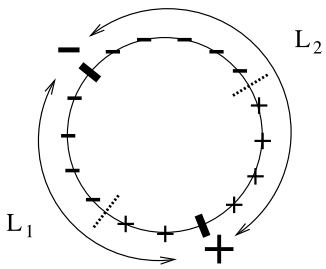

Figure 4. A ring divided into two independent segments by oppositely oriented zealots (thick lines). Also shown is the state of each voter and the domain wall in each segment at long times (dotted lines).

4.1. Two zealots

Suppose that two zealots of opposite opinion are randomly placed on a periodic ring of length L. The ring is thus split into two independent segments of lengths L_1 and L_2, with $L = L_1 + L_2 + 2$ (figure 4). We take the ring to be large so that we may write $L \approx L_1 + L_2$. As shown in figure 4, the voters in each segment coarsen and eventually there remains one domain of + voters that is separated from one domain of − voters by a single domain wall. Each domain wall performs a free random walk and the walk is reflected upon reaching the end of its segment. A basic fact from the theory of random walks [18] is that each domain wall is equiprobably located within the interval in the long-time limit. We now exploit this property to determine the magnetization distribution.

doi:10.1088/1742-5468/2007/08/P08029

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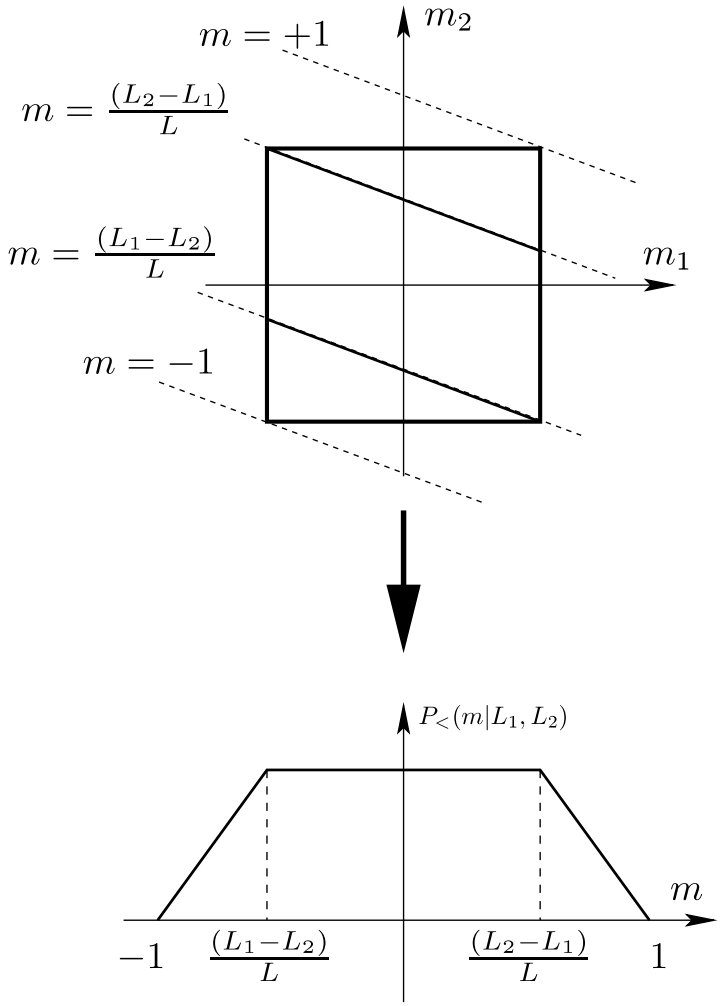

Figure 5. (Top) Rays of fixed magnetization (dashed) for the case $L_1 < L_2$. The probability for a given value of m is proportional to the length of the ray corresponding to this m value within the unit square (solid). (Bottom) The resulting magnetization distribution $P_<(m|L_1, L_2)$ for a given L_1 and $L_1 < L_2$.

For interval lengths L_1 and L_2 and respective magnetizations m_1 and m_2, the magnetization m of the entire ring is given by $mL = m_1L_1 + m_2L_2$. Thus a given value of m is achieved if m_1 and m_2 are related by (figure 5)

$$m_2 = \frac{mL}{L_2} - \frac{m_1L_1}{L_2}. \tag{13}$$

Then the probability $P(m|L_1, L_2)$ for a system of two segments with lengths L_1 and L_2 to have magnetization equal to m is proportional to the length of the ray defined by equation (13) that lies within the unit square in the m_1–m_2 plane. As illustrated in figure 5, the distribution $P_<(m|L_1, L_2)$, where the subscript $<$ now signifies the range $L_1 < L_2$, increases linearly with m for $-1 < m < (L_1 - L_2)/L$, is constant for $(L_1 - L_2)/L < m < (L_2 - L_1)/L$, and then decreases linearly with m for $(L_2 - L_1)/L < m < 1$.

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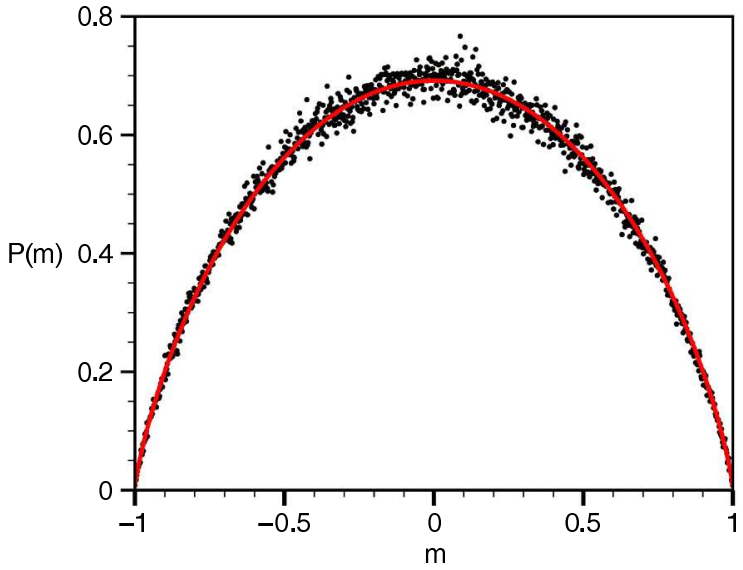

Figure 6. Comparison the analytic magnetization distribution for two zealots on the ring (equation (15)) and simulation results (points).

Using this m dependence of $P_<(m|L_1, L_2)$ and also imposing normalization, we thus find the magnetization distribution for fixed L_1, L_2 with $L_1 < L_2$ to be:

$$P_<(m|L_1, L_2) = \begin{cases} \dfrac{L^2(1+m)}{4L_1L_2} & -1 < m < \frac{L_1-L_2}{L} \\ \dfrac{L}{2L_2} & |m| < \frac{L_2-L_1}{L} \\ \dfrac{L^2(1-m)}{4L_1L_2} & \frac{L_2-L_1}{L} < m < 1. \end{cases} \tag{14}$$

The complementary distribution $P_>(m|L_1, L_2)$ for $L_1 > L_2$ is obtained from equation (14) by interchanging the roles of L_1 and L_2.

Now we integrate over all values of L_1 to find the configuration-averaged magnetization distribution $P(m)$. The details of this calculation are a bit tedious and are given in appendix A. The final result is

$$\begin{aligned} P(m) &= \frac{1}{L}\left[\int_0^{L/2} P_<(m|L_1, L_2)\,\mathrm{d}L_1 + \int_{L/2}^{L} P_>(m|L_1, L_2)\,\mathrm{d}L_1\right] \\ &= \left(\frac{1-|m|}{2}\right)\ln\left(\frac{1+|m|}{1-|m|}\right) - \ln\left(\frac{1+|m|}{2}\right). \end{aligned} \tag{15}$$

As shown in figure 6, the agreement between equation (15) and simulations is excellent.

4.2. Many zealots

We now study the magnetization distribution when many zealots are randomly distributed on the ring, with the restriction of equal numbers of each type of zealot ($Z_+ = Z_- = Z$). Two distinct possibilities can arise:

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(1) A segment of consecutive susceptible voters is surrounded by two zealots of the same sign. With voter model dynamics, these segments eventually align with the state of the confining zealots so that the segment freezes.

(2) A segment of consecutive susceptible voters is surrounded by two zealots of opposite opinion. Eventually a single domain wall remains that diffuses freely within the segment.

We first consider the simpler case where equal numbers of + and − zealots are randomly but alternately placed around the ring so that no frozen segments arise. The segment lengths $\{L_i\}$ with $i = 1, 2, \ldots, Z$, obey the constraint $\sum_i L_i = L$ (ignoring the space occupied by the zealots themselves).

To find the magnetization distribution, we map the state of the voters onto an equivalent random walk as follows. In a segment of length L_i, the difference in the number of + and − voters at long times is uniformly distributed in $[-L_i, L_i]$. We define this difference as the unnormalized magnetization M_i. We now make the following approximations that apply when $L, Z \to \infty$ such that each L_i is also large. In this limit, we may assume that each L_i is independent and identically distributed. As a result, the sum of the unnormalized magnetizations over all intervals is equivalent to the displacement of a random walk of Z steps with each step uniformly distributed in $[-L_i, L_i]$.

To solve this random walk problem, we use the basic fact that the Fourier transform for the probability distribution of the entire walk $\mathcal{P}(k)$ is simply the product of the Fourier transforms of the single-step distributions [5, 18]. Since the Fourier transform of a uniform single-step distribution over the range $[-L_i, L_i]$ is $(\sin kL_i)/kL_i$, we then have

$$\mathcal{P}(k) = \prod_{i=1}^{Z} \frac{\sin kL_i}{kL_i}. \tag{16}$$

Since we are interested in the asymptotic limit where the unnormalized magnetization becomes large, we study the limit of $\mathcal{P}(k)$ for small k. Thus we expand each factor in $\mathcal{P}(k)$ in a Taylor series to first order, and then re-exponentiate to yield

$$\begin{aligned}\mathcal{P}(k) &\approx \prod_{i=1}^{Z}(1 - k^2 L_i^2/6) \\ &\sim 1 - \sum_{i}^{Z} k^2 L_i^2/6 \sim \mathrm{e}^{-k^2 \sum_i L_i^2/6}.\end{aligned}$$

We now invert this Fourier transform to give the distribution of the unnormalized magnetization

$$\begin{aligned}P(M) &= \frac{1}{2\pi}\int \mathrm{e}^{-k^2 \sum_i L_i^2/6}\, \mathrm{e}^{-\mathrm{i}kM}\, \mathrm{d}k \\ &= \frac{1}{\sqrt{2\pi\sigma_M^2}}\, \mathrm{e}^{-M^2/2\sigma_M^2},\end{aligned} \tag{17}$$

with $\sigma_M^2 = \sum_i L_i^2/3$.

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What we want, however, is the magnetization distribution; this is related to $P(M)$ by $P(m)\,\mathrm{d}m = P(M)\,\mathrm{d}M$. We thus find

$$P(m) = \frac{1}{\sqrt{2\pi\sigma_m^2}}\,\mathrm{e}^{-m^2/2\sigma_m^2}, \tag{18}$$

where $\sigma_m^2 = \sum_i L_i^2/3L^2$. If the number of intervals is large, then each L_i is approximately L/Z, from which we obtain $\sigma_m^2 \approx 1/3Z$. (The result $\sigma_m^2 = 1/3Z$ is exact if all interval lengths are equal.) As in the mean-field limit, the width of the magnetization distribution is controlled by the *number* of zealots and not their concentration, so that a small number of zealots is effective in maintaining the magnetization close to zero.

A similar approach applies in the case where the spatial ordering of the zealots is uncorrelated. In this case, approximately half of all segments will be terminated by oppositely oriented zealots and half by zealots of the same species. For the latter type of segments, the unnormalized magnetization will equal $\pm L_i$ equiprobably. Under the assumption that exactly half of the segments are frozen and half contain a single freely diffusing domain wall, the analogue of equation (16) is

$$\mathcal{P}(k) = \prod_{i=1}^{Z/2} \frac{\sin kL_i}{kL_i} \prod_{i=1}^{Z/2} \cos kL_i. \tag{19}$$

The second product accounts for frozen segments in which the unnormalized magnetization equals $\pm L_i$ equiprobably. For these segments the Fourier transform of the single-step probability for a random walk whose steps length are $\pm L_i$ equals $\cos kL_i$. Following the same steps that led to equation (18), we again obtain a Gaussian magnetization distribution, but with σ_m^2 given by $\sigma_m^2 = \sum_i 2L_i^2/3L^2 \to 2/3Z$.

5. Two dimensions

In the classical voter model, the two-dimensional system is at the critical dimension so that its behavior deviates from that of the mean-field system by logarithmic corrections. In the presence of zealots, however, the behaviors in two dimensions and in mean field are quite close, as illustrated in figure 7.

Our results for two dimensions are based on numerical simulations. In our simulations, we pick a random voter and apply the update rules of section 2. The unit of time is defined so that a time increment $\mathrm{d}t = 1$ corresponds to N update events, so that each voter is updated once on average. The system is initialized with each voter equally likely to be in the $+$ or the $-$ states. From the N_+ voters in the $+$ state, Z_+ of them are designated as zealots, and similarly for voters in the $-$ state. After the system reaches the steady state, we measure steady-state properties at time intervals ΔT. The delay time T to reach the steady state depends on the lattice dimension and the zealot density, while ΔT is the correlation time for the system in the steady state. By making measurements every ΔT steps, we obtain data for effectively uncorrelated systems. Typically, for a given initial condition, we made 100 measurements and then averaged over many configurations of zealots.

The resulting data for the magnetization distribution is typically noisy, and we employ a Gaussian averaging of nearby points to smooth the data. If m_i denotes the

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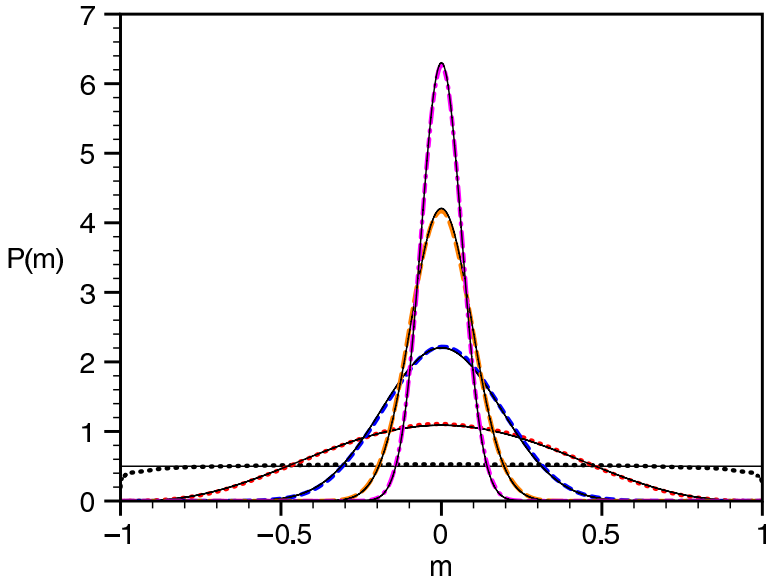

Figure 7. Comparison of simulations for the magnetization distribution in two dimensions (dashed) with the mean-field results (solid curves). The simulations are for 1000 voters with $2, 8, 32, 128$ and 512 total zealots, with equal numbers of each type.

ith magnetization value, then the smoothed magnetization distribution at m_i is defined as

$$\overline{P(m_i)} = \frac{1}{\sqrt{\pi d^2}} \sum_{k=-d}^{d} P(m_{i+k})\, \mathrm{e}^{-(k/d)^2},$$

where the sum includes the initial point, as well as the d points to the left and to the right of the initial point, with d typically in the range 20–40. Such a smoothed distribution is the quantity that is actually plotted in figures 2, 3, 7, and in the spatially averaged distribution in figure 8.

6. Discussion

We have shown that a small number of zealots in a population of voters is quite effective in maintaining a steady state in which consensus is never achieved. When there are equal numbers of zealots of each type, the steady-state fraction of democrats and republicans equals 1/2; equivalently, the magnetization equals zero. For unequal densities of the two types of zealots, the steady-state magnetization is simply $m^* = (Z_+ - Z_-)/(Z_+ + Z_-)$, where Z_+ and Z_- are the number of zealots of each type. The magnetization distribution is generically Gaussian, $P(m) \propto \mathrm{e}^{-(m-m^*)^2/2\sigma^2}$, with $\sigma \propto 1/\sqrt{Z}$, and $Z = Z_+ + Z_-$ is the total number of zealots. A Gaussian magnetization distribution arises universally in one dimension, on the square lattice (two dimensions), and on the complete graph (mean-field limit). One basic consequence of this distribution is that as the total number of voters N increases, the fraction of zealots needed to keep the magnetization less than a specified level vanishes as $1/\sqrt{N}$.

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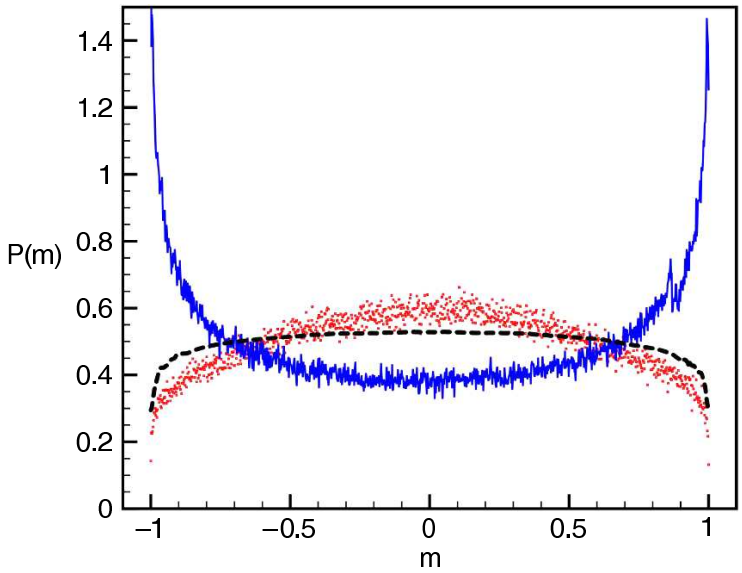

Figure 8. Comparison of simulations for the magnetization distribution in two dimensions when the two zealots are adjacent (curve with peaks near ± 1), maximally separated (dots), and averaged over many different zealot configurations (dashed).

There are several additional aspects of the influence that zealots have on the voter model that are worth pointing out. Although the time to reach consensus is infinite because this state can never be achieved, one can ask for the time until a specified plurality is first achieved. Equivalently, we can ask for the probability that the magnetization first reaches a value m, when the system is initialized with $m = m_0$. From the above generic Gaussian form of the magnetization distribution, we expect that the mean time for a symmetric system to first reach a magnetization m will thus scale as $e^{am^2 Z}$, where a is a constant of order one. Thus one must wait an extremely long time before the system achieves even a modest deviation away from the zero-magnetization state when the number of zealots becomes appreciable. Perhaps this trivial fact is the underlying reason why so many democratic countries are characterized by small majorities in governance.

Another interesting feature is the role of the zealots' spatial positions on the steady state. For example, if there are only two zealots that are adjacent, one might expect that the effect of this 'dipole' would be weaker than that of two separated monopoles. This is precisely the effect that is observed in figure 8. When the two zealots are adjacent, their effects are substantially screened and the magnetization distribution is peaked near $m = \pm 1$. That is, the voters show a preference for consensus in spite of the zealots. On the other hand, when the zealots are maximally separated, the magnetization distribution is close to the distribution that arises when averaging over possible positions of the two zealots.

Zealots are also quite effective in reducing the total number of opinion changes in the system. If the population is close to zero magnetization, each voter typically has equal numbers of neighbors of each type. If the voters are not strongly correlated, each voter would change its state at a rate that is approximately equal to 1/2. However, simulations

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on the square lattice show that the flip rate of each susceptible voter is considerably smaller. For example, for 1000 voters with 10 zealots (5 of each type), the rate of opinion changes of the susceptibles is around 1/5 and this rate decreases as the density of zealots decreases.

Finally, a slight embellishment of our model could apply to real voting patterns in a democracy with strong regional differences. Here it is natural to partition a population into enclaves, with an imbalance of one type of zealot over the other in each enclave. Such a spatial distribution would correspond to red (republican) and blue (democrat) states in the parlance of US electoral politics. It would be interesting to study if such an extension can actually account for election results.

Acknowledgments

We gratefully acknowledge the support of the Alexander von Humboldt Foundation through the fellowship IV-SCZ/1119205 (MM) and NSF grant DMR0535503 (AP and SR).

Appendix. Magnetization distribution for two zealots

We want to compute the integral

$$P(m) = \frac{1}{L}\left[\int_0^{L/2} P_<(m|L_1, L_2)\,\mathrm{d}L_1 + \int_{L/2}^{L} P_>(m|L_1, L_2)\,\mathrm{d}L_1\right]. \tag{A.1}$$

Since $P_<(m|L_1, L_2)$ and $P_>(m|L_1, L_2)$ have different forms in different parts of the interval $[-1, 1]$, each of the above integrals needs to be split into two parts. For $P_<(m|L_1, L_2)$ and assuming that $m > 0$, the linear ramp part of the probability distribution needs to be used for $(L_2 - L_1)/L < m$, which translates for $L_1 > L(1-m)/2$. Similarly, for $P_>(m|L_1, L_2)$ and again for $m > 0$, the linear ramp must be used when $(L_1 - L_2)/L < m$, or $L_1 < L(1+m)/2$. Thus the above integral becomes

$$P(m) = \frac{1}{L}\left[\int_0^{(L/2)(1-m)} \frac{L\,\mathrm{d}L_1}{2(L-L_1)} + \int_{(L/2)(1-m)}^{L/2} \frac{(1-m)L^2\,\mathrm{d}L_1}{4L_1(L-L_1)} + \int_{L/2}^{(L/2)(1+m)} \frac{(1-m)L^2\,\mathrm{d}L_1}{4L_1(L-L_1)} + \int_{(L/2)(1+m)}^{L} \frac{L\,\mathrm{d}L_1}{2L_1}\right]. \tag{A.2}$$

Each of these integrals is then elementary. We also obtain the result for $m < 0$ by reflecting the result of the above integral about $m = 0$ to give equation (15).

Note added. As this manuscript was being written, we became aware of a very recent eprint by Chinellato *et al* [19]; they study essentially the same model as in this work, but with a somewhat different focus than ours.

References

[1] Liggett T M, 1999 *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes* (New York: Springer)
[2] Glauber R J, 1963 *J. Math. Phys.* **4** 294
[3] Krapivsky P L, 1992 *Phys. Rev.* A **45** 1067

J. Stat. Mech. (2007) P08029

[4] Sood V and Redner S, 2005 *Phys. Rev. Lett.* **94** 178701
[5] See e.g. Redner S, 2001 *A Guide to First-Passage Processes* (New York: Cambridge University Press) chapter 4
[6] Mobilia M, 2003 *Phys. Rev. Lett.* **91** 028701
[7] Mobilia M and Georgiev I T, 2005 *Phys. Rev.* E **71** 046102
[8] Galam S and Jacobs F, 2007 *Physica* A **381** 366
[9] See e.g. http://www.uselectionatlas.org/RESULTS/index.html.
[10] Axelrod R, 1977 *J. Conflict Res.* **41** 203
Axtell R, Axelrod R, Epstein J and Cohen M D, 1996 *Comput. Math. Org. Theory* **1** 123
[11] Weisbuch G, Deffuant G, Amblard F and Nadal J P, 2002 *Complexity* **7** 55
Ben-Naim E, Krapivsky P L and Redner S, 2003 *Physica* D **183** 190
[12] Vazquez F and Redner S, 2004 *J. Phys. A: Math. Gen.* **37** 8479
[13] Gardiner C W, 1985 *Handbook of Stochastic Methods* 2nd edn (Berlin: Springer)
Van Kampen N, 1992 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
[14] Ben-Naim E and Krapivsky P L, 2004 *Phys. Rev.* E **69** 046113
[15] Reichenbach T, Mobilia M and Frey E, 2006 *Phys. Rev.* E **74** 051907
Reichenbach T, Mobilia M and Frey E, 2007 *Nature* in press doi:10.1038/nature06095
[16] Considine D, Redner S and Takayasu H, 1989 *Phys. Rev. Lett.* **63** 2857
[17] McKane A J and Newman T J, 2005 *Phys. Rev. Lett.* **94** 218102
[18] See e.g. Weiss G H, 1994 *Aspects and Application of Random Walk* (Amsterdam: North-Holland)
[19] Chinellato D D, de Aguiar M A M, Epstein I R, Braha D and Bar-Yam Y, 2007 *Preprint* 0705.4607 [nlin.SI]

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