Social Balance on Networks: The Dynamics of Friendship and Hatred

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Basic question:

How do social networks evolve when both friendly and unfriendly relationships exist?

Partial answer: *(Heider 1944, Wasserman & Faust 1994)*

Social balanced defined; balanced states on a complete graph must be either paradise or bipolar.

This work:

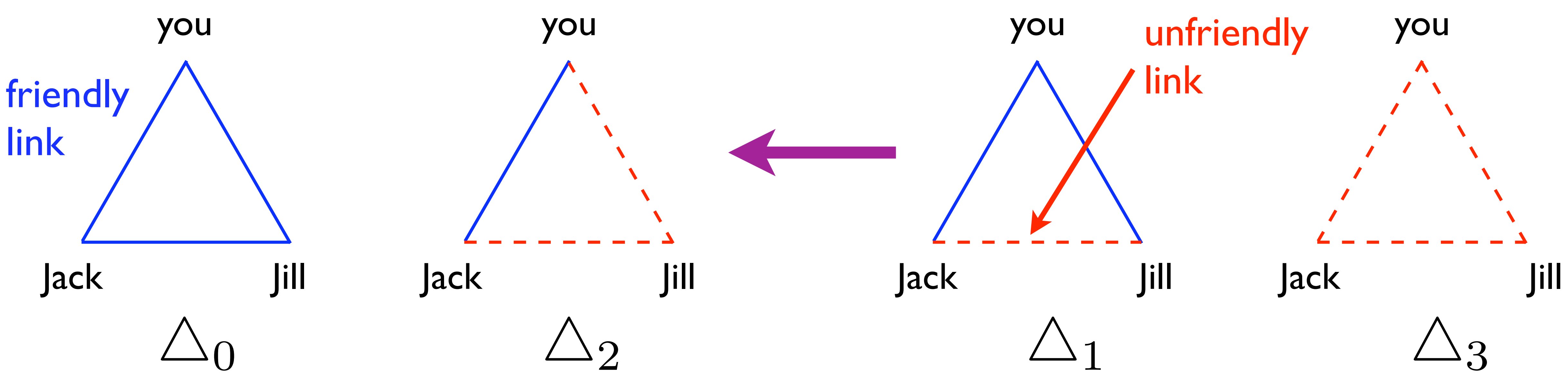
Endow a network with the simplest dynamics and investigate evolution of relationships.

related work: Kulakowsi et al.

Main result:

Dynamical phase transition between bipolarity and paradise.

Socially Balanced States



unfrustrated/balanced

frustrated/imbalanced

Social Balance

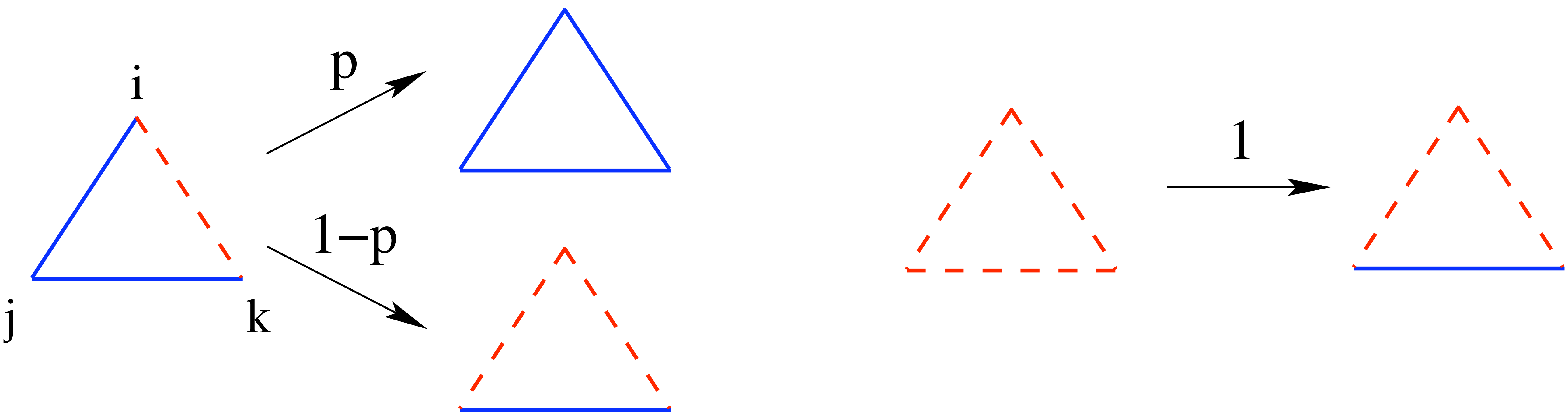
a friend of my friend is my friend;
an enemy of my enemy is my friend;
a friend of my enemy is my enemy;
an enemy of my friend is my enemy.

Local Triad Dynamics on Arbitrary Networks

(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad
2. Reverse a single link so that the triad becomes balanced

   probability p: unfriendly → friendly; probability 1-p: friendly → unfriendly



Fundamental parameter p:

p=1/3: flip a random link in the triad equiprobably

p>1/3: predisposition toward tranquility

p<1/3: predisposition toward hostility

Triad Evolution on the Complete Graph

Basic graph characteristics:

N nodes

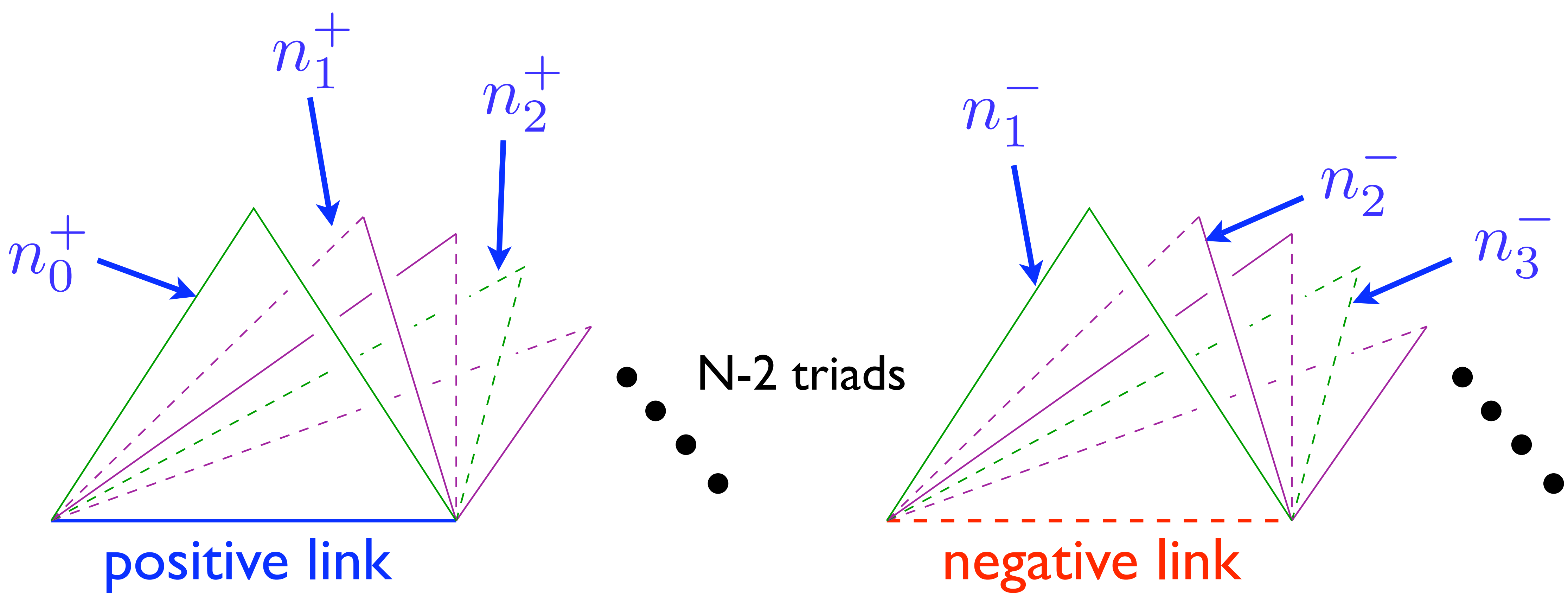
$\frac{N(N-1)}{2}$ links

$\frac{N(N-1)(N-2)}{6}$ triads

$\rho =$ friendly link density

$n_k =$ density of triads of type k

$n_k^{\pm} =$ density of triads of type k attached to a $\pm$ link



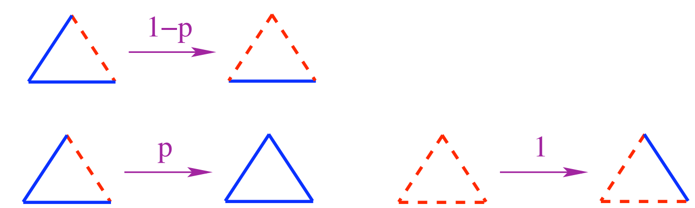
Triad Evolution on the Complete Graph

n_k = density of triads of type k

$n_k^{\pm}$ = density of triads of type k attached to a $\pm$ link

$\pi^+ = (1-p)\, n_1$ flip rate $+ \;\rightarrow\; -$

$\pi^- = p\, n_1 + n_3$ flip rate $- \;\rightarrow\; +$



Master equations:

$\triangle_1 \rightarrow \triangle_0$ $\quad$ $\triangle_0 \rightarrow \triangle_1$

$$\frac{dn_0}{dt} = \pi^- n_1^- - \pi^+ n_0^+ \,,$$

$$\frac{dn_1}{dt} = \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+ \,,$$

$$\frac{dn_2}{dt} = \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+ \,,$$

$$\frac{dn_3}{dt} = \pi^+ n_2^+ - \pi^- n_3^- \,.$$

Steady State Solution

$$\begin{aligned}
\frac{dn_0}{dt} &= \pi^- n_1^- - \pi^+ n_0^+ \,, \\
\frac{dn_1}{dt} &= \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+ \,, \\
\frac{dn_2}{dt} &= \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+ \,, \\
\frac{dn_3}{dt} &= \pi^+ n_2^+ - \pi^- n_3^- \,.
\end{aligned}$$

impose $\dot{n}_i$ and $\pi^+ = \pi^-$

gives $n_k^+ = n_{k+1}^-$

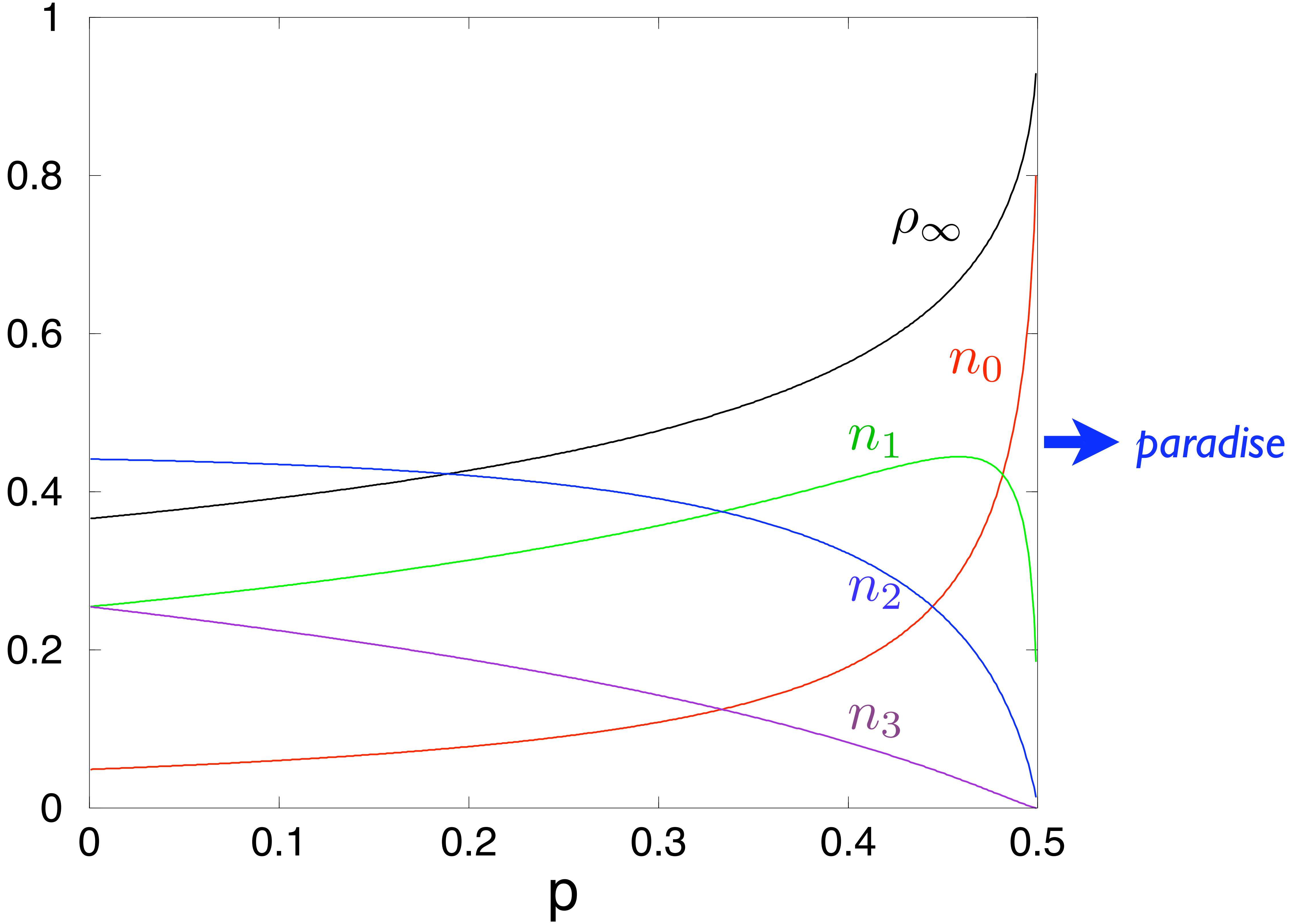
finally, use $n_k^\pm = \begin{cases} \frac{(3-k)n_k}{3n_0+2n_1+n_2} \\ \\ \frac{kn_k}{n_1+2n_2+3n_3} \end{cases}$

$$n_j = \binom{3}{j} \rho_\infty^{3-j} (1-\rho_\infty)^j,$$

$$\rho_\infty = \begin{cases} 1/[\sqrt{3(1-2p)}+1] & p \le 1/2; \\ \\ 1 & p \ge 1/2 \end{cases}$$

Steady State Triad Densities

steady state only for p≤½



The Evolving State

rate equation for the friendly link density:

$- \to +$ in $\triangle_1$ $\quad + \to -$ in $\triangle_1$ $\quad - \to +$ in $\triangle_3$

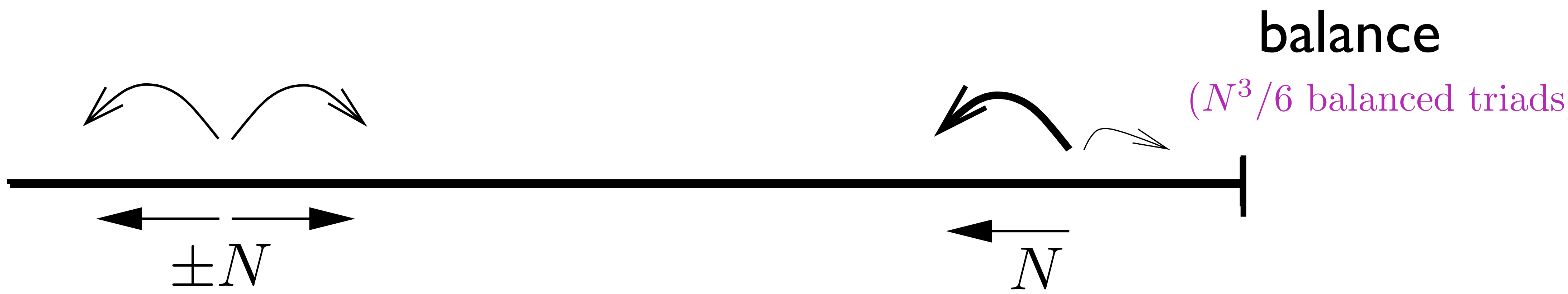
$$\frac{d\rho}{dt} = 3\rho^2(1-\rho)[p-(1-p)] + (1-\rho)^3$$
$$= 3(2p-1)\rho^2(1-\rho) + (1-\rho)^3$$

$$\rho(t) \sim \begin{cases} \rho_\infty + Ae^{-Ct} & p < 1/2; \\ 1 - \dfrac{1-\rho_0}{\sqrt{1+2(1-\rho_0)^2 t}} & p = 1/2; \\ 1 - e^{-3(2p-1)t} & p > 1/2. \end{cases}$$

- $p < 1/2$: rapid onset of frustration
- $p = 1/2$: slow relaxation to paradise
- $p > 1/2$: rapid attainment of paradise

Fate of a Finite Society

p<1/2: effective random walk picture



$D \propto N^2$ $\qquad$ $v \propto N$

$$\rightarrow T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2}$$

p>1/2: inversion of the rate equation

$$u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim \frac{\ln N}{2p-1}$$

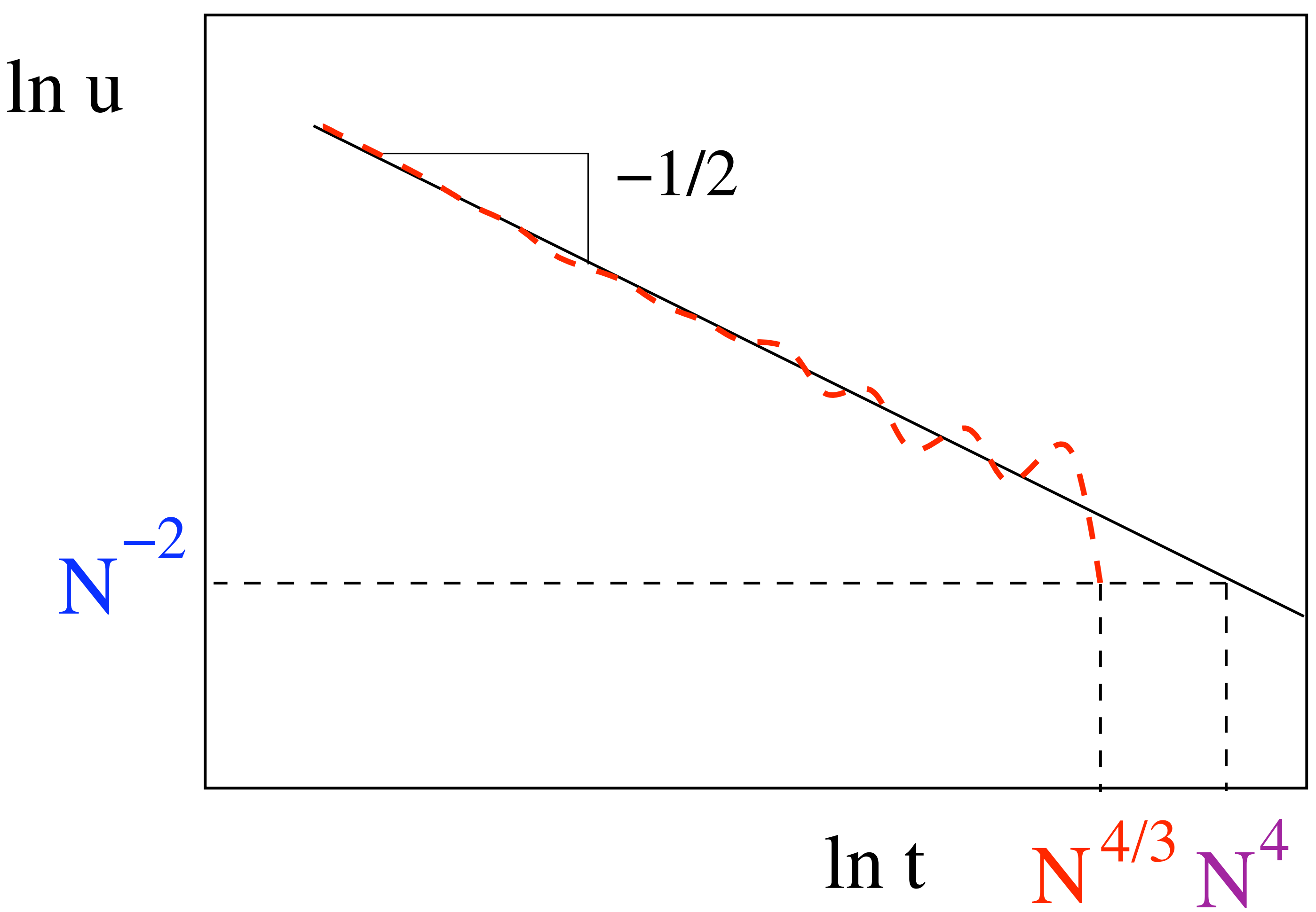
u=1-ρ is the density of unfriendly links

p=1/2

naive rate equation estimate:

$$u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \qquad \rightarrow \; T_N \sim N^4$$

incorporating fluctuations as balance is approached:

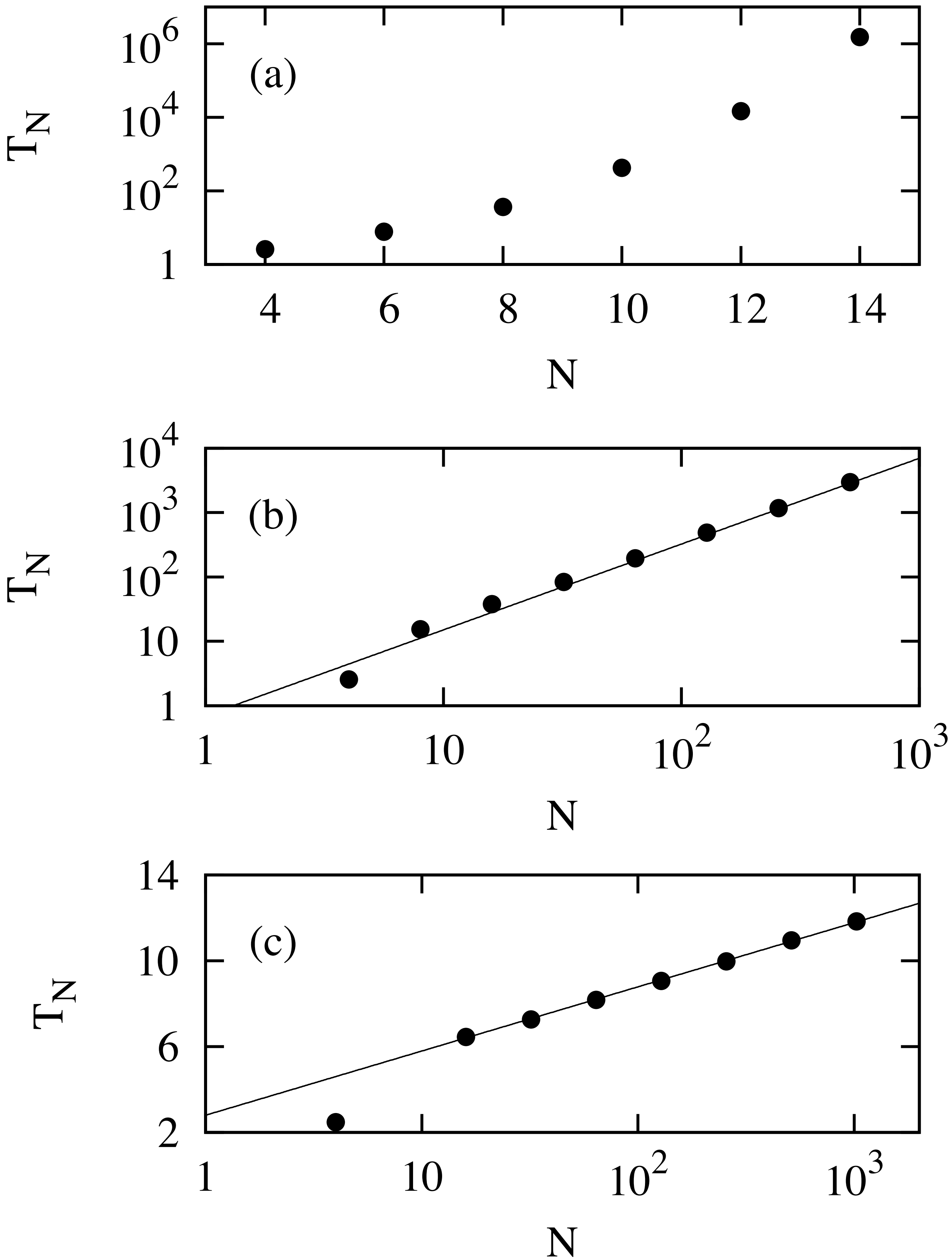


$$\begin{aligned} U &= Lu + \sqrt{L}\,\eta \\ &\sim \frac{L}{\sqrt{t}} + \sqrt{L}\, t^{1/4} \end{aligned}$$

equating the 2 terms in U:

$$T_N \sim L^{2/3} \sim N^{4/3}$$

Simulations for a Finite Society



$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$

$$p = \frac{1}{2}, \qquad T_N \sim N^{4/3}$$

$$p > \frac{1}{2}, \qquad T_N \sim \frac{\ln N}{2p - 1}$$

Constrained (Socially Aware) Triad Dynamics

1. Pick a random imbalanced (frustrated) triad

2. Reverse a random link (p=⅓) to eliminate a frustrated triad
   only if the total number of frustrated triads does not increase

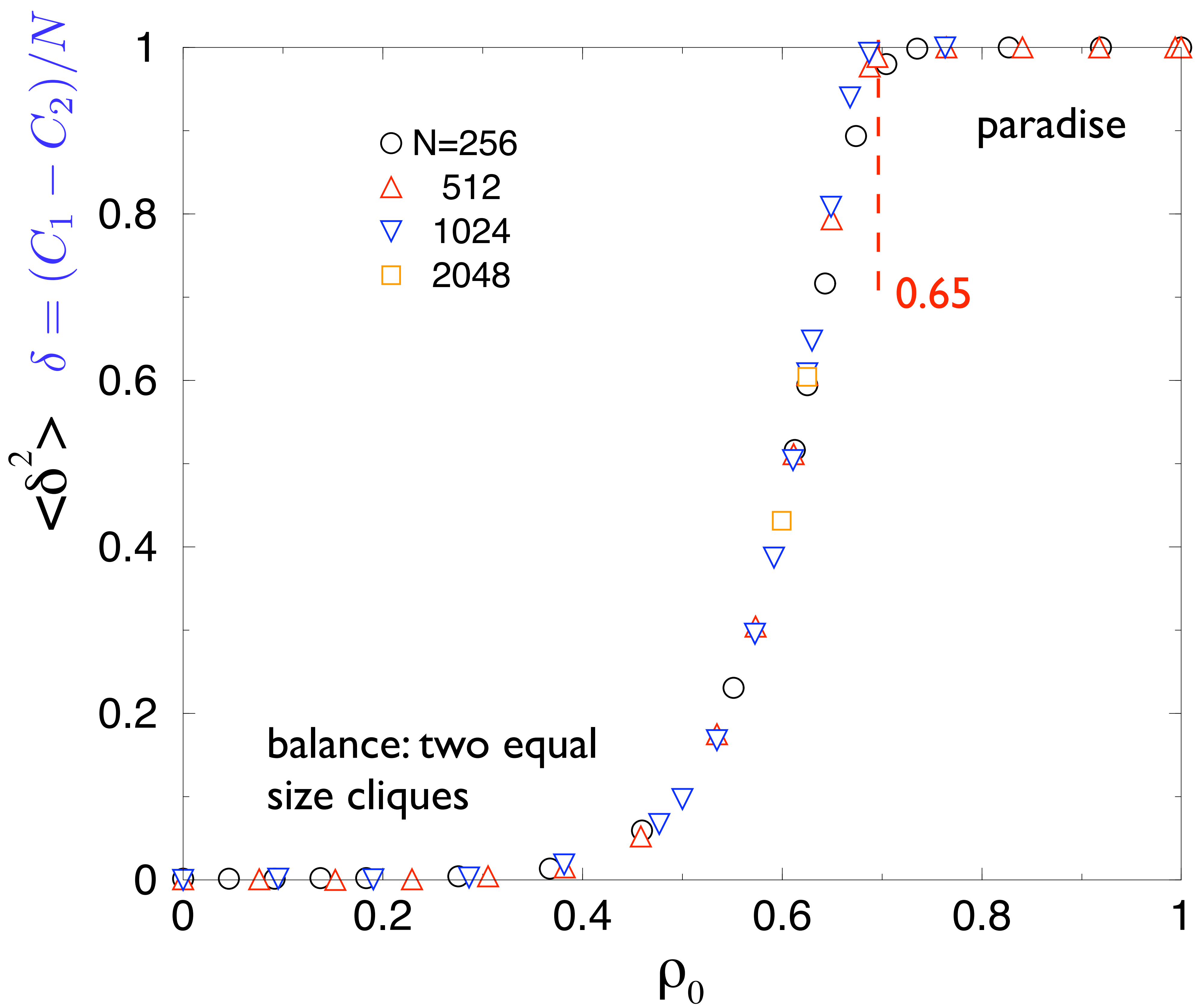
Outcome: Quick approach to a final static state

Typically: $T_N \sim \ln N$

Final state is almost always balanced even though *jammed states* are much more numerous.

Jammed state: Imbalanced triads exist, but any update only increases the number of imbalanced triads.

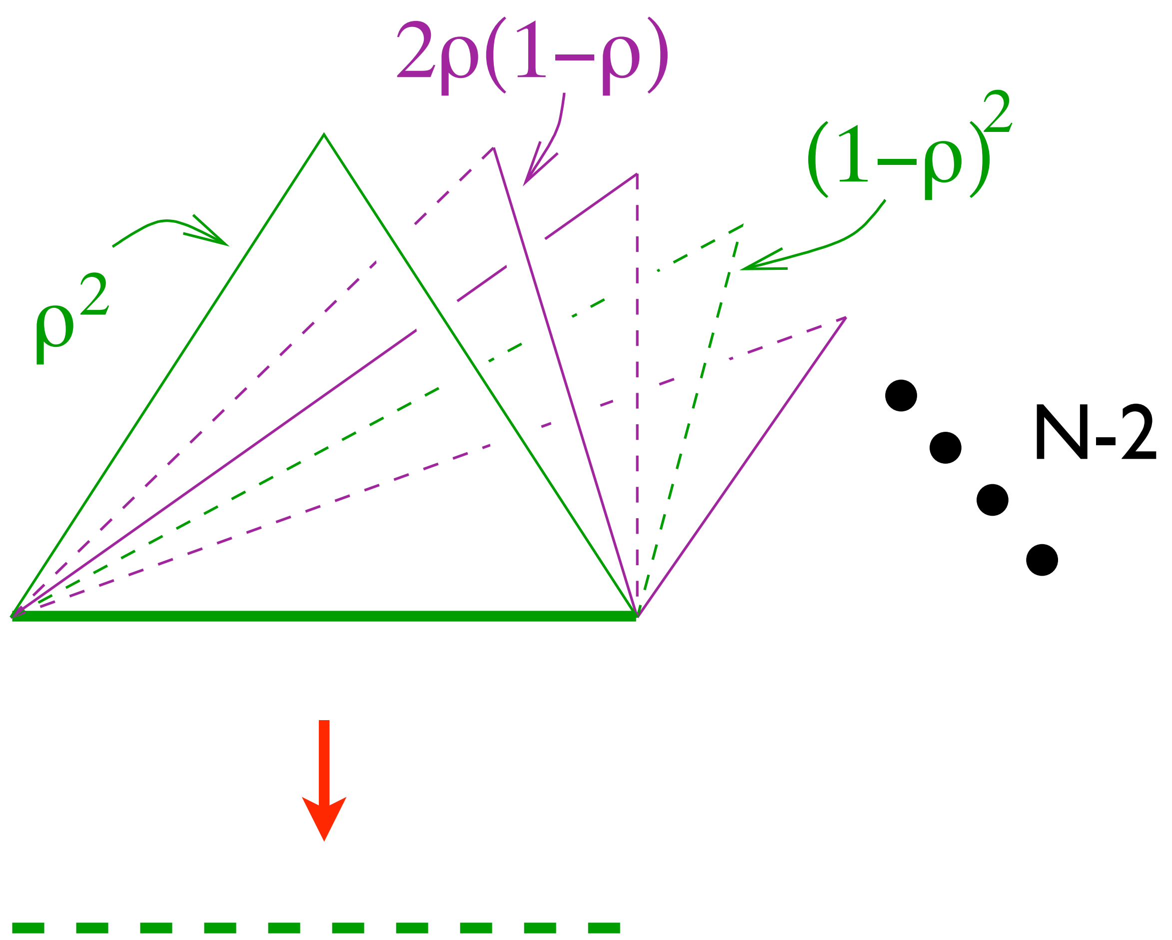
Final Clique Sizes



Origin of the Balance/Paradise Transition

First consider evolution of an uncorrelated network:

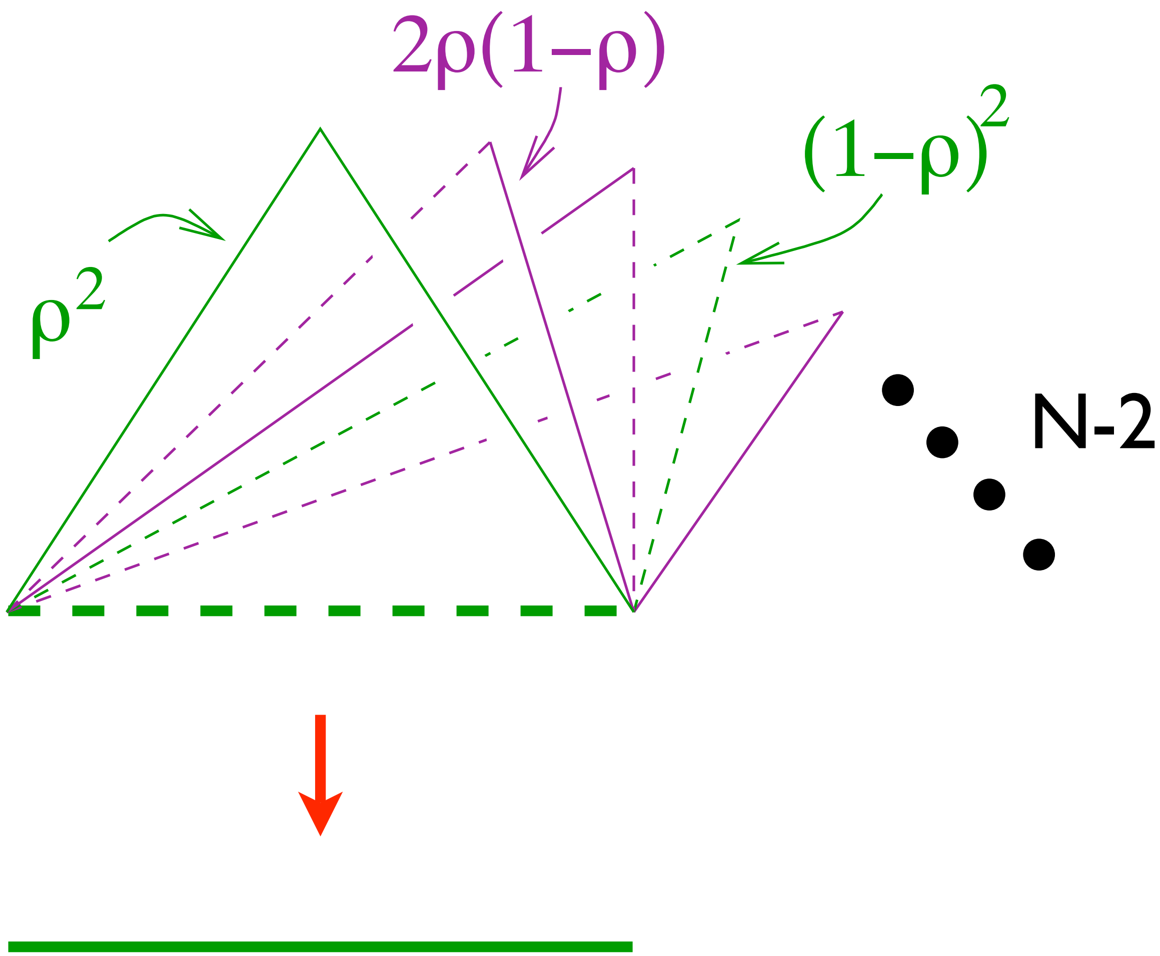
for + → −



we need:

$$\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}\ ,\ \text{with } \vec{n}_+ = [\rho^2, 2\rho(1-\rho), (1-\rho)^2, 0]$$

$\rightarrow 1 - 4\rho(1-\rho) < 0,$ impossible, so + links never flip for − → +



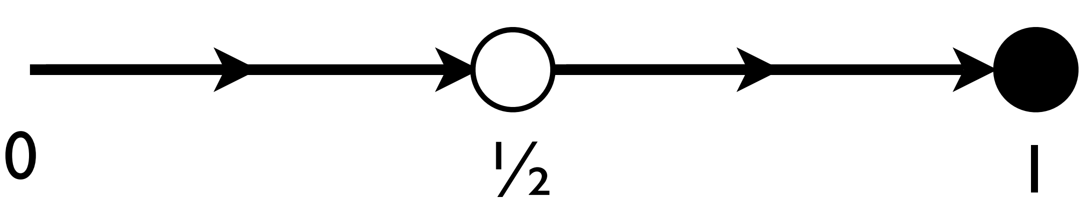
we need:

$$\underbrace{n_1^- + n_3^-}_{\text{frustrated}} > \underbrace{n_0^- + n_2^-}_{\text{unfrustrated}}, \text{ with } \vec{n}_- = [0, \rho^2, 2\rho(1-\rho), (1-\rho)^2]$$

$$\rightarrow 1 - 4\rho(1-\rho) > 0, \text{ valid when } \rho \neq 1/2$$

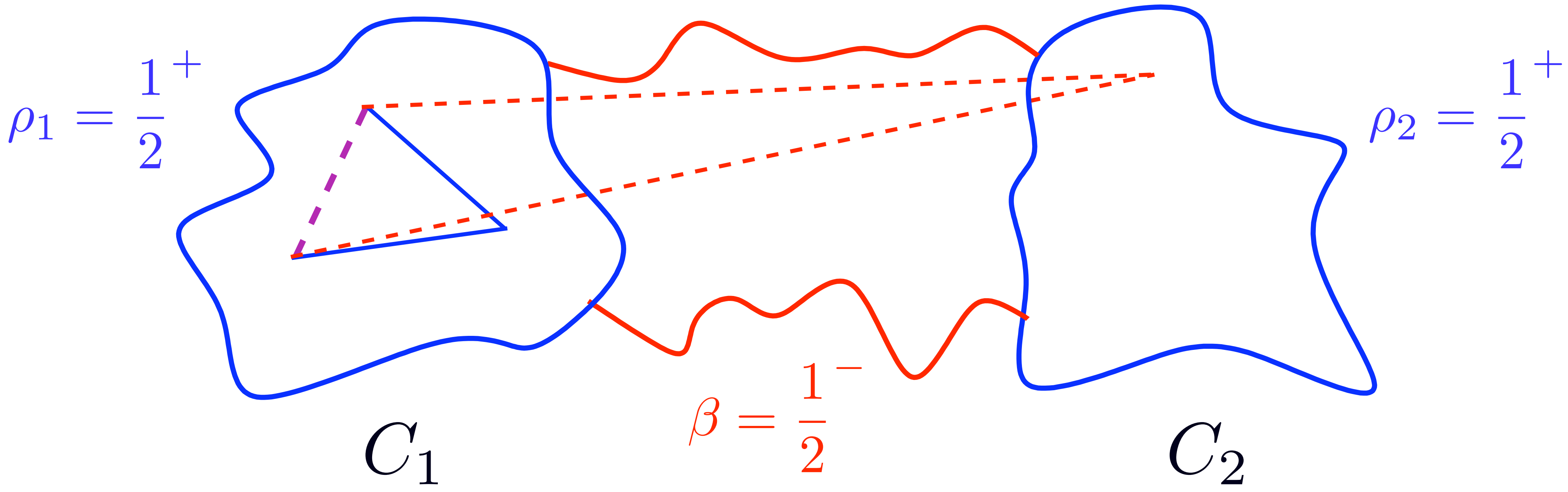
Conclusion: only negative links flip, except when ρ→½

flow diagram for ρ:



Instability near ρ=½

intraclique relationship evolution



for $-\to+$, we need:

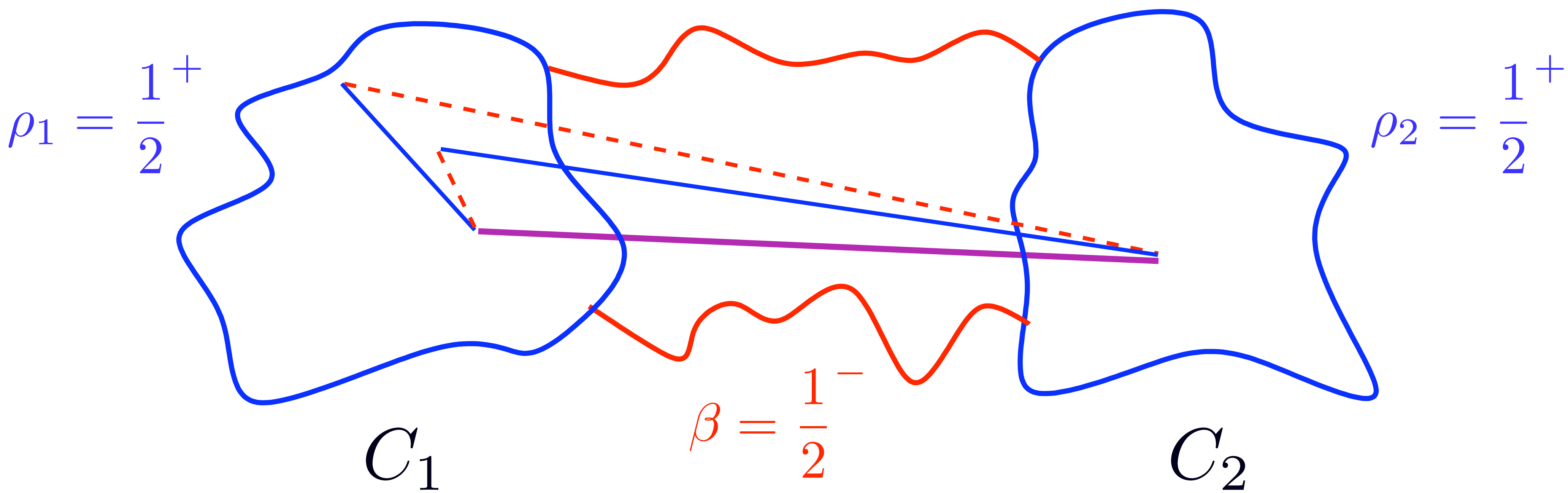
$$\underbrace{n_1^- + n_3^-}_{\text{frustrated}} > \underbrace{n_0^- + n_2^-}_{\text{unfrustrated}}, \text{ with } \vec{n}_- = \begin{cases} [0, \rho_i^2, 2\rho_i(1-\rho_i), (1-\rho_i)^2] & \text{intraclique} \\ [0, \beta^2, 2\beta(1-\beta), (1-\beta)^2] & \text{interclique} \end{cases}$$

$$\to C_1[1 - 4\rho_i(1-\rho_i)] + C_2[1 - 4\beta(1-\beta)] > 0, \text{ always true}$$

negative intraclique links disappear

increased cohesiveness within cliques

interclique relationship evolution



for + → – , we need:

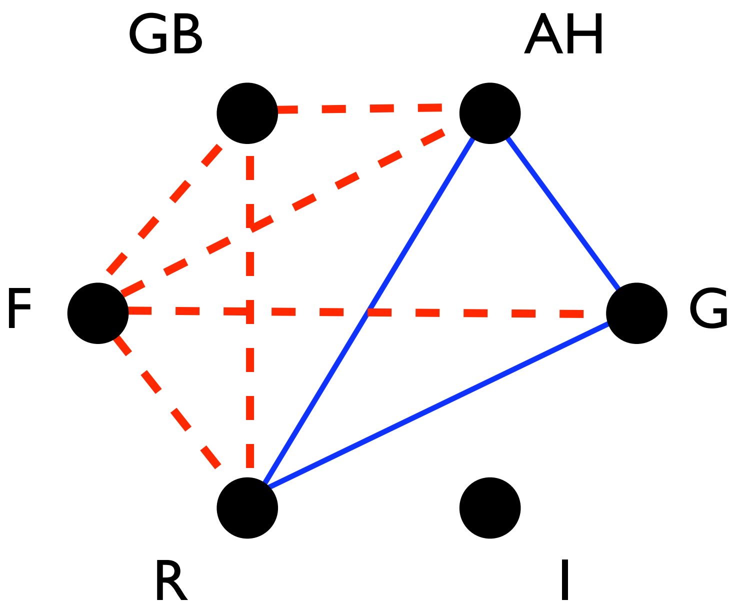
$$\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}, \text{ with } \vec{n}_+ = [\beta\rho_i, \beta(1-\rho_i) + \rho_i(1-\beta), (1-\beta)(1-\rho_i), 0]$$

$$\to [C_1(2\rho_1 - 1) + C_2(2\rho_2 - 1)](1 - 2\beta) > 0, \quad \text{true if } \rho_1, \rho_2 > 1/2, \beta < 1/2$$

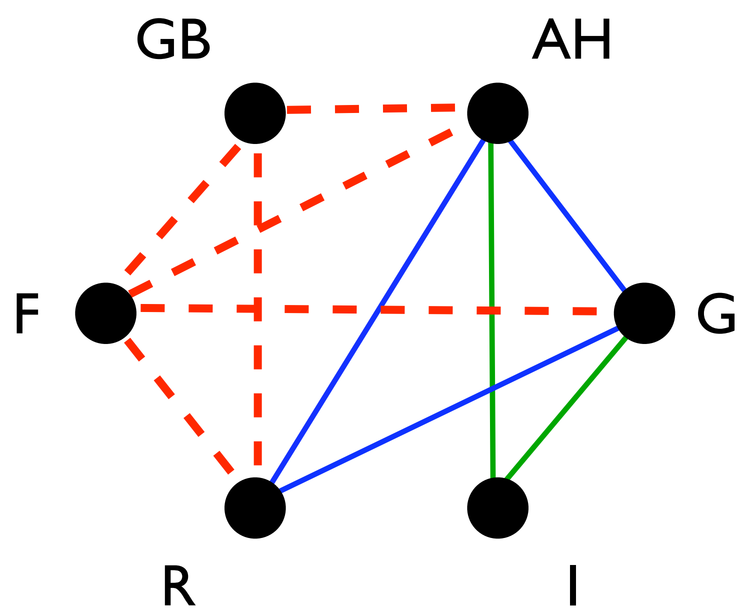
positive interclique links disappear

increased emnity between cliques

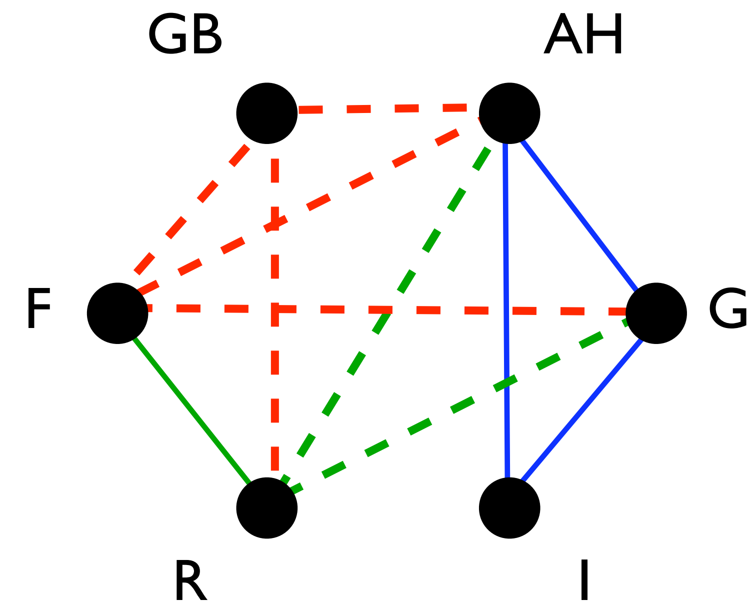
A Historical Lesson



3 Emperor's League 1872-81

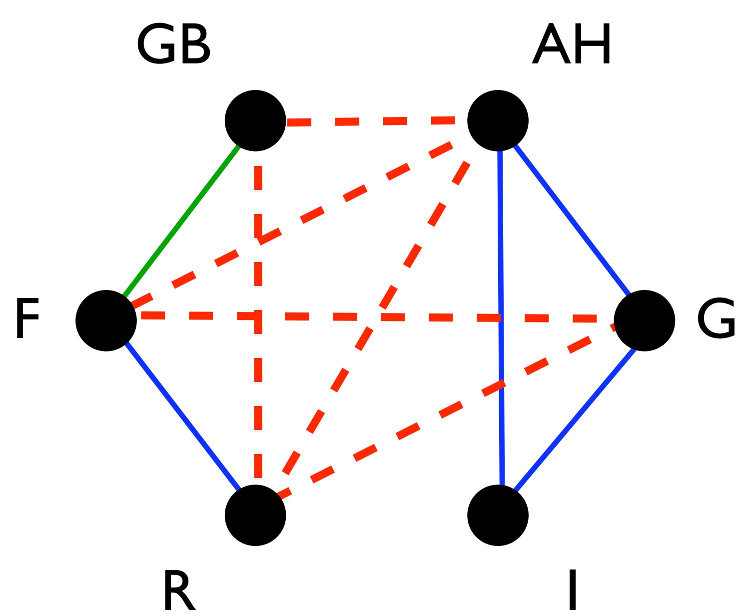


Triple Alliance 1882

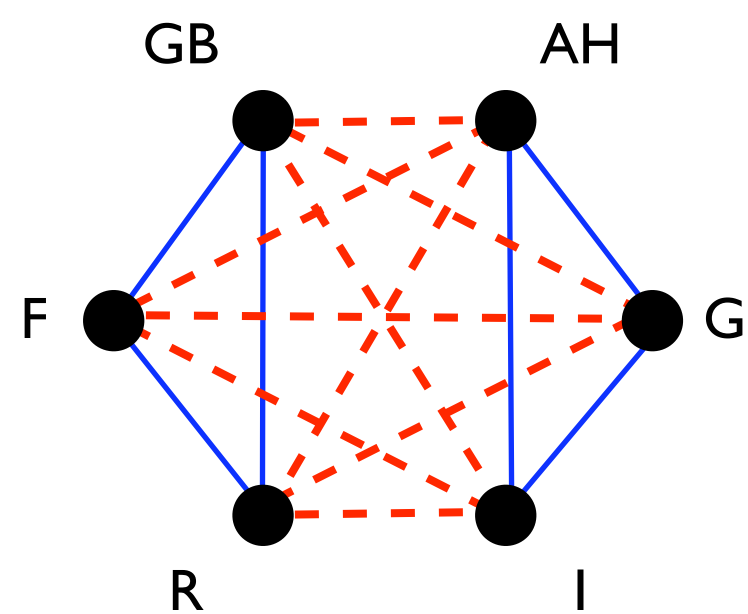


German-Russian Lapse 1890

French-Russian Alliance 1891-94



Entente Cordiale 1904



British-Russian Alliance 1907

Summary & Outlook

If we can't all love each other → *social balance*

Local triad dynamics:

finite network: social balance, with the time until balance strongly dependent on p

infinite network: phase transition between paradise and social balance at p=½

Global triad dynamics (p=⅓):

jammed states possible but never occur

infinite network: two cliques always emerge, with paradise when $\rho_0 \cong 0.65$ (rough argument gives $\rho_0 = ½$)

Open questions:

incomplete graphs, indifference, continuous interactions

asymmetric relations

allow