The Dynamics of Consensus and Clash

Annual CNLS Conference 2006

Question: How do generic opinion dynamics models with (quasi)-ferromagnetic interactions evolve?

Models:

- Voter model on heterogeneous graphs — *V. Sood*
- Bounded compromise — *E. Ben-Naim & P. Krapivsky*
- Axelrod model of cultural diversity — *F. Vazquez*

Results:

- Heterogeneous voter model: *fast consensus*
- Bounded compromise: *self-similar fragmentation*
- Axelrod model: *non-monotonicity and long time scale*

Voter Model

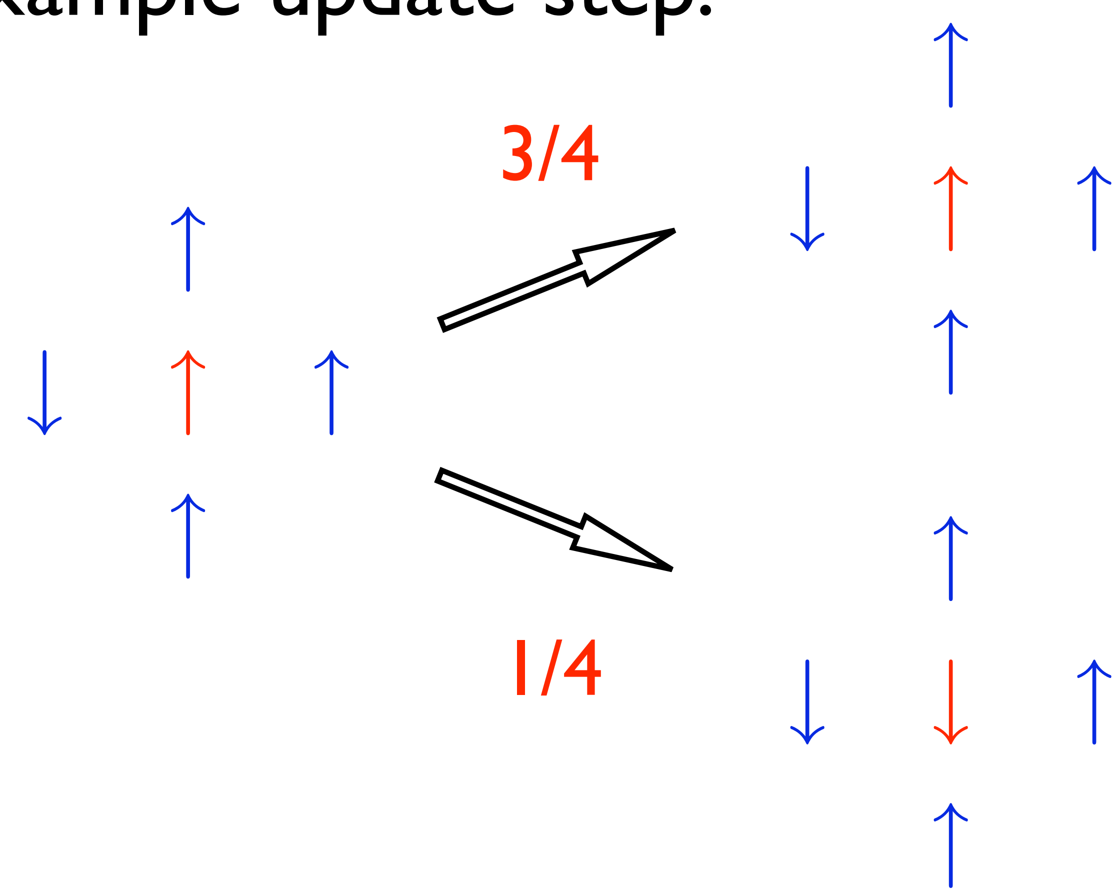
Liggett (1985)

0. Binary spin variable at each site i, $\sigma_i = \pm 1$
1. Pick a random spin
2. Assume state of randomly-selected neighbor

   each individual has zero self-confidence and adopts state of randomly-chosen neighbor

3. Repeat 1 & 2 until consensus *necessarily* occurs

Example update step:



encoded by flip rate

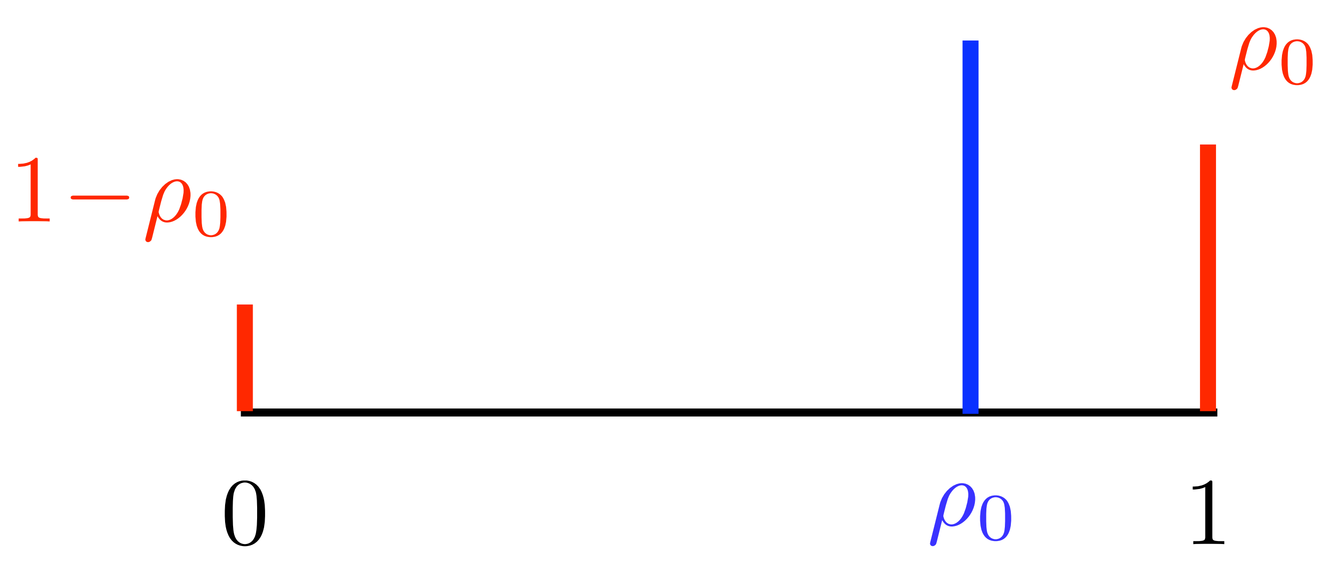
$$w\big(\{\sigma\}_i \to \{\sigma\}'_i\big) = \frac{1}{2}\Big(1 - \frac{\sigma_i}{z}\sum_{k\,\mathrm{nn}\,i}\sigma_k\Big)$$

1. Final State (Exit) Probabilities $E_{\pm}(\rho_0) = \rho_0$

Equation of Motion for single spin:

mean spin $s_i = \langle \sigma_i \rangle$: $\dfrac{ds_i}{dt} = -2\langle \sigma_i w_i \rangle = -s_i + \dfrac{1}{z} \sum_{k \text{ nn } i} s_k$

$$\rightarrow \dot{m} \equiv \sum_i \dot{s}_i = 0$$



2. Spatial Dependence of 2-Spin Correlations

(infinite system)

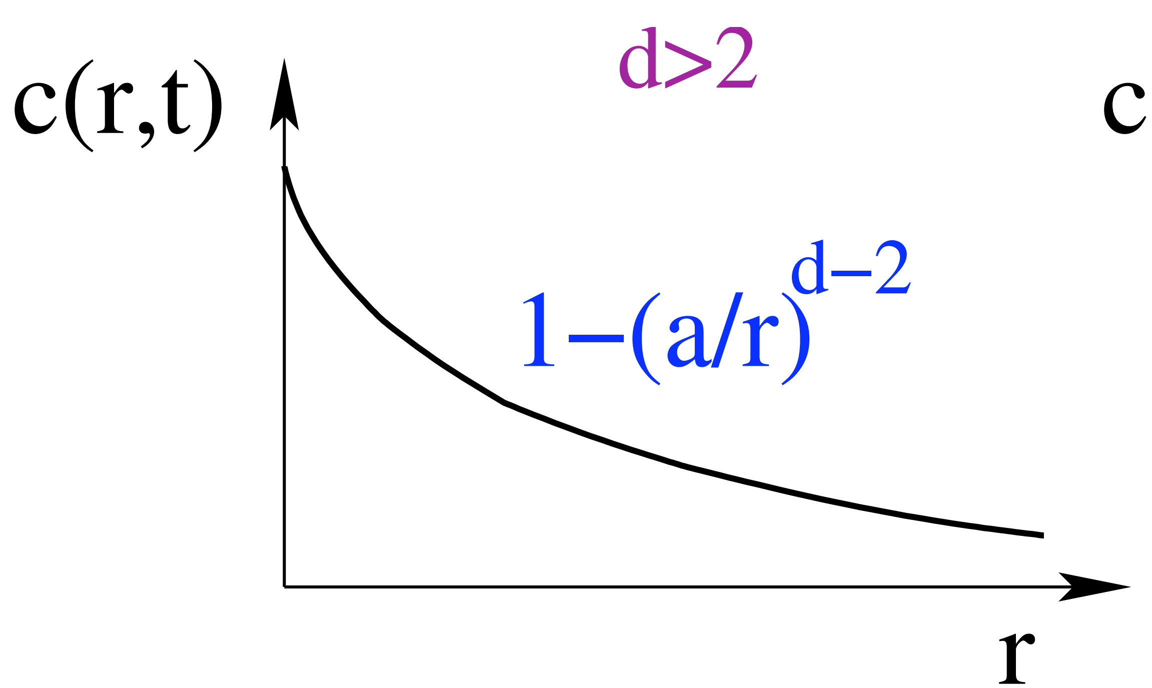
Equation for 2-spin correlation function:

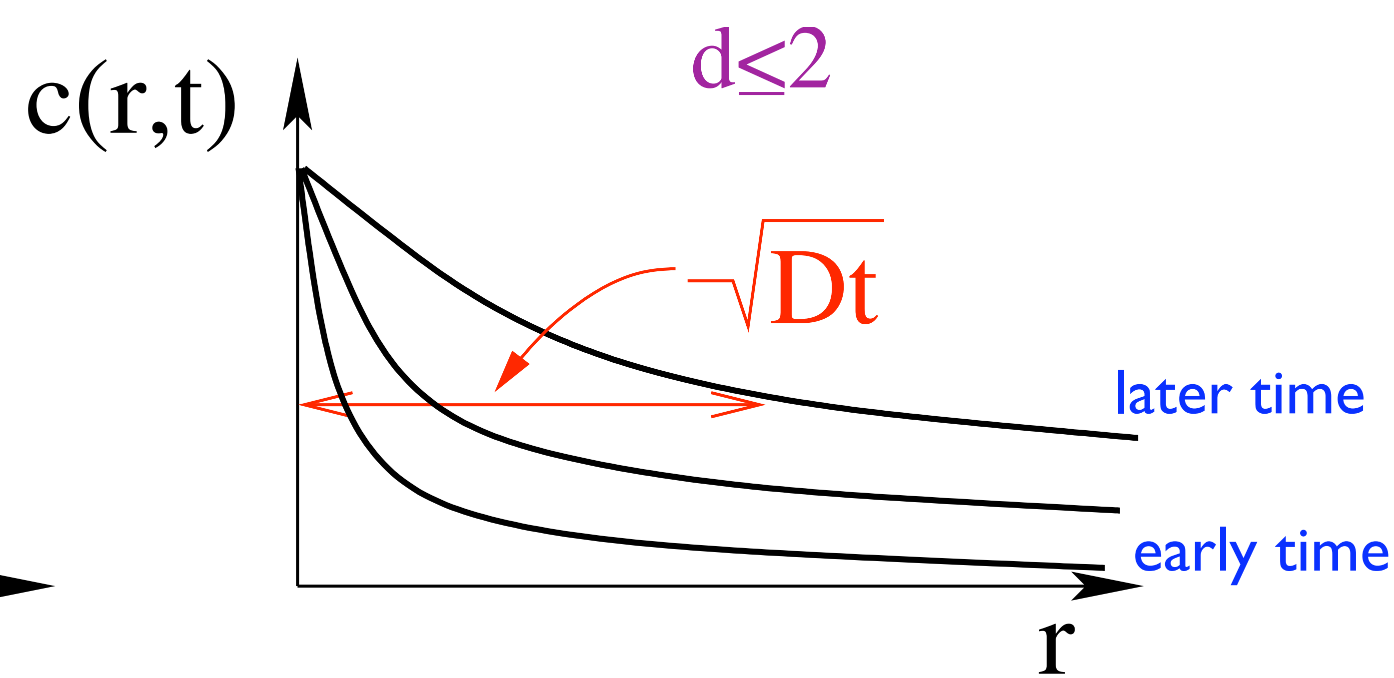
$$\frac{d(s_i s_j)}{dt} = -2\langle \sigma_i \sigma_j (w_i + w_j)\rangle \rightarrow \frac{\partial c_2(r,t)}{\partial t} = \nabla^2 c_2(r,t)$$

$$c(r=0,t) = 0;\ \ c(r,t=0) = \delta(r)$$

Asymptotic solution:

$$c(r,t) \sim \begin{cases} 1 - \dfrac{1-(\frac{a}{r})^{d-2}}{1-(\frac{a}{\sqrt{Dt}})^{d-2}} & d \neq 2 \\ \dfrac{1-\frac{\ln r}{\ln a}}{1-\frac{\ln\sqrt{Dt}}{\ln a}} & d = 2 \end{cases}$$





3. System Size Dependence of Consensus Time

Liggett (1985), Krapivsky (1992)

$$\int^{\sqrt{Dt}} c(r,t) r^{d-1}\, dr = N$$

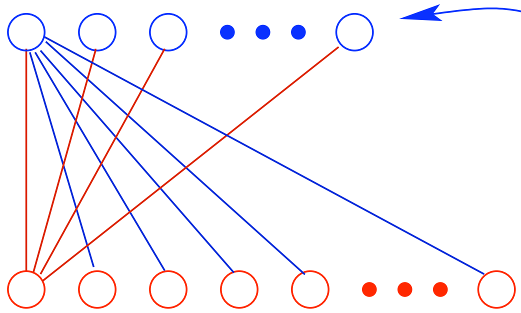
dimension	consensus time
1	N^2
2	N ln N
>2	N

Voter Model on Heterogeneous Graphs

Castellano et al (2003)
Suchecki et al (2004)
Sood & SR (2005)

illustrative example: *complete bipartite graph*

pick site on **a** sublattice — pick ↓ on **a** — pick ↑ on ***b*** sublattice



a sites
degree b

b sites
degree a

$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\,\frac{N_b}{b} - \frac{N_a}{a}\,\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\,\frac{N_a}{a} - \frac{N_b}{b}\,\frac{a-N_a}{a}\right]$$

Subgraph densities: $\rho_a = N_a/a,\ \rho_b = N_b/b \quad dt = 1/(a+b)$

$$\rho_{a,b}(t) = \frac{1}{2}[\rho_{a,b}(0) - \rho_{b,a}(0)]\, e^{-2t} + \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

$$\rightarrow \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

N.B.: magnetization is ***not*** conserved

Exit probabilities:

$$E_+ = 1 - E_- = \frac{1}{2}[\rho_a(0) + \rho_b(0)].$$

Exit probabilities

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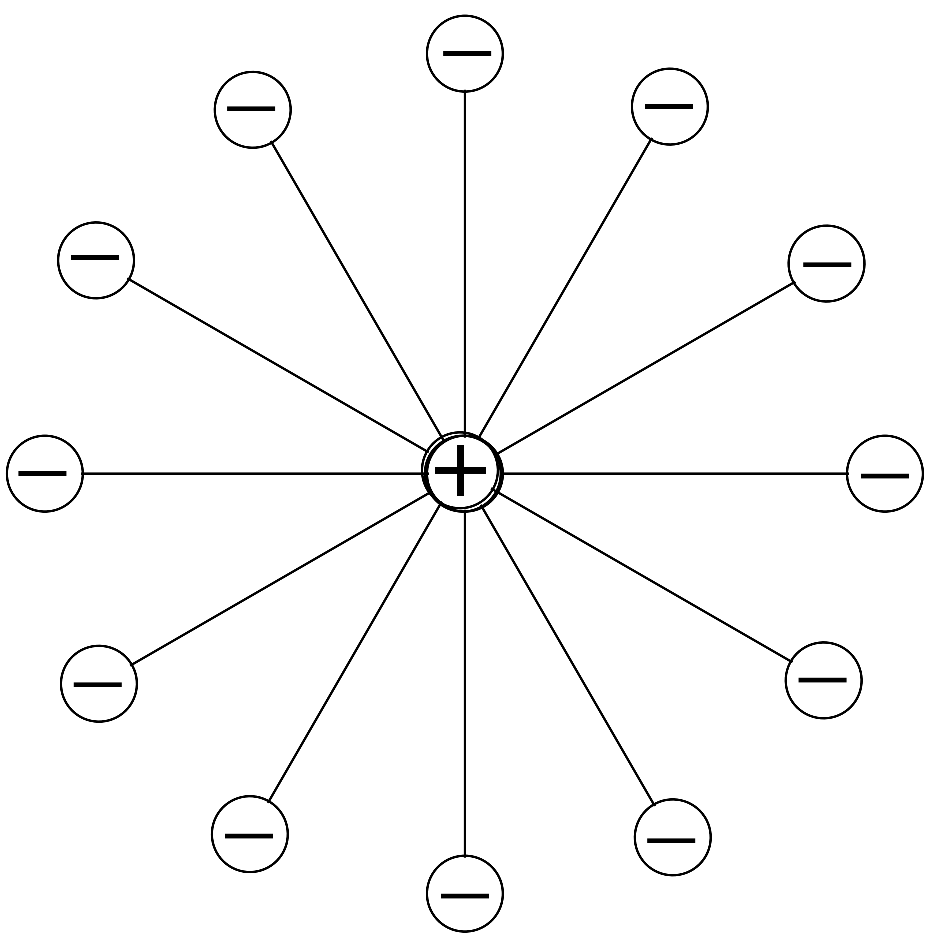
Extreme case: star graph

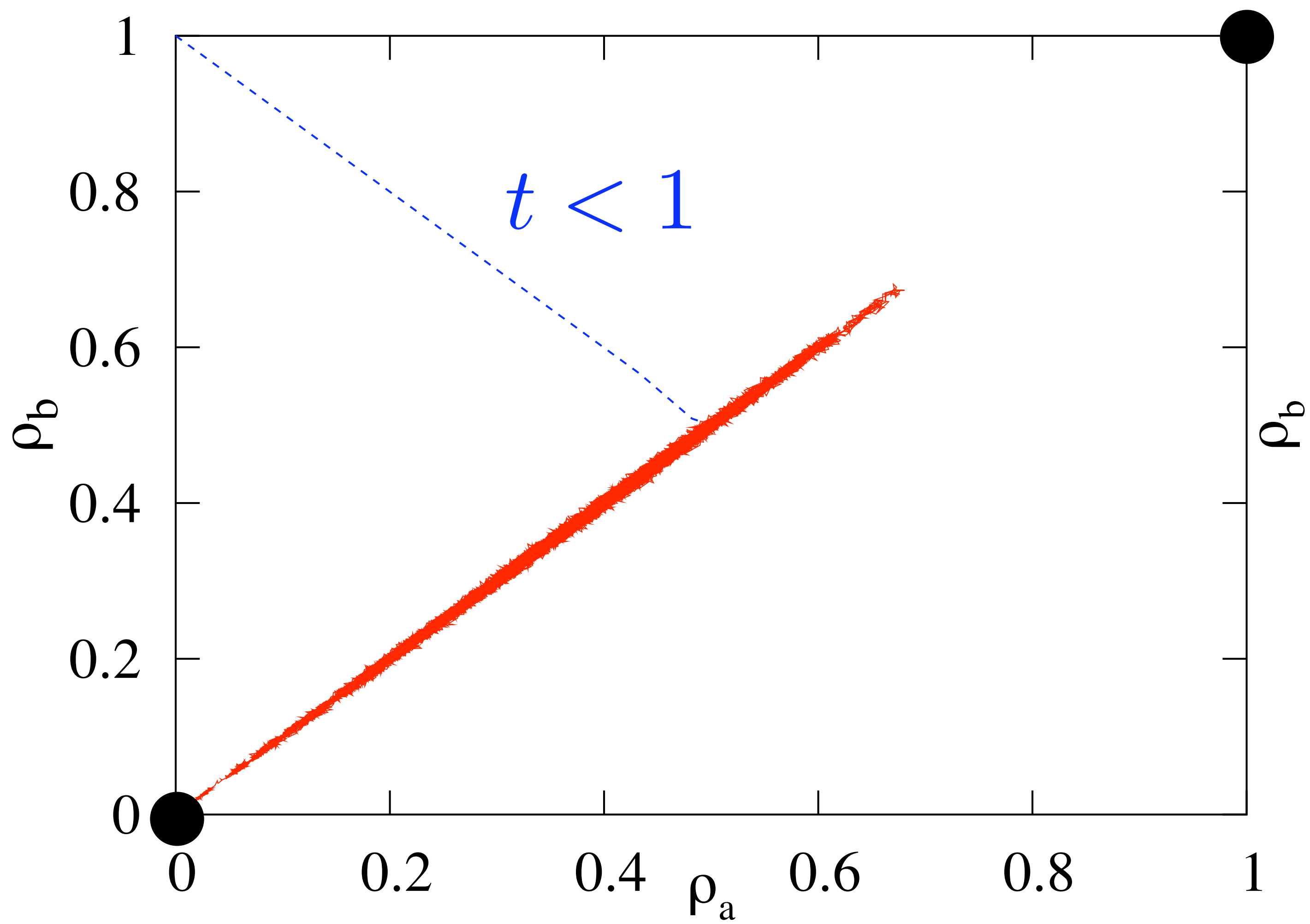


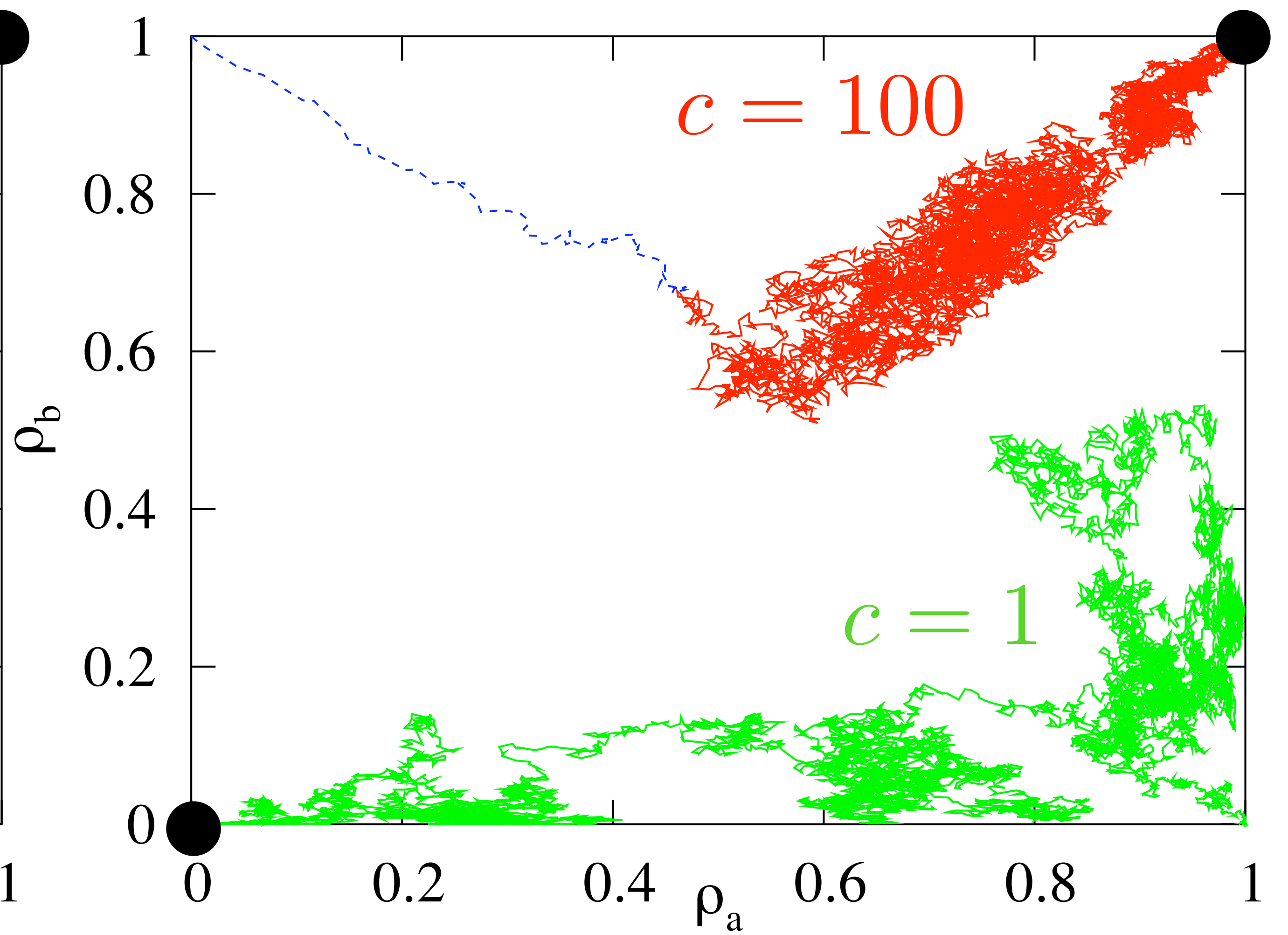
Initial state: 1 plus, N minus

Final state: all + probability 1/2!

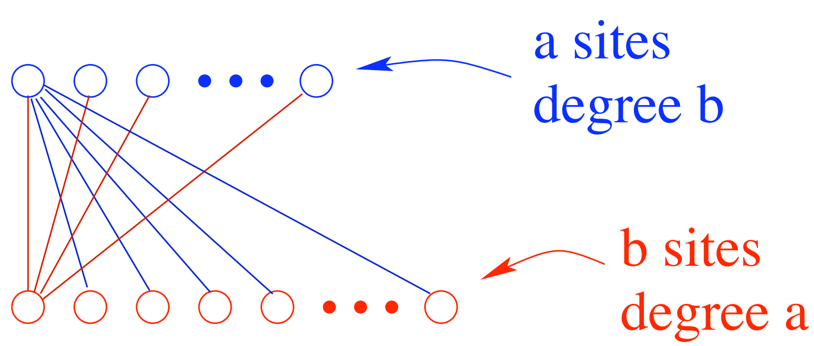
Route to Consensus

trajectory of a single realization

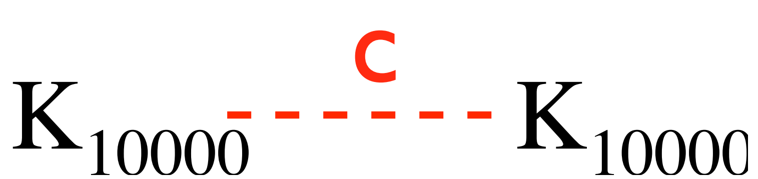




complete bipartite graph



two-clique graph



Mean Consensus Time

pick site on the *a* sublattice

pick ↓ on *a*

pick ↑ on ***b*** sublattice

consensus time from new state

$$\begin{aligned}
T(\rho_a,\rho_b) \quad &= \quad \frac{a}{a+b}(1-\rho_a)\rho_b\left[T(\rho_a+\frac{1}{a},\rho_b)+\delta t\right] \\
&+ \quad \frac{a}{a+b}\rho_a(1-\rho_b)\left[T(\rho_a-\frac{1}{a},\rho_b)+\delta t\right] \\
&+ \quad \frac{b}{a+b}(1-\rho_b)\rho_a\left[T(\rho_a,\rho_b+\frac{1}{b})+\delta t\right] \\
&+ \quad \frac{b}{a+b}\rho_b(1-\rho_a)\left[T(\rho_a,\rho_b-\frac{1}{b})+\delta t\right] \\
&+ \quad (1-\rho_a-\rho_b+2\rho_a\rho_b)\left[T(\rho_a,\rho_b)+\delta t\right],
\end{aligned}$$

continuum limit:

$$\begin{aligned}
N\delta t \quad &= \quad (\rho_a-\rho_b)(\partial_a-\partial_b)T_N(\rho_a,\rho_b) \\
&\quad -\frac{1}{2}(\rho_a+\rho_b-2\rho_a\rho_b)\left(\frac{1}{a}\partial_a^2+\frac{1}{b}\partial_b^2\right)T_N(\rho_a,\rho_b)
\end{aligned}$$

assuming $\rho_a = \rho_b$ and $\rho = (\rho_a + \rho_b)/2$:

equation of motion for T becomes:

$$\frac{1}{4}\rho(1-\rho)\left(\frac{1}{a}+\frac{1}{b}\right)\partial^2 T = -1$$

with solution:

$$T_{ab}(\rho) = -\frac{4ab}{a+b}\left[(1-\rho)\ln(1-\rho) + \rho\ln\rho\right]$$

implications:

$$a = \mathcal{O}(1),\ b = \mathcal{O}(N)\ \text{(star graph)},\ T = \mathcal{O}(1)$$

$$a = \mathcal{O}(N),\ b = \mathcal{O}(N)\ \text{(symmetric graph)},\ T = \mathcal{O}(N)$$

Power-Law Degree Distribution Network

$$n_j = \text{fraction of nodes with degree } j$$

$$\mu_m = \sum_j j^m\, n_j = m^{\text{th}} \text{ moment of degree distribution}$$

$$\omega = \frac{1}{\mu_1} \sum_j j n_j \rho_j = \text{degree-weighted up spin density}$$

Basic result: $$T_N(\omega) = -N \frac{\mu_1^2}{\mu_2} \left[(1-\omega)\ln(1-\omega) + \omega \ln \omega\right]$$

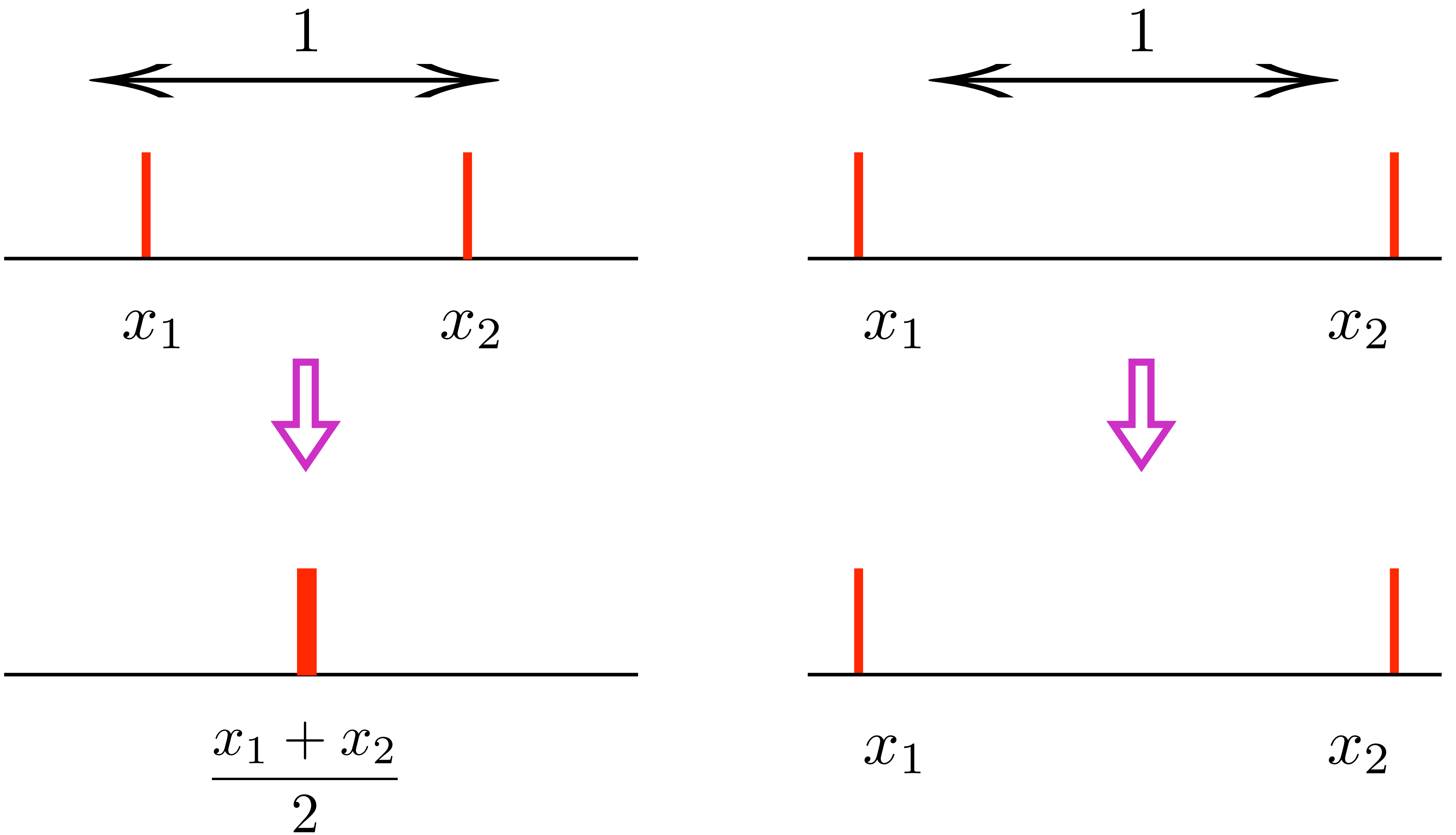
For power-law network: $(n_j \sim j^{-\nu})$

$$T_N \sim \begin{cases} N & \nu > 3, \\ N/\ln N & \nu = 3, \\ N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\ (\ln N)^2 & \nu = 2, \\ \mathcal{O}(1) & \nu < 2. \end{cases}$$

] quick consensus!

Bounded Compromise Model

Deffuant et al (2000)



If $|x_2 - x_1| < 1$ compromise

If $|x_2 - x_1| > 1$ no interaction

The Opinion Distribution

Basic observable: P(x,t) = probability that agent has opinion x

Fundamental parameter: Δ, the initial opinion range

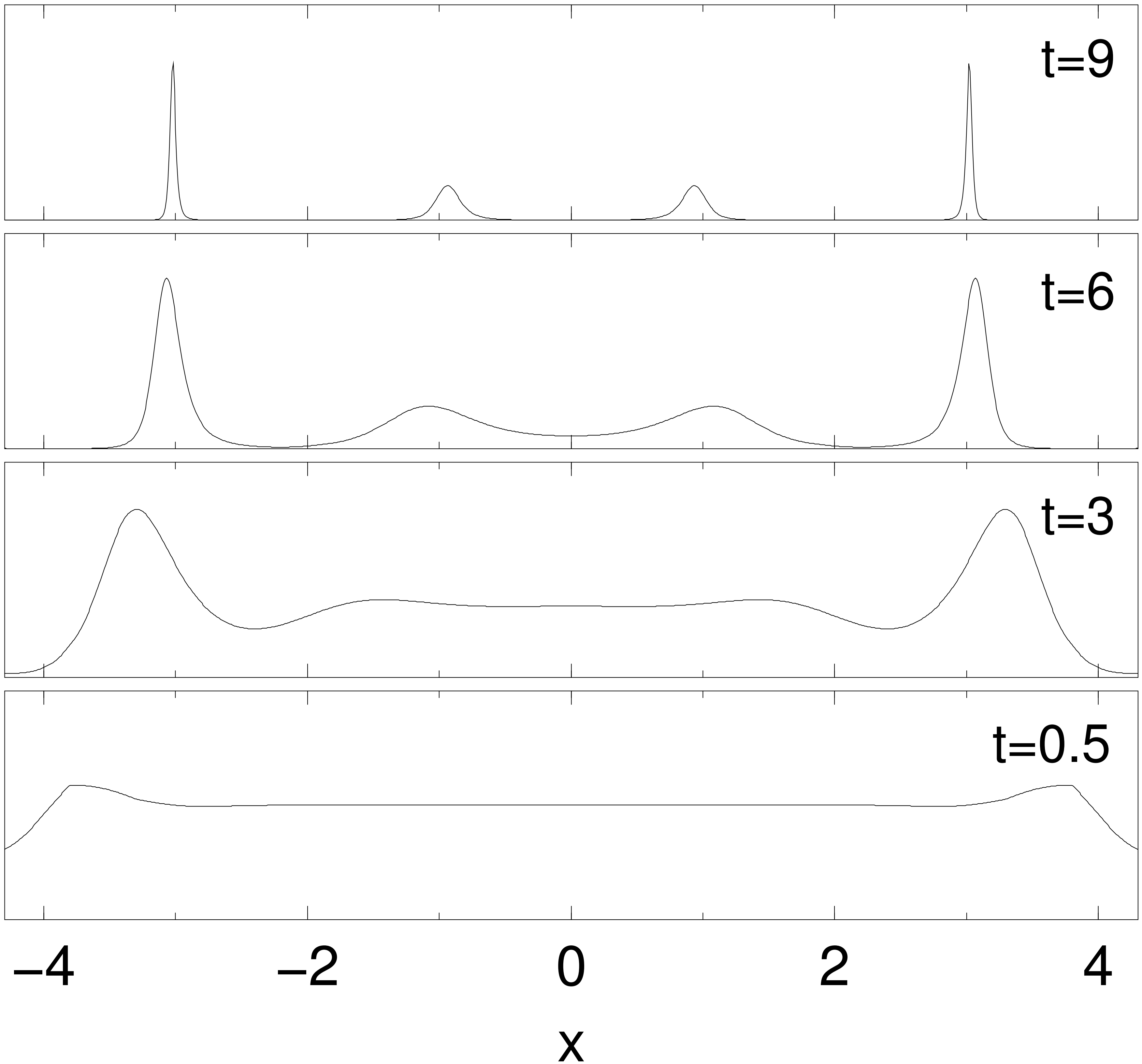
Master equation:

$$\frac{\partial P(x,t)}{\partial t} = \int\int_{|x_1-x_2|<1} dx_1\, dx_2\, P(x_1,t)P(x_2,t) \times \left[\delta\left(x - \frac{x_1+x_2}{2}\right) - \delta(x-x_1)\right]$$

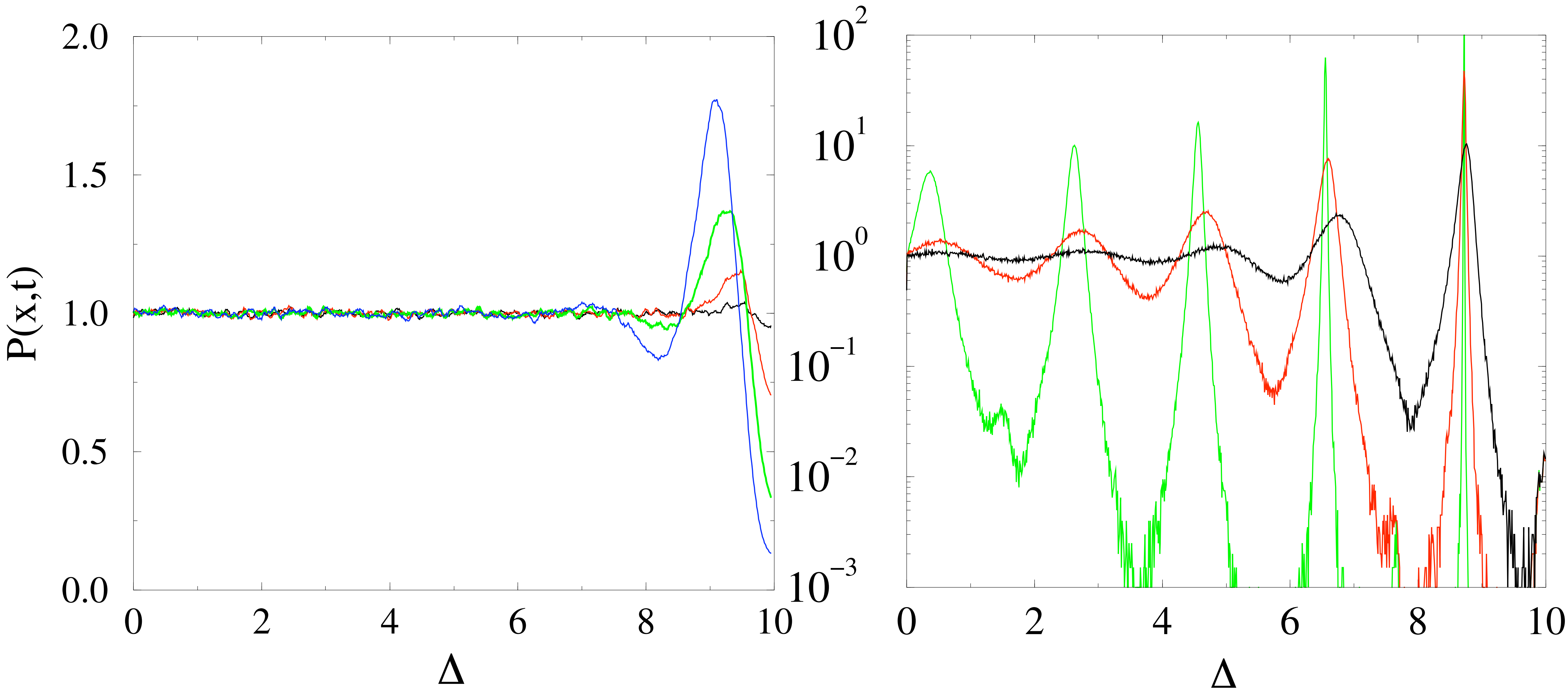
Δ<1: eventual consensus

Δ>1: disjoint "parties"

Early time evolution (for Δ=4)



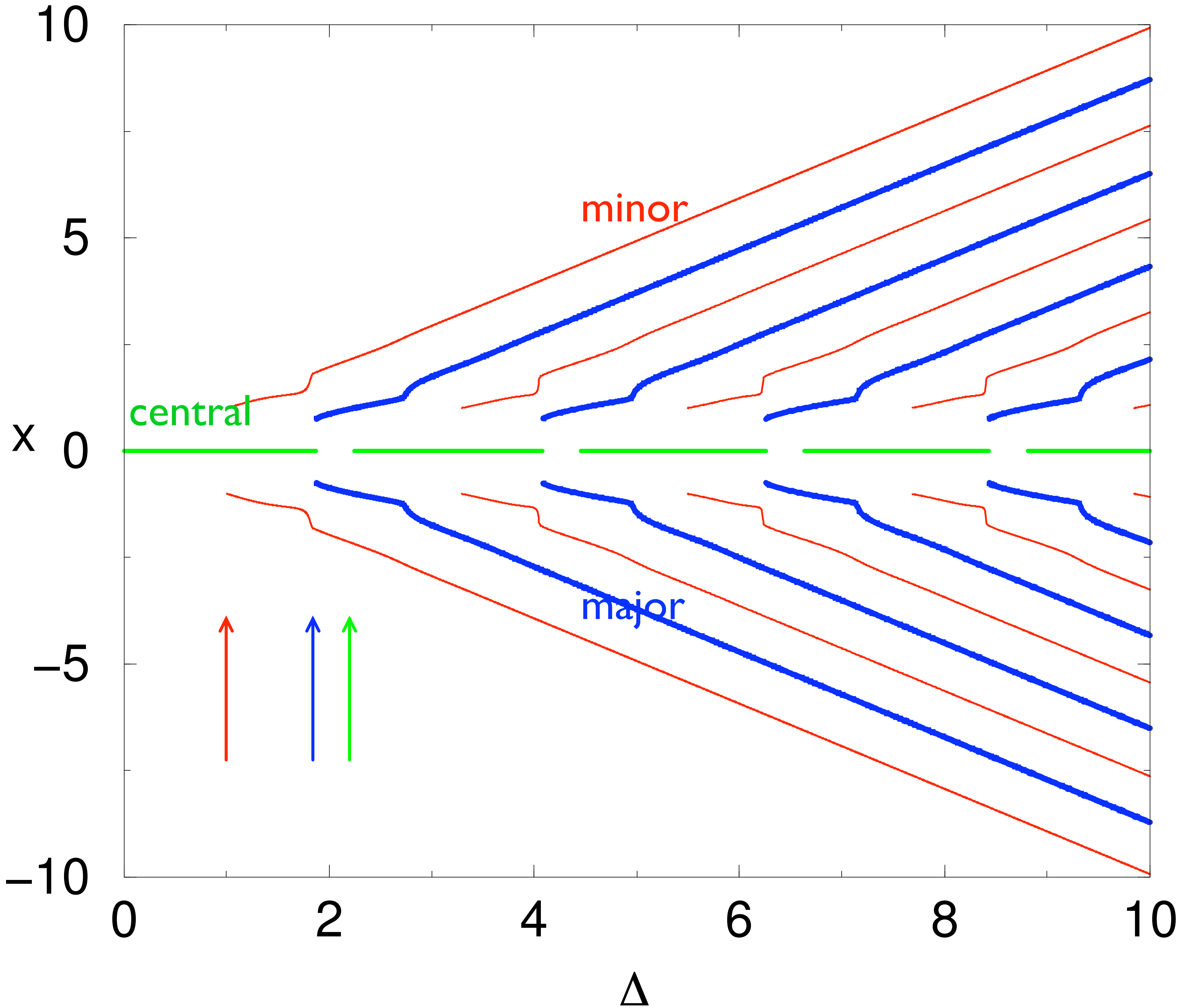
Time Evolution (for Δ=10)



early time

late time

Bifurcation Sequence



Axelrod Model

Axelrod (1997)
Castellano et al (2000)
Klemm et al (2003)
Vazquez & SR (2006)

culture = (car, food, politics, job,.....)

F features

q states

SUV
economy
sports
luxury
⋮

Typical interaction:

(SUV, steak, GOP, cop) (SUV, vegan, dem, lawyer)

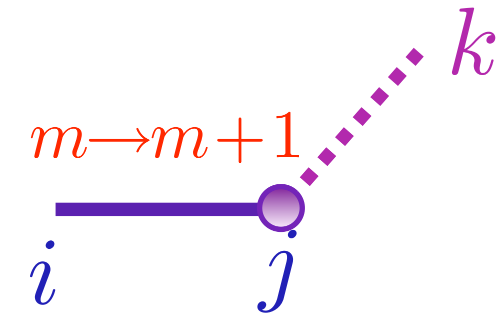
(SUV, steak, GOP, cop) (SUV, steak, dem, lawyer)

Basic question: *consensus or cultural fragmentation?*

A Minimalist (*ersatz* Mean-Field) Description

$P_m(t) \equiv$ fraction of links with m shared features

Master equation:



direct interaction

gain — loss

$$\frac{dP_m(t)}{dt} = \frac{2}{\eta+1}\left[\frac{m-1}{F}P_{m-1} - \frac{m}{F}P_m\right]$$

gain -- — gain +

$$+\frac{2\eta}{\eta+1}\sum_{k=1}^{F-1}\frac{k}{F}P_k\left[\frac{m+1}{F}P_{m+1} + \lambda\left(1-\frac{m-1}{F}\right)P_{m-1}\right.$$

loss

$$\left. - \left[\lambda\left(1-\frac{m}{F}\right) + \frac{m}{F}\right]P_m\right]$$

total activity of indirect bonds

indirect interaction

$\eta + 1 =$ coordination number

$\lambda =$ prob. that i & k share 1 feature not shared by $j = (q-1)^{-1}$

Special Case: F=2, varying q

qualitatively similar to general (F,q)

$$
\begin{aligned}
\frac{dP_0}{dt} &= \frac{\eta}{\eta+1} P_1 \left[-\lambda P_0 + \frac{1}{2} P_1\right] \\
\frac{dP_1}{dt} &= -\frac{P_1}{\eta+1} + \frac{\eta}{\eta+1} P_1 \left[\lambda P_0 - \frac{1+\lambda}{2} P_1 + P_2\right] \\
\frac{dP_2}{dt} &= \frac{P_1}{\eta+1} + \frac{\eta}{\eta+1} P_1 \left[\frac{\lambda}{2} P_1 - P_2\right]
\end{aligned}
$$

Formal Solution:

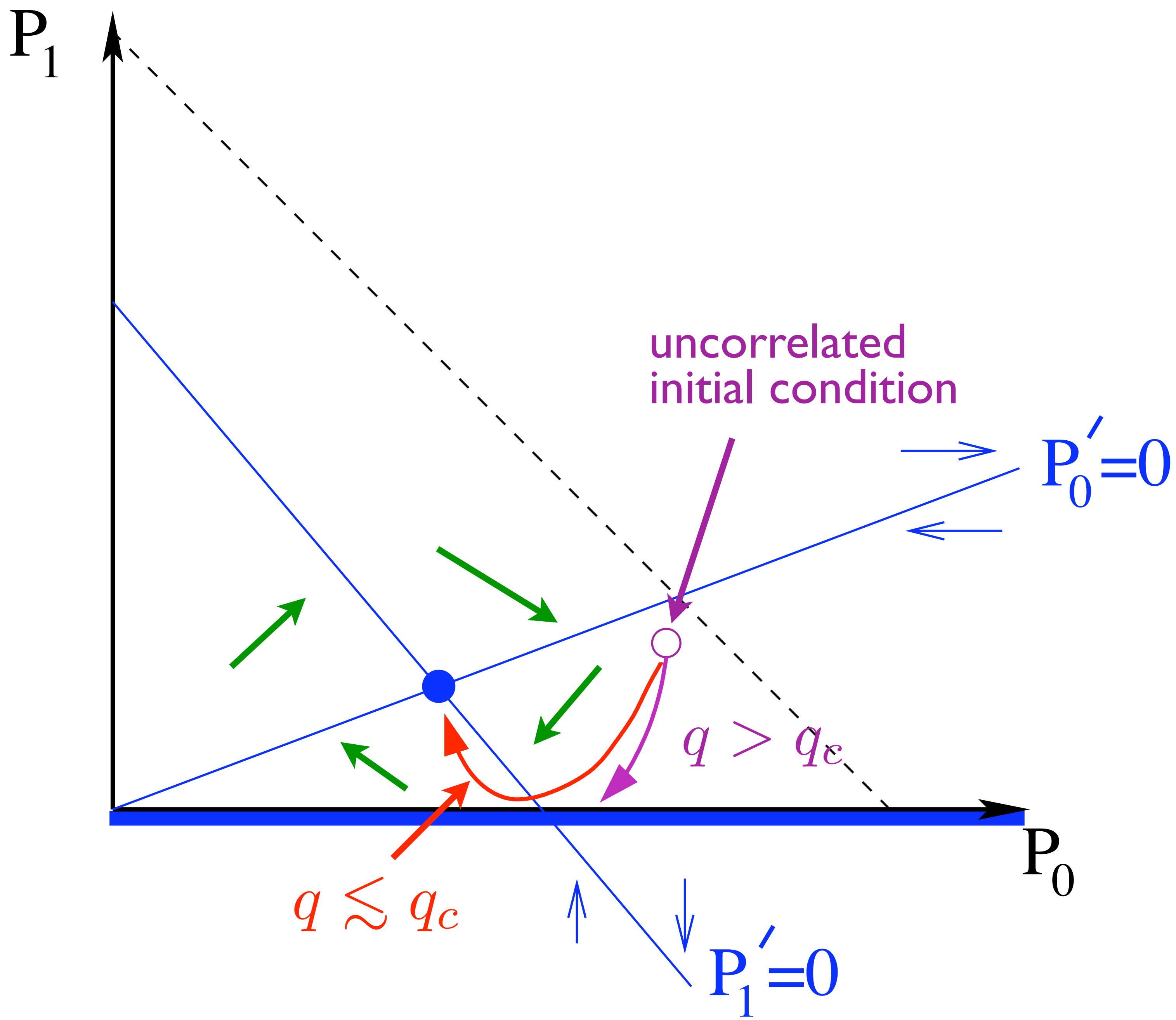
$$
\tau = \frac{1}{4\lambda(\eta-1)} \left[\ln\left(\frac{S+\Delta}{\eta\lambda(1-\lambda)^2}\right) - 2\ln\left(1 \pm \frac{\sqrt{\Delta}}{1-\lambda}\right) + \frac{1-\lambda}{\sqrt{-S}} \ln\left(\frac{(\sqrt{-S}-1-\lambda)(\sqrt{-S}\pm\sqrt{\Delta})}{(\sqrt{-S}+1+\lambda)(\sqrt{-S}\mp\sqrt{\Delta})}\right)\right]
$$

$$
\begin{aligned}
S &= 4\eta\lambda - (1+\lambda)^2 \\
\Delta &= 2\eta(1+\lambda)^2 P_1 - S \\
\tau &= t/[2(1+\eta)(1+\lambda)]
\end{aligned}
$$

Dynamical Analysis

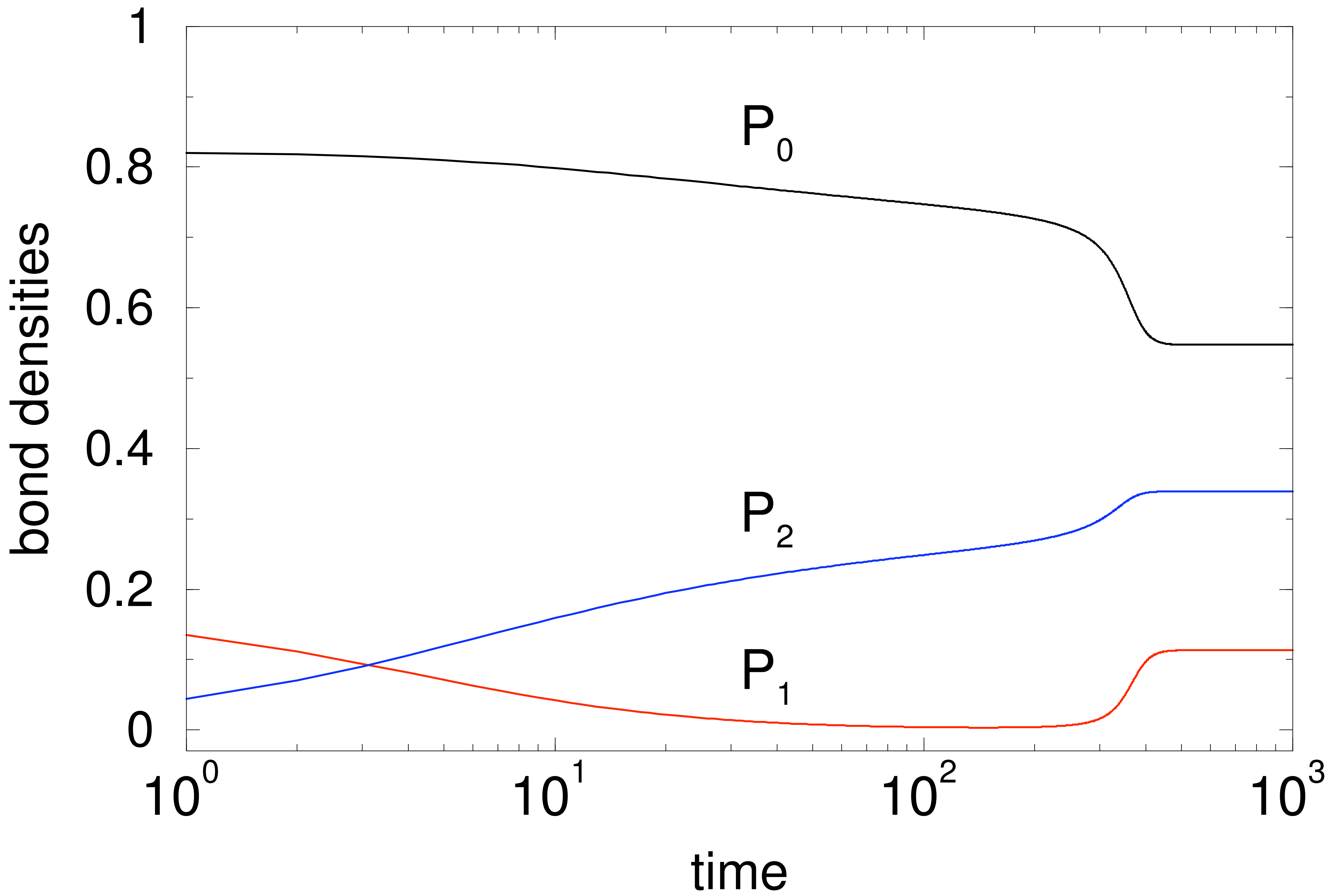
$$\frac{dP_0}{dx} = -\lambda P_0 + \frac{1}{2} P_1$$
$$\frac{dP_1}{dx} = \left(1 - \frac{1}{\eta}\right) + (\lambda - 1) P_0 - \left(\frac{3+\lambda}{2}\right) P_1$$

$$dx = \frac{\eta}{\eta+1} P_1 dt$$
$$P_2 = 1 - P_0 - P_1$$



Simulation Results

$$q = q_c - \frac{1}{4} \qquad q_c = 10.898\ldots$$



Outlook & Open Questions

1. Heterogeneous voter model: fast consensus

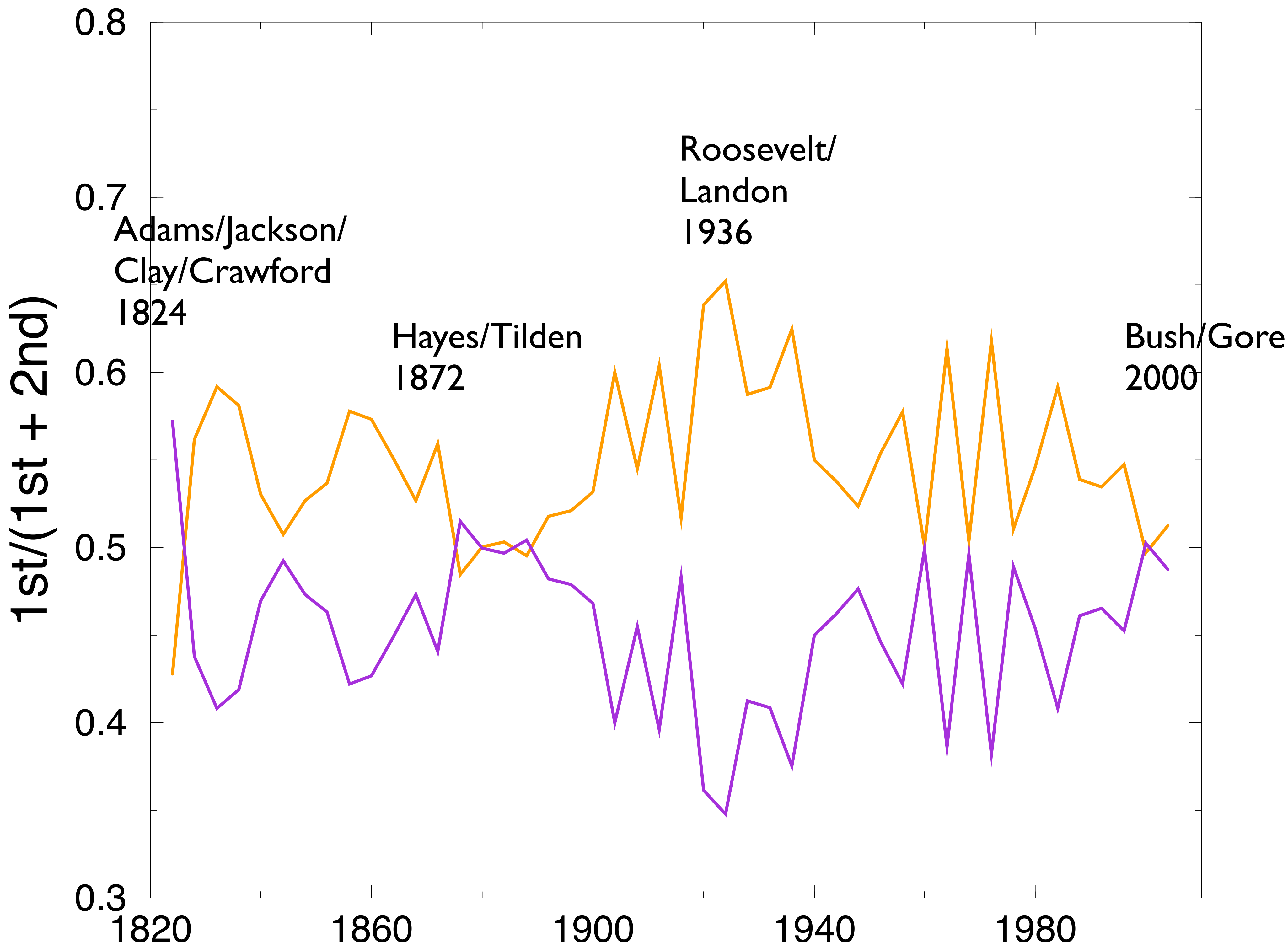
   What is the route to consensus?

   Role of fluctuations?

   Role of the correlations?

   Application to real voting?

History of US Presidential Elections



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   Application to real voting? *permanence/impermanence*

2. Bounded Compromise: fragmentation a natural outcome

   Is threshold an appropriate mechanism
   for fragmentation?

A Possible Realization

1993 Canadian Federal Election

year	BQ	NDP	L	PC	SC	R/CA
1979		26	114	136	6	
1980		32	147	103		
1984		30	40	211		
1988		43	83	169		
1993	54	9	177	2		52
1997	44	21	155	20		60

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    What is the route to consensus?

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    Application to real voting? *permanence/impermanence*

2. Bounded Compromise: fragmentation a natural outcome

    Is threshold the right mechanism for
    lack of consensus and fragmentation?

3. Axelrod model: slow non-monotonic dynamics

    Spatially local interactions? *more complex than coarsening*

    Why is the dynamics so slow?

    Why is there non-monotonicity?