Cutting Corners

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13th C. Itzykson Conference: *Puzzles of Growth*

Question: *What is the shape of a manifold whose surface dynamically erodes?*

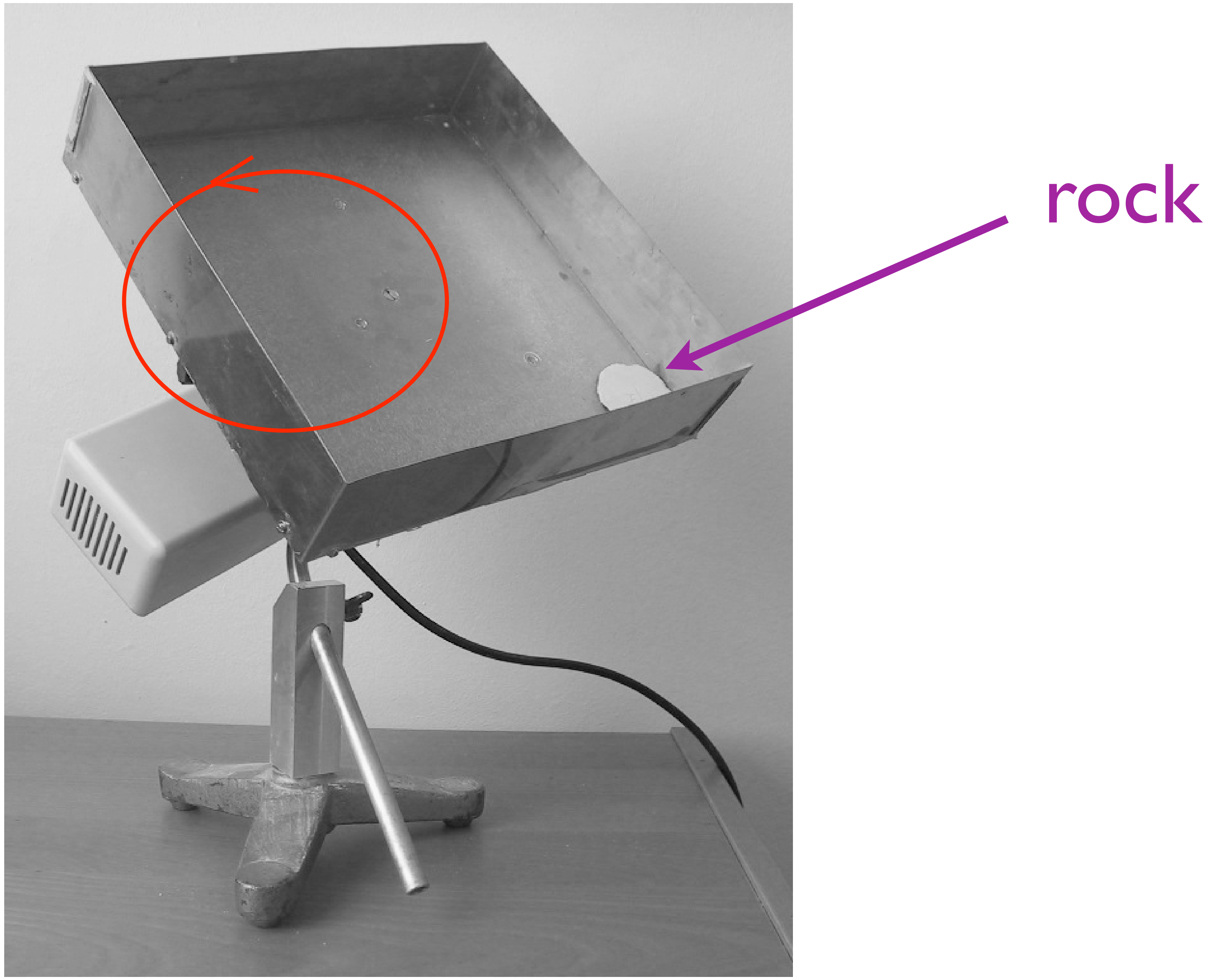
Examples: *Boulders → pebbles*
Ising interface evolution

Results: *Chipping model → not-quite round shape*
Corner interface ↔ ASEP ↔ integer partitions

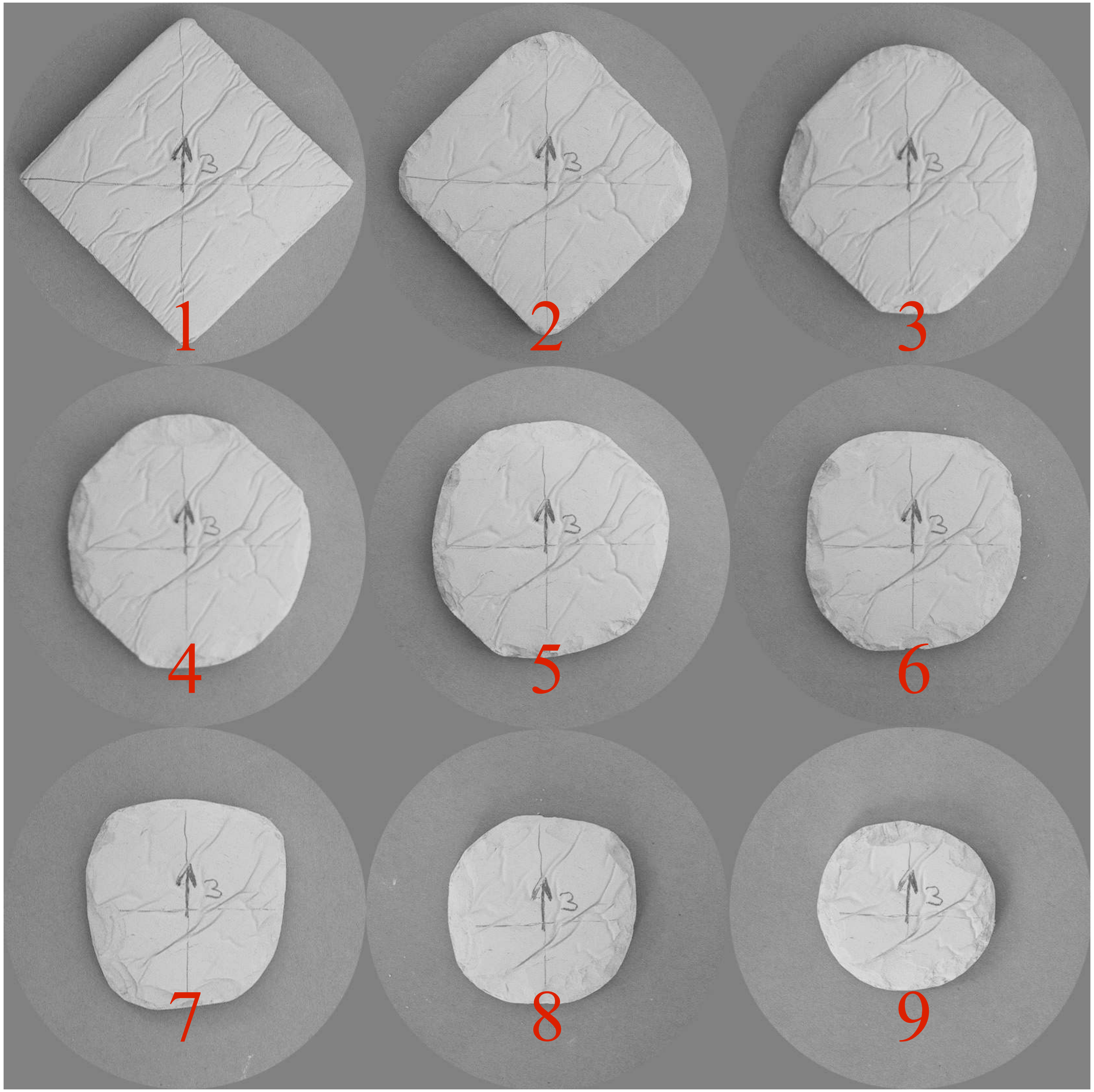
Challenges: *3 dimensions*
does a paramagnet reach the ground state?

Doug Durian's Erosion Machine

Durian et al., PRL **97**, 028001 (2006);
PRE **75**, 021301 (2007)



Evolution of a Square Rock



Expectations

Aristotle (~350 BC):

Rounding due to faster erosion at exposed corners and extremities.

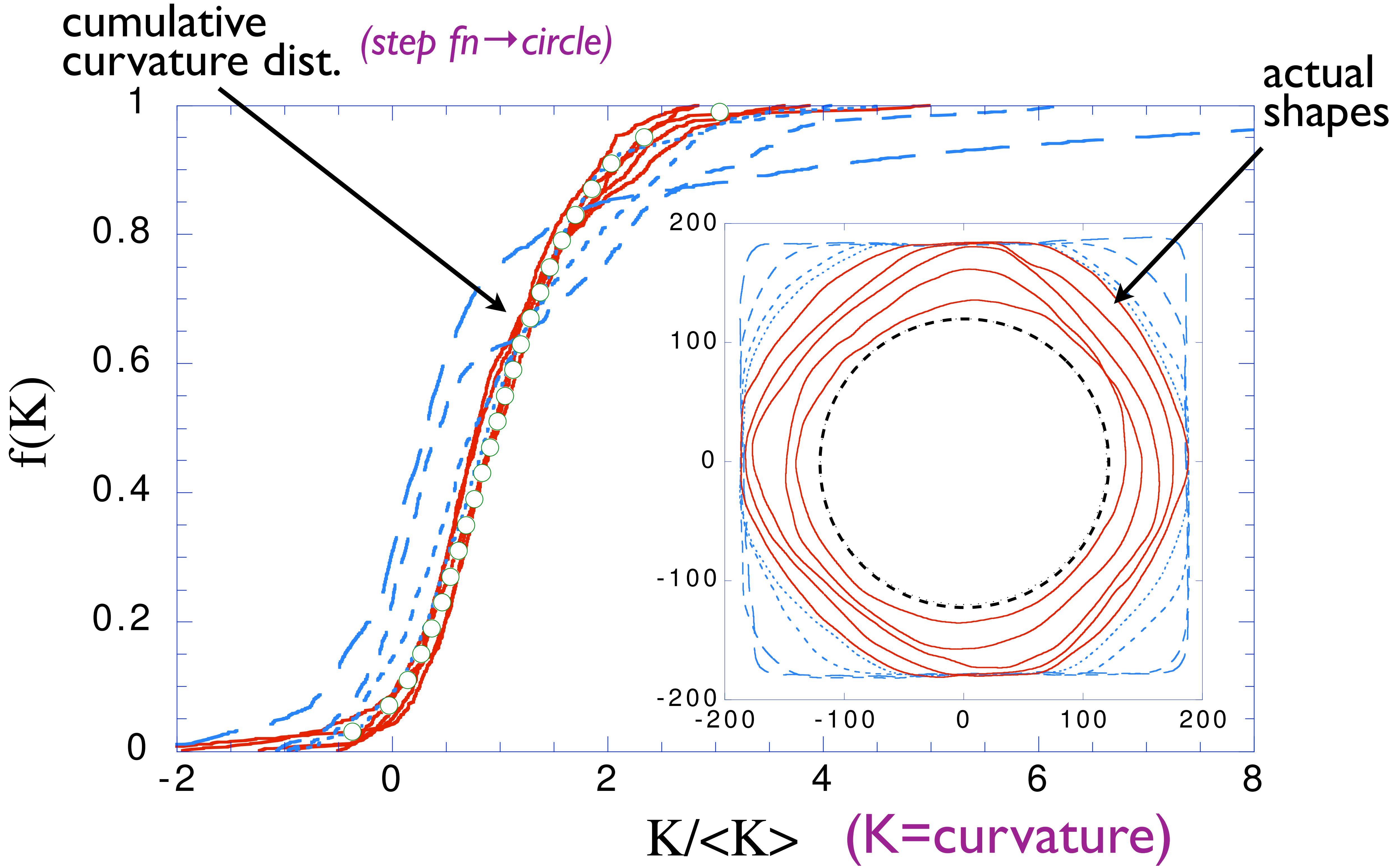
Mullins (1956): & many differentrial geometry pubs

If $v_{\mathrm{interface}} \propto$ local curvature,

$\rightarrow$ circular limiting shape for $d = 2$

non-circular limiting shape for $d > 2$

Rock Chipping: Final Shape ***not*** Circular


cumulative curvature dist.

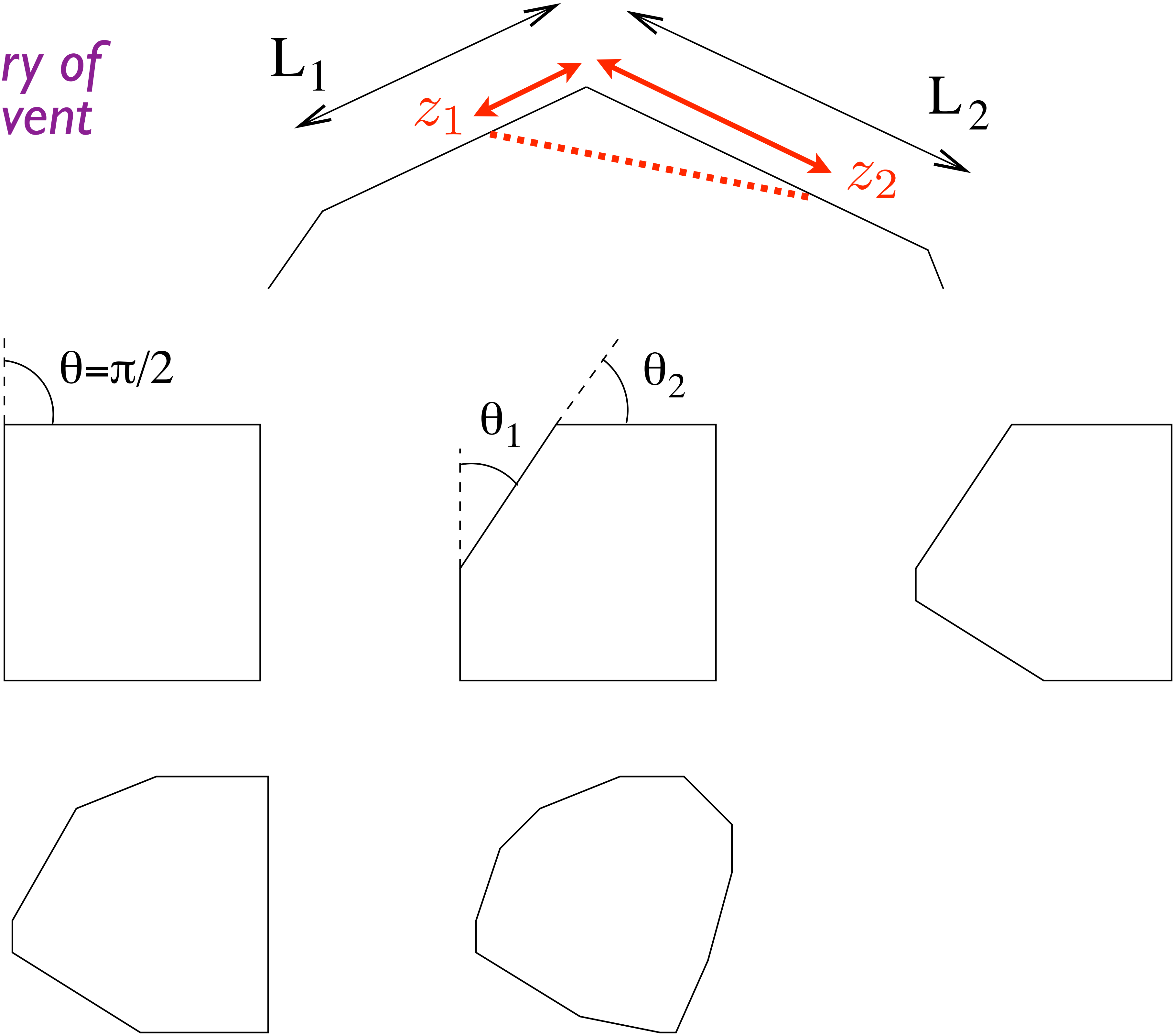
(step fn→circle)

actual shapes

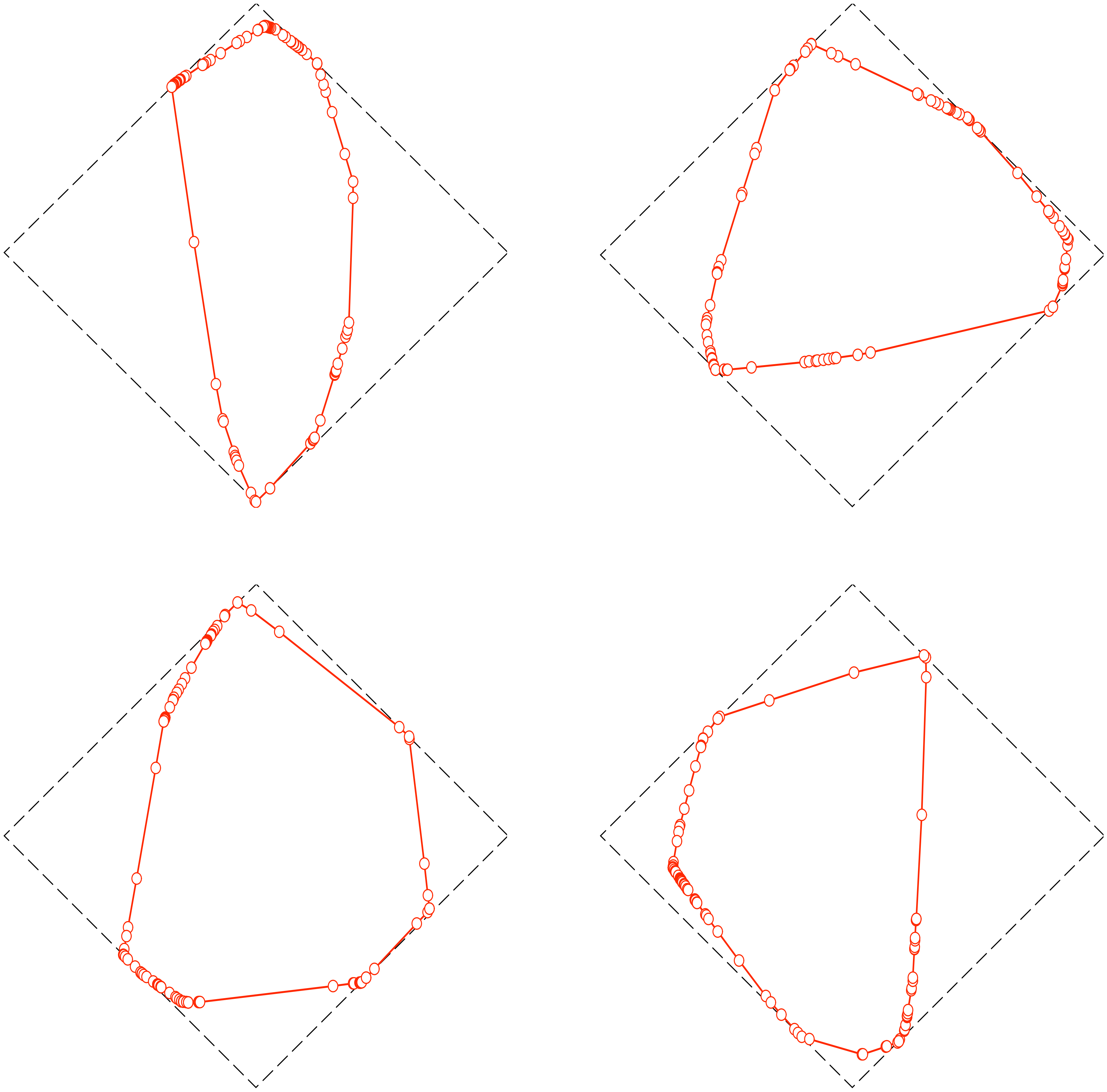
Chipping Model

PLK & SR
PRE **75**, 031119 (2007)

geometry of single event



Numerical Realizations (100 corners)



Angle Evolution for Bisection

$n_k \equiv \#$ corners with "angle" k

$$\begin{aligned} k &\equiv -\ln_2(2\theta/\pi) \\ &= \text{number of halvings} \end{aligned}$$

Master equation: *(start with square; t+4 corners at time t)*

$$n_k(t+1) - n_k(t) = -\overset{\text{lose a } k\text{-corner}}{\searrow}\frac{1}{t+4}\, n_k(t) + \frac{2}{t+4}\overset{\text{bisect a } (k-1)\text{-corner}}{\swarrow} n_{k-1}(t)$$

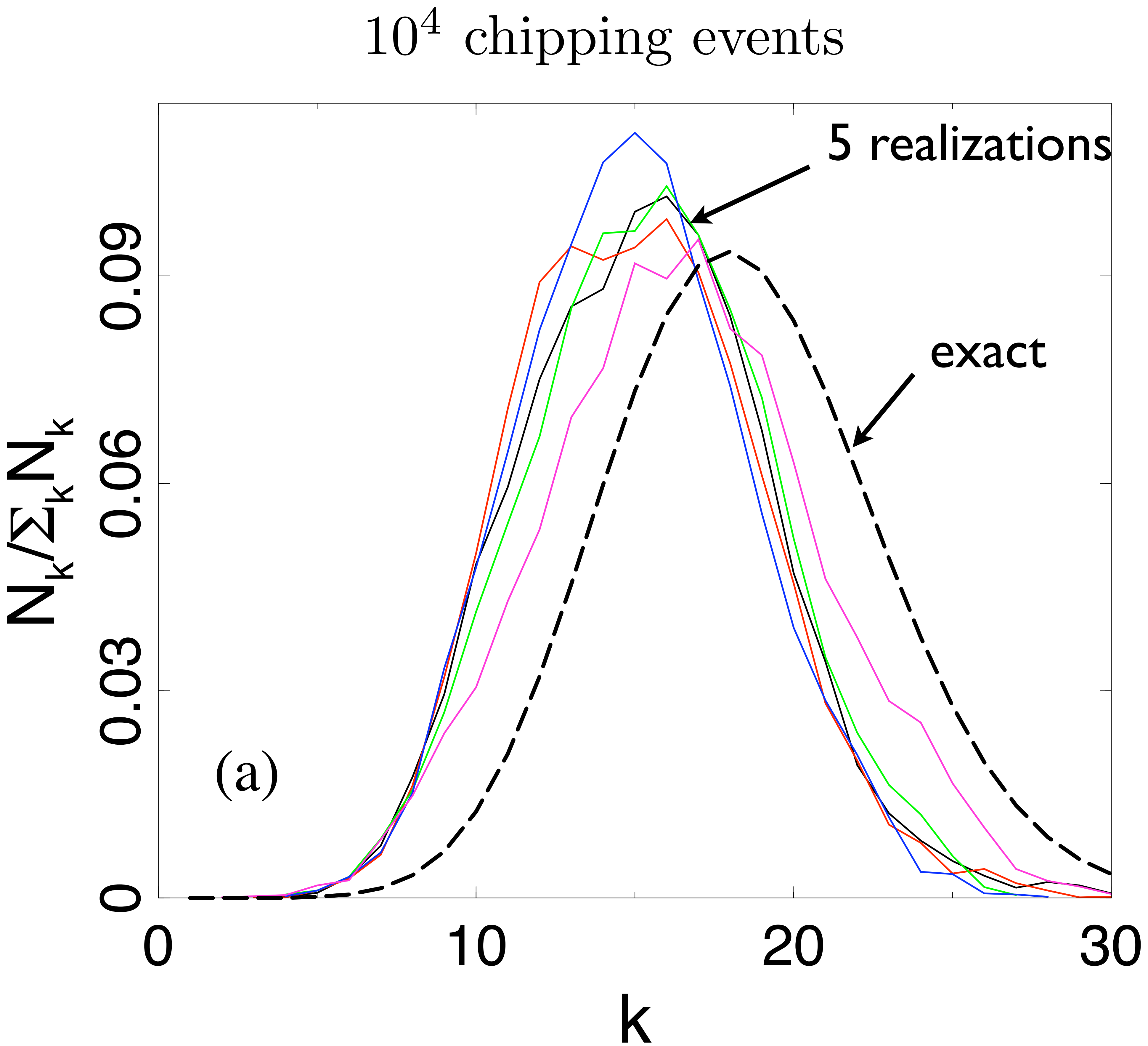
Continuum limit:

$$\frac{dn_k}{dt} = -\frac{n_k}{t} + \frac{2}{t}\, n_{k-1}$$

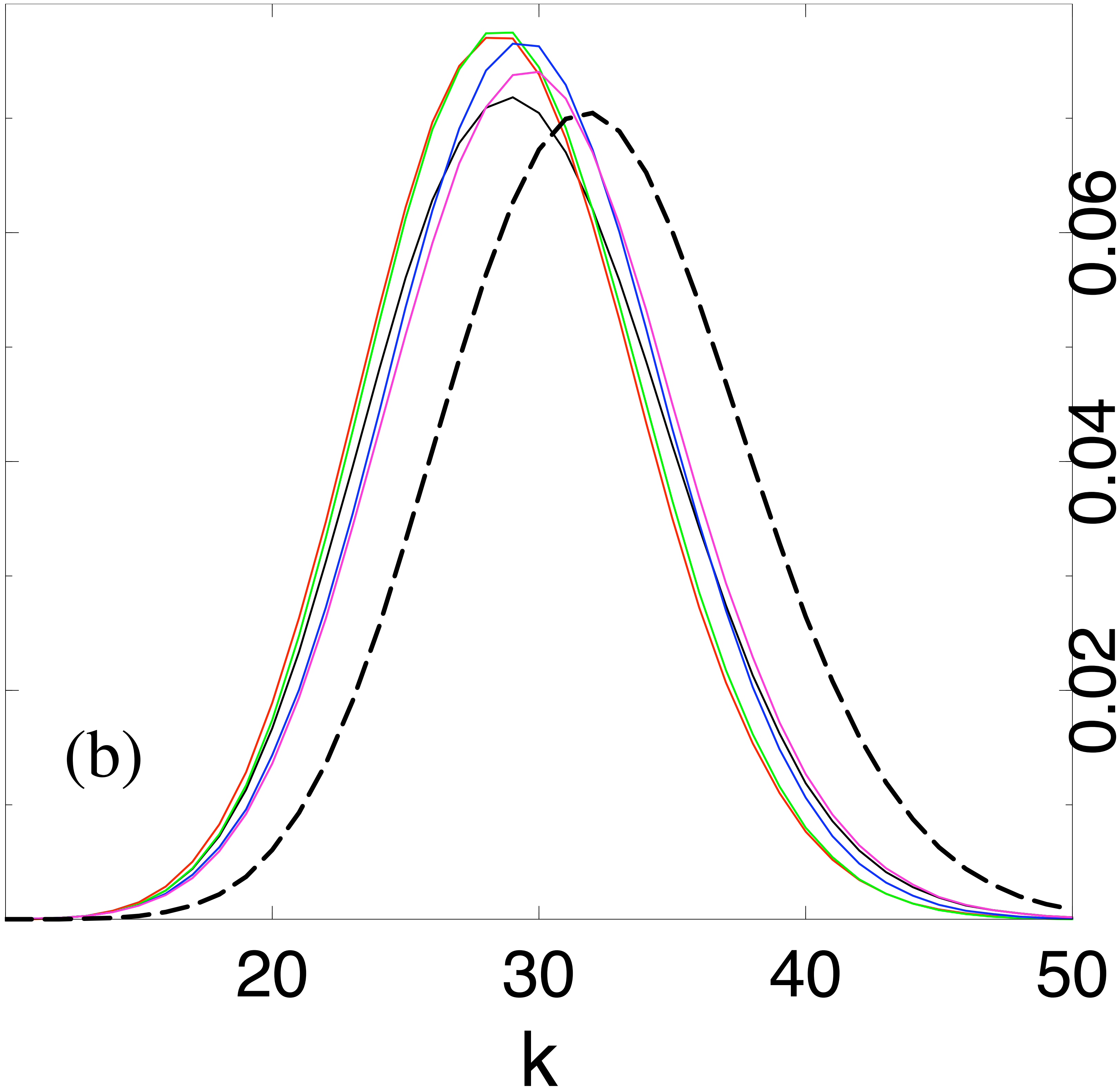
Result:

$$n_k(t) = \frac{12}{t}\,\frac{(2\ln t)^k}{k!}$$

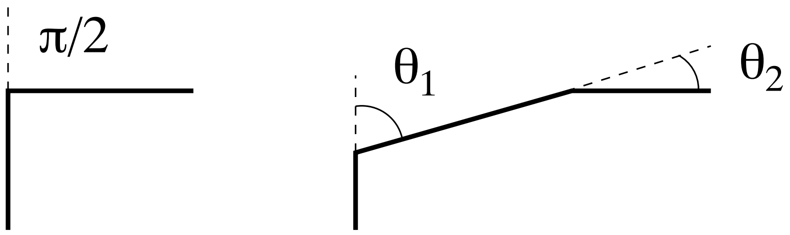
Angle Distribution for Bisection



10^7 chipping events



Angle Evolution for General Angles



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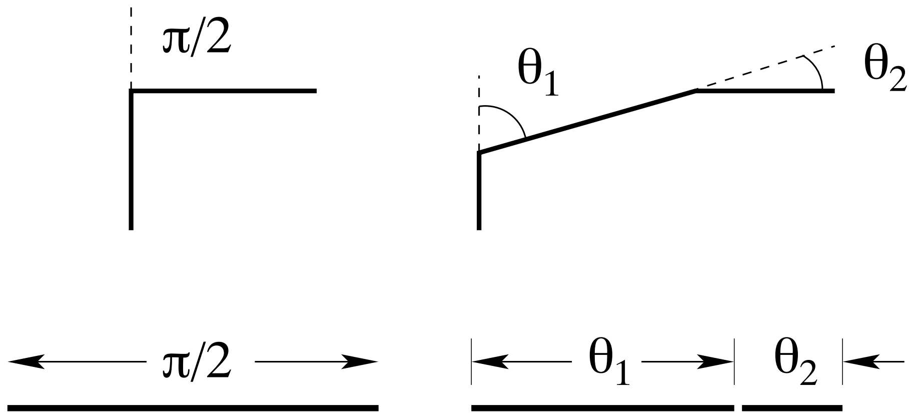


↔ *segment fragmentation*

$$c(x,t) = \text{fraction of angles } x = \theta/2\pi$$

$$\frac{\partial c(x,t)}{\partial t} = -c(x,t) + 2\int_x^1 c(y,t)\,\frac{dy}{y}$$

compare with

$$\frac{dn_k}{dt} = -\frac{n_k}{t} + \frac{2}{t}\,n_{k-1}$$

$$c(\theta,t) \;=\; \frac{8}{\pi}\sqrt{\frac{2t}{\ln(\pi/2\theta)}}\,e^{-t}I_1\left(\sqrt{8t\ln(\pi/2\theta)}\right) + \frac{8}{\pi}e^{-t}\delta\left(\theta - \frac{\pi}{2}\right),$$

$$\sim \quad e^{\sqrt{-t\ln\theta}}$$

Ziff & McGrady (1985); Ziff (1992)

→*broad distribution of angles*

Asymmetry

$$X^2(N) = \frac{1}{N}\sum_{i=1}^{N} x_i^2 \qquad Y^2(N) = \frac{1}{N}\sum_{i=1}^{N} y_i^2$$

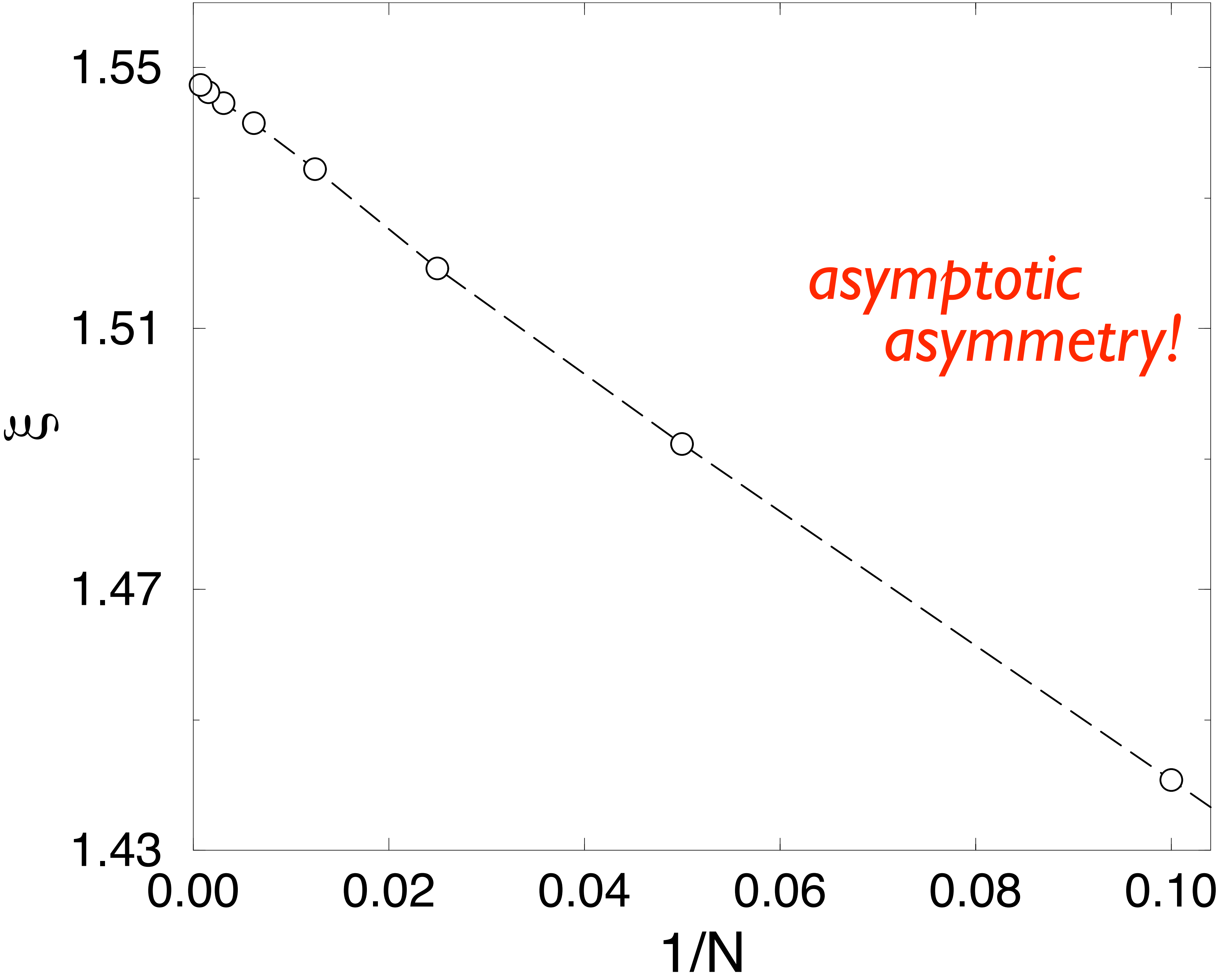
$$\begin{aligned} R_+^2(N) &= \max(X^2(N), Y^2(N)) \\ R_-^2(N) &= \min(X^2(N), Y^2(N)) \end{aligned}$$

for each realization

$$\xi(N) \equiv \sqrt{\langle R_+^2(N)\rangle}/\sqrt{\langle R_-^2(N)\rangle}$$

average over all realizations

Simulation Results



asymptotic asymmetry!

Dynamics of Ising Interfaces

V. Spirin, PLK, SR, PRE **63**, 036118 (2001);
PRE **65**, 016119 (2002)

Ferromagnetic Ising model

Even coordinated lattices

Periodic boundary conditions

Zero-temperature Glauber dynamics:

Pick a random spin and compute energy change ΔE if the spin were to flip:

$$\begin{array}{lll} \text{if} & \Delta E < 0 & \text{do it} \\ \text{if} & \Delta E > 0 & \text{don't do it} \\ \text{if} & \Delta E = 0 & \text{do it with prob. } 1/2 \end{array}$$

Dynamics of Ising Interfaces

V. Spirin, PLK, SR, PRE **63**, 036118 (2001);
PRE **65**, 016119 (2002)

Start with each spin ↑ or ↓ with probability ½.

Impose T=0 Glauber dynamics.

What is the final state?

d=1: ground state is *always* reached

d=2: ground state is *sometimes* reached

d>2: ground state is *never* reached

Many interacting interfaces are complicated

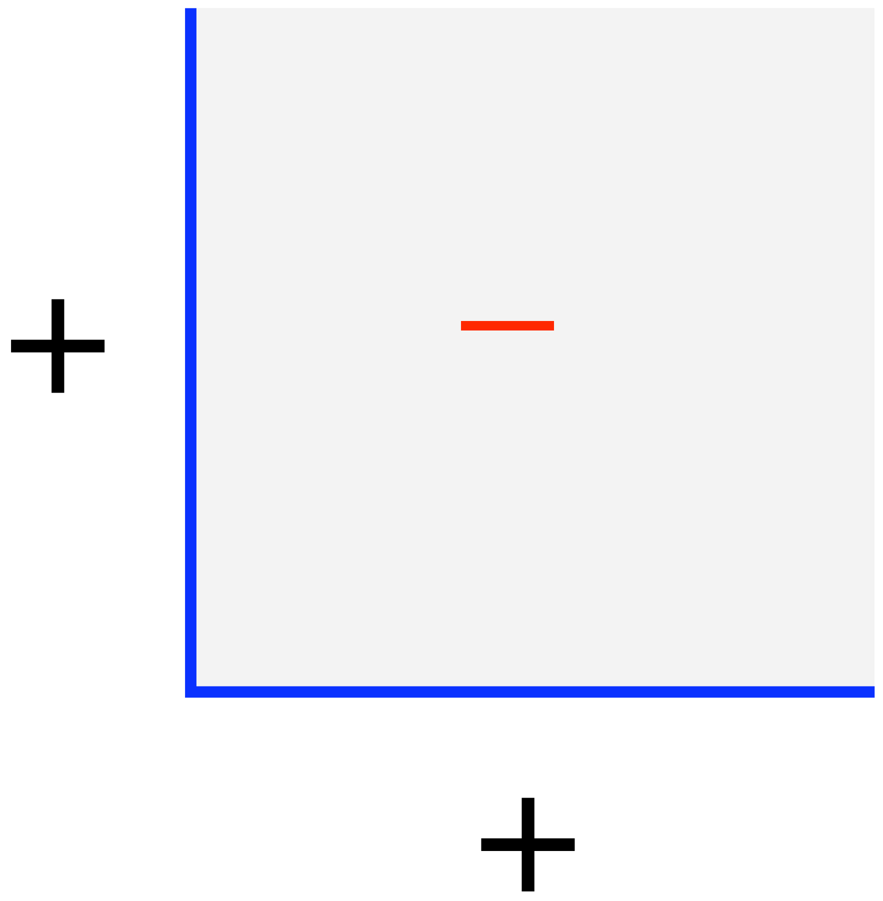
→ study dynamics of a *single* interface

Simplest Ising Interface

J. Tailleur, PLK, & SR
PRE **69**, 026125 (2004)

a straight interface is stable: too simple

an evolving interface must have curvature:
→ *single corner interface*



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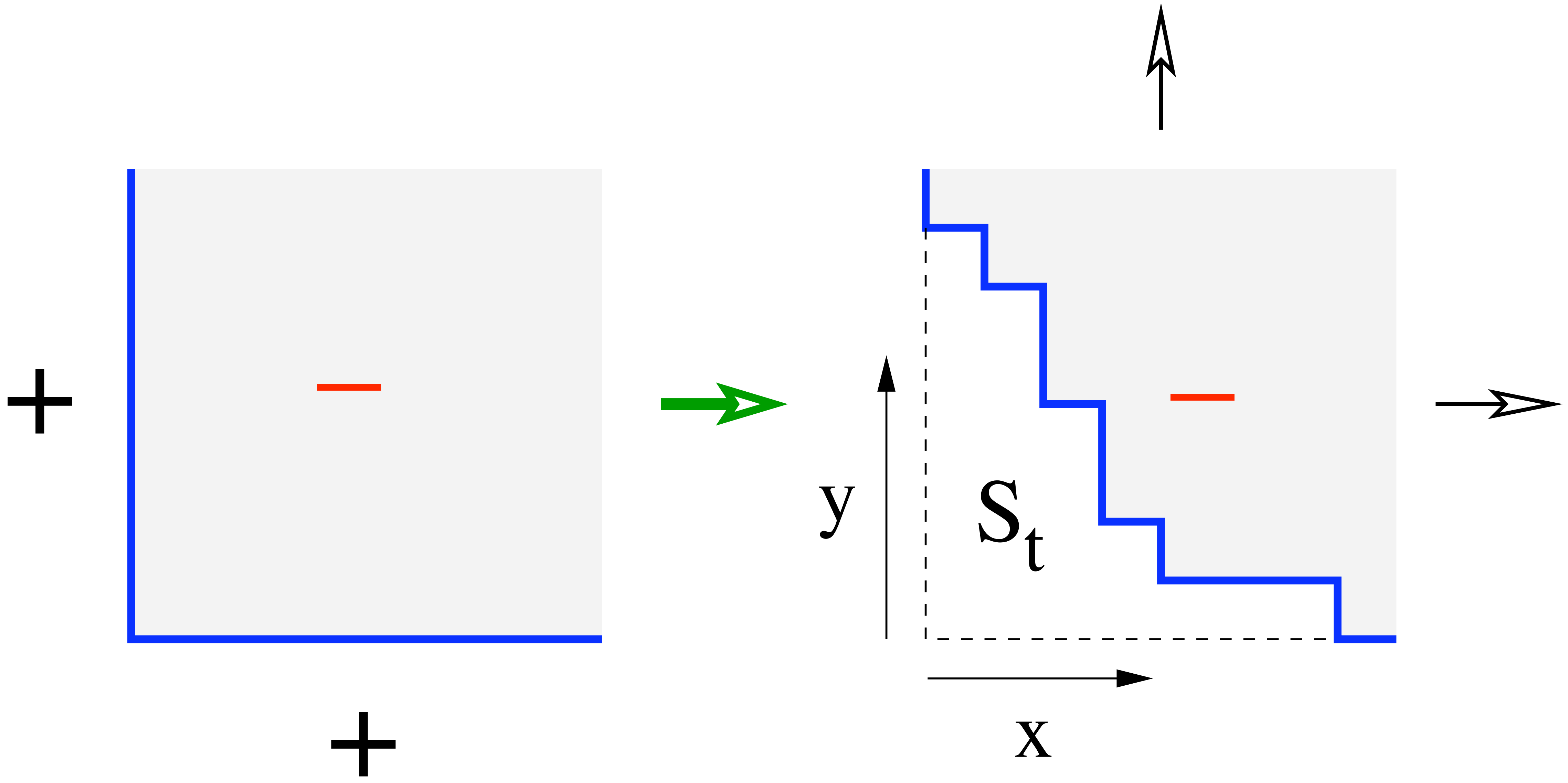

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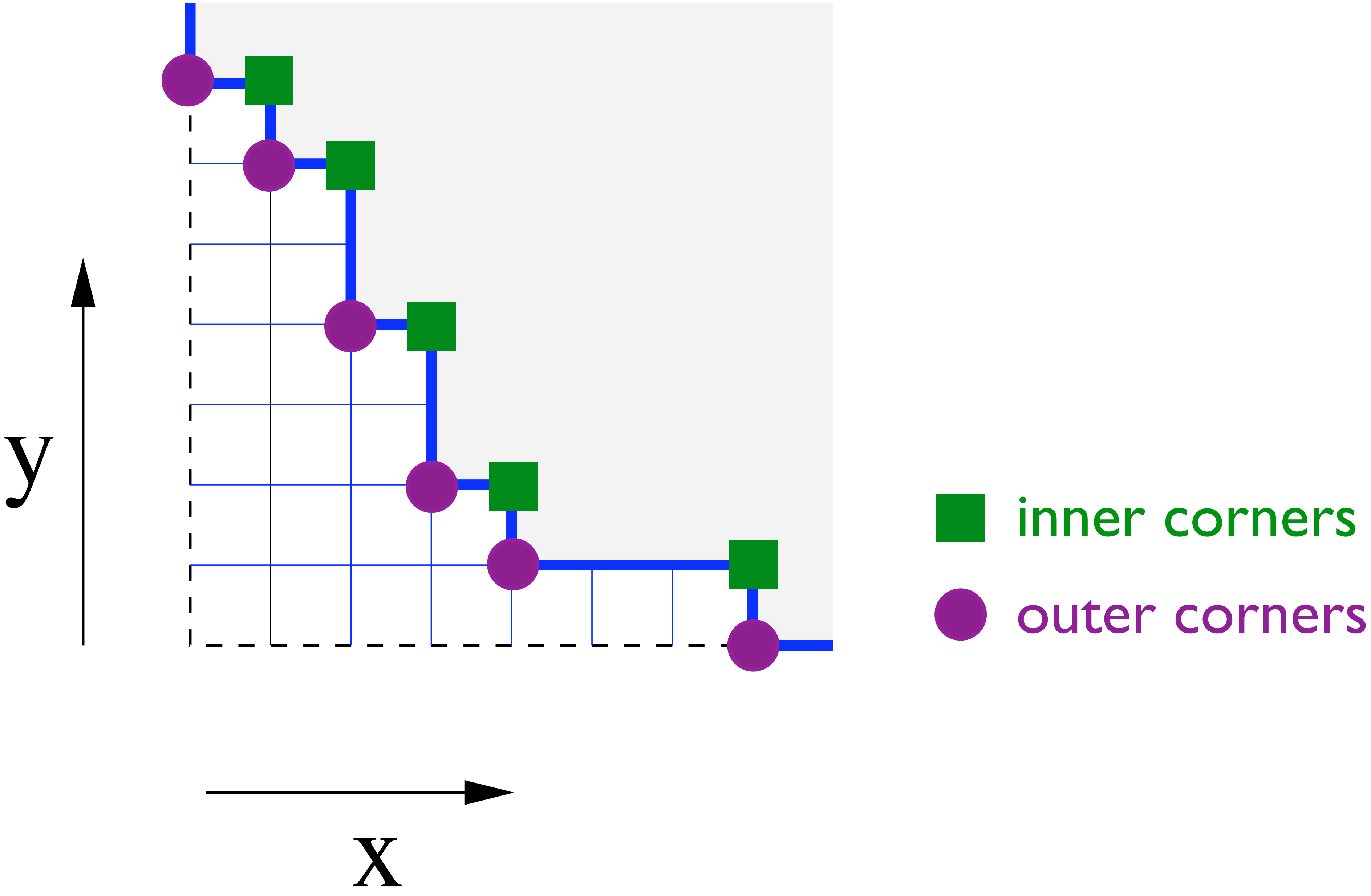
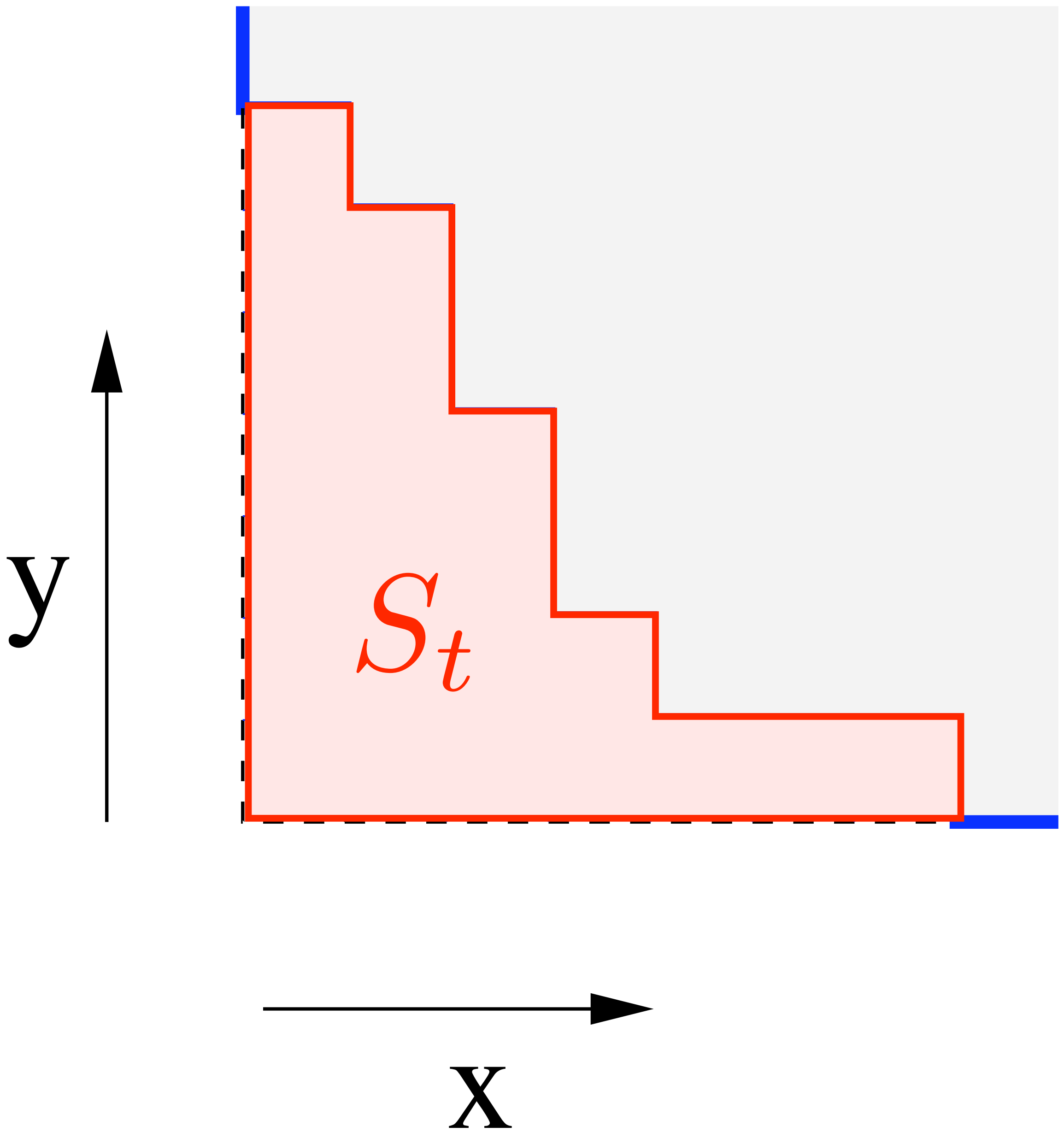


one more "outer" corner than "inner" corner

Simplest Ising Interface



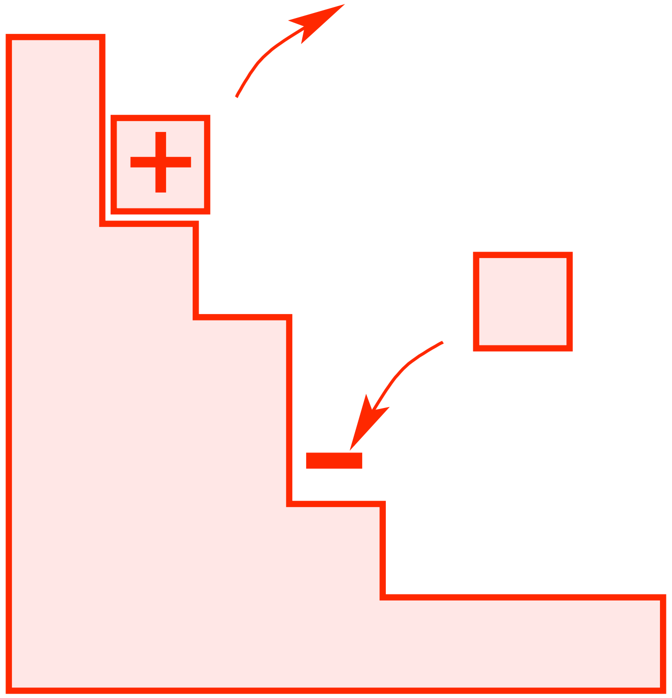
one more “outer” corner than “inner” corner

$$\rightarrow S_t = t \quad x, t \sim \sqrt{t}$$

interface recedes diffusively

Three Basic Models

1. all corners flip equiprobably
   evaporation = deposition
   → Ising interface



Three Basic Models

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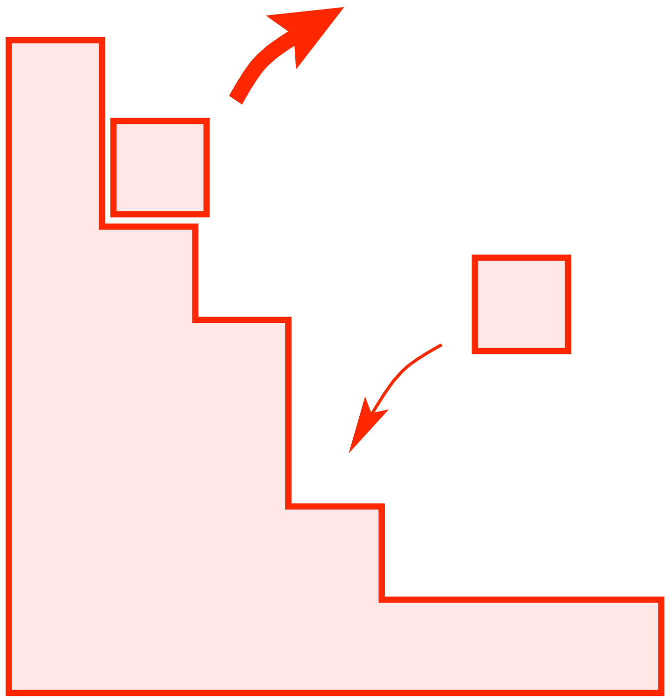
2. evaporation > deposition
   → equilibrium integer partitions

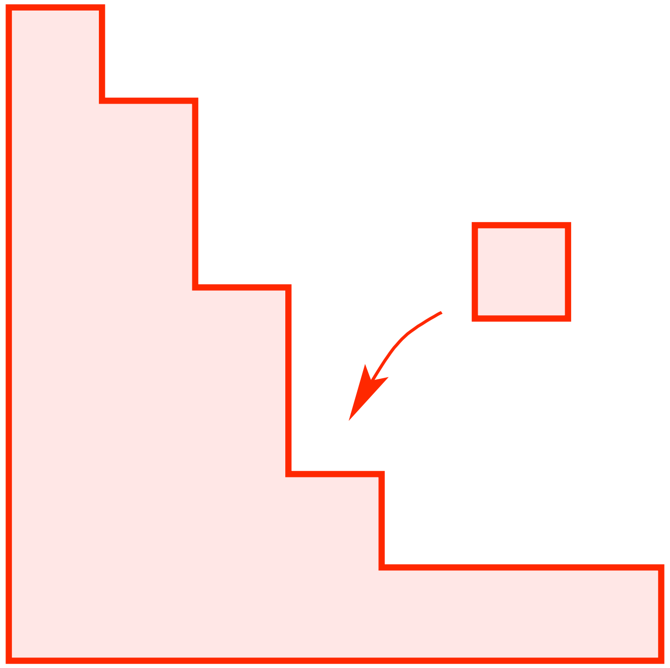


Three Basic Models

1. all corners flip equiprobably
   evaporation = deposition
   → Ising interface

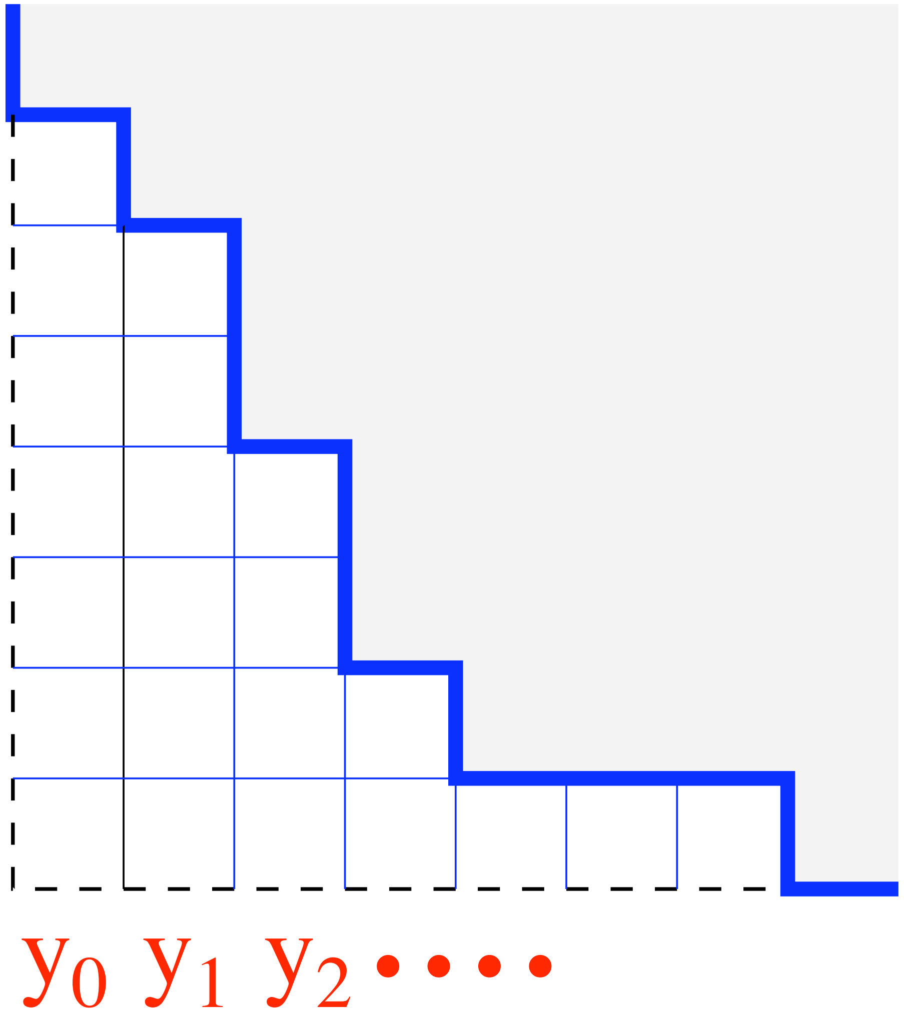
2. evaporation > deposition
   → equilibrium integer partitions

3. deposition only ($H>0^+$)
   → growing integer partitions



2. Equilibrium Integer Partitions

Equivalence of corner interface & Young diagram



$$22 = \{7, 6, 4, 2, 1, 1, 1\}$$

2. Equilibrium Integer Partitions

number of distinct partitions of t:

$$p(t) \sim \frac{1}{4\sqrt{3}\,t}\, e^{\sqrt{2\pi^2 t/3}}$$

Hardy & Ramanujan (1918)

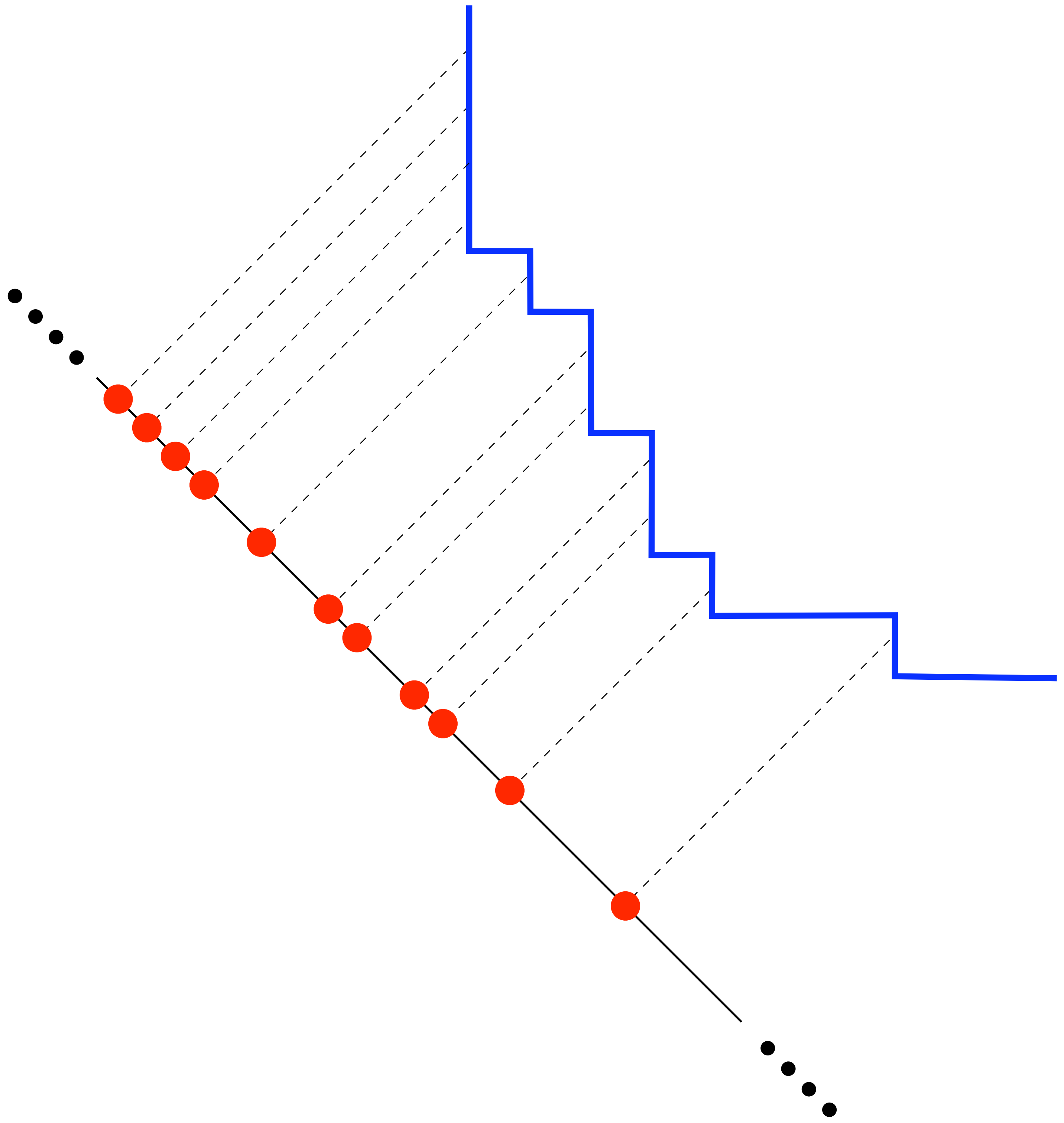
if each partition equiprobable, limiting interface shape is:

$$e^{-\lambda x} + e^{-\lambda y} = 1 \qquad \lambda = \frac{\pi}{\sqrt{6t}}$$

Temperley (1952)

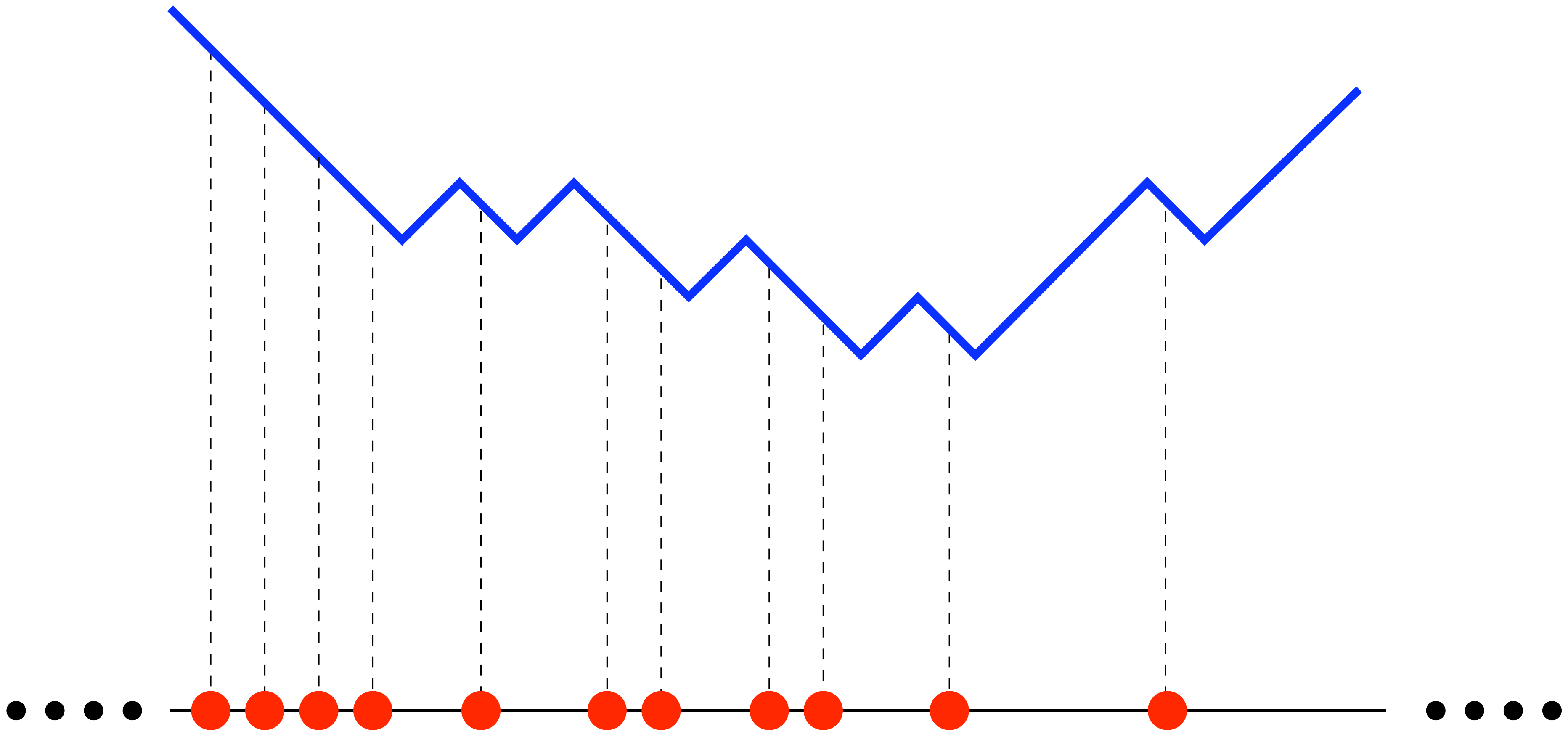
3. Growing Integer Partitions

Ising interface with $H>0^+$



Equivalent to *Asymmetric* Exclusion Process

downslope → particle
upslope → hole



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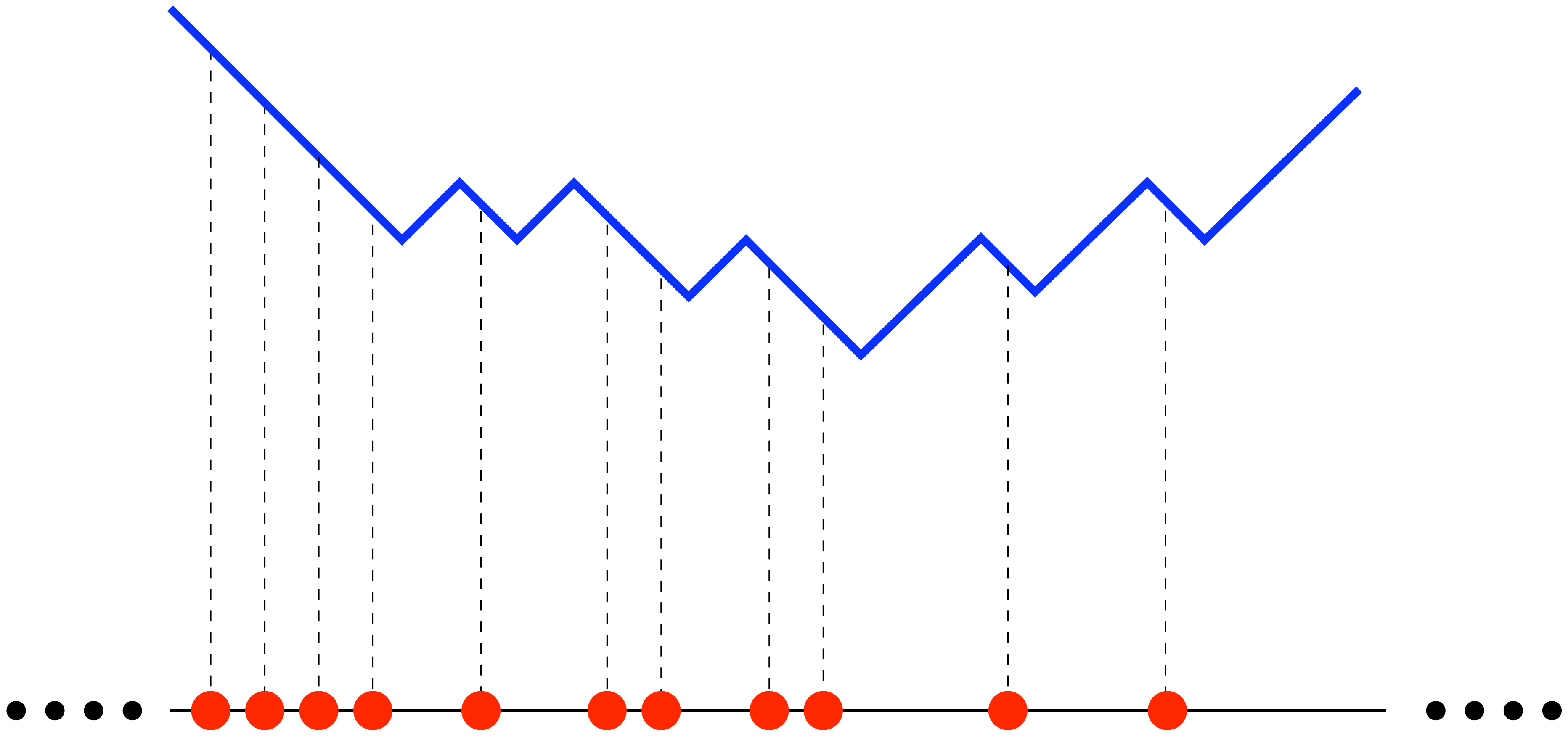

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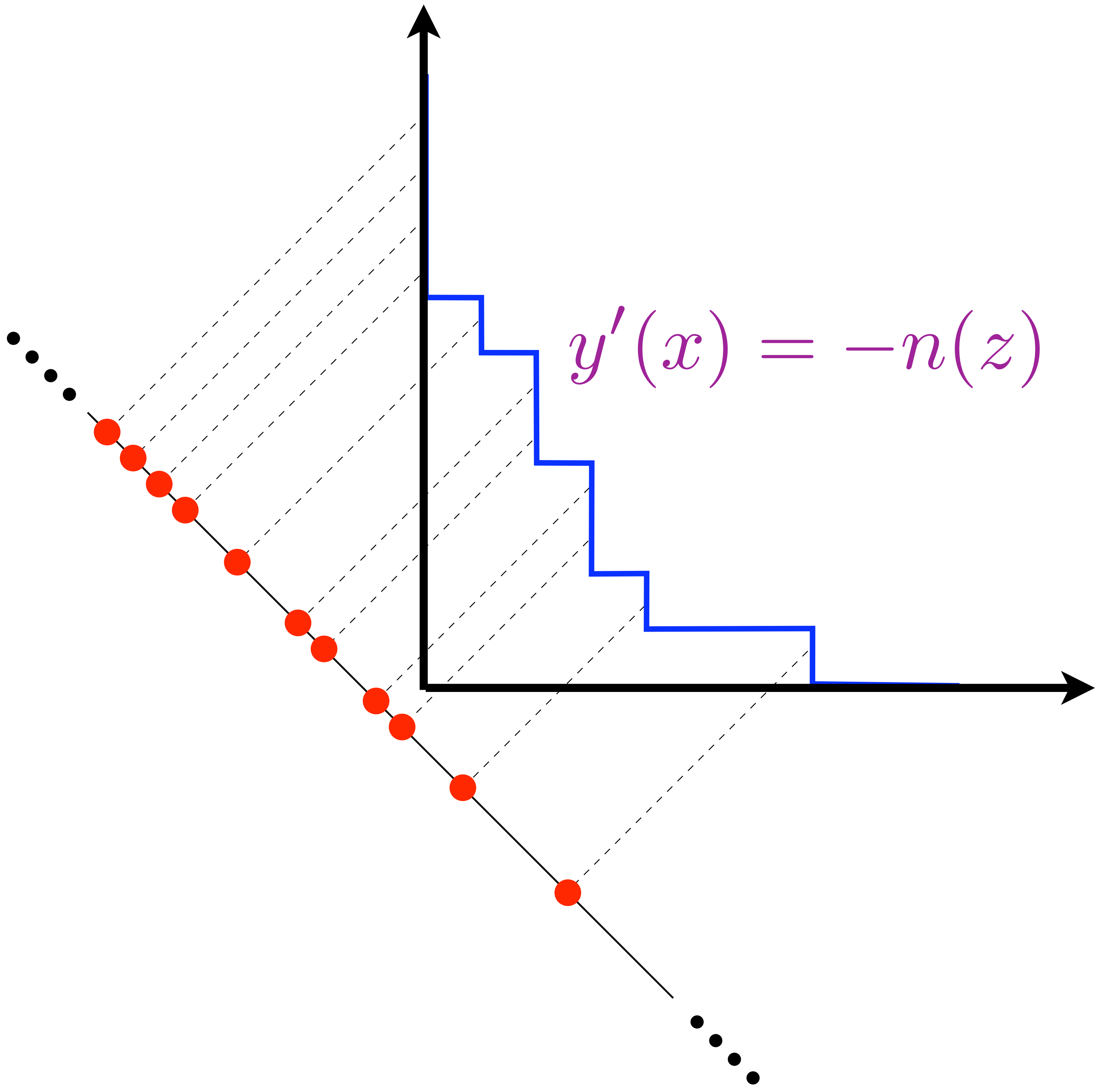

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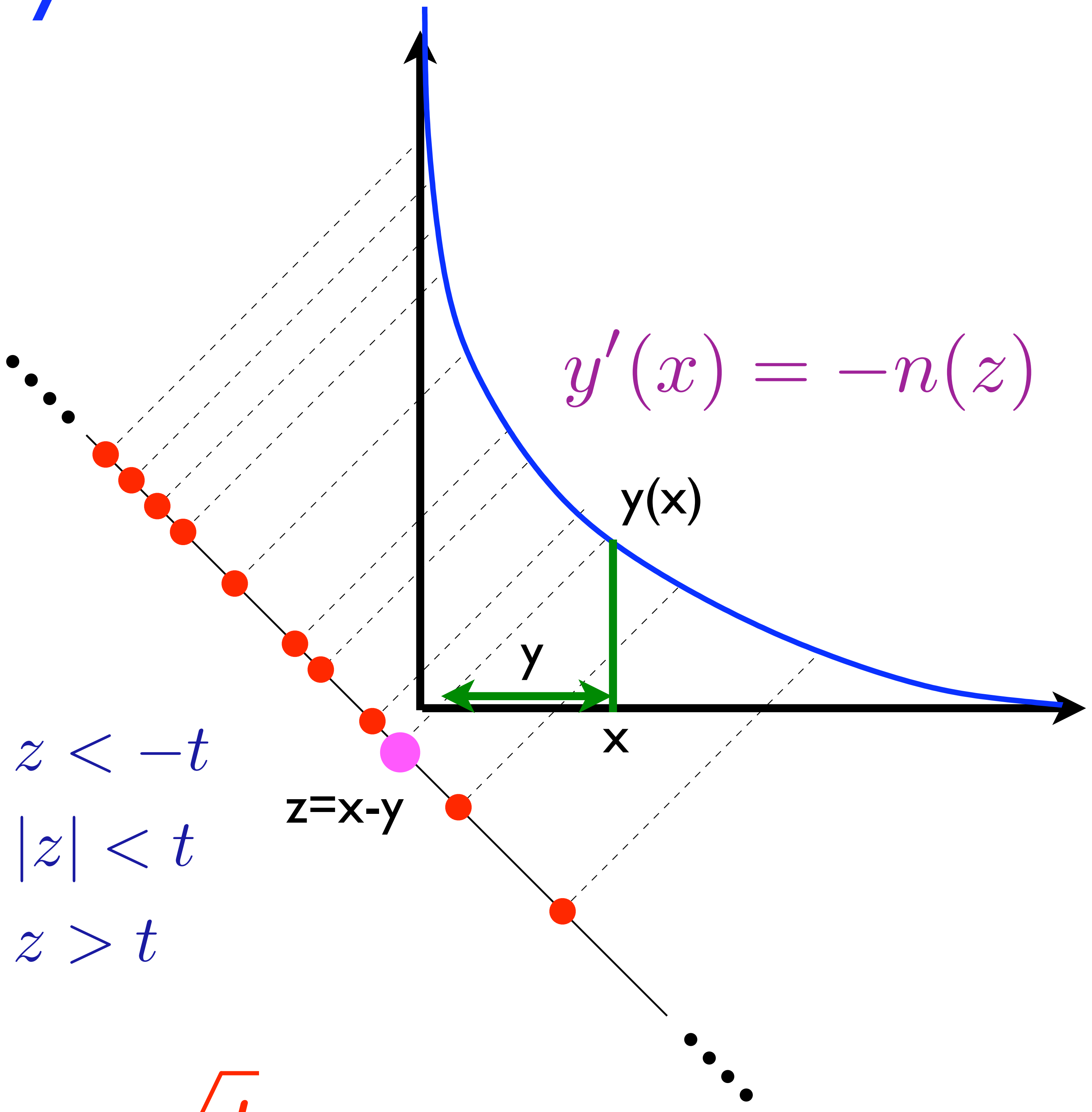


$$n_t + [n(1-n)]_z = 0$$

$$n(z,t) \to \begin{cases} 1 & z < -t \\ \frac{1}{2}\left(1 - \frac{z}{t}\right) & |z| < t \\ 0 & z > t \end{cases}$$

$$\to \sqrt{x} + \sqrt{y} = \sqrt{t}$$

Rost (1981)

1. Ising Interface

Macroscopic Allen-Cahn equation for limiting shape:

$v_n = -D \, \nabla \cdot \mathbf{n}$ *interface velocity ∝ local curvature*

for the corner geometry: $y_t = D \, \dfrac{y_{xx}}{1+y_x^2}$

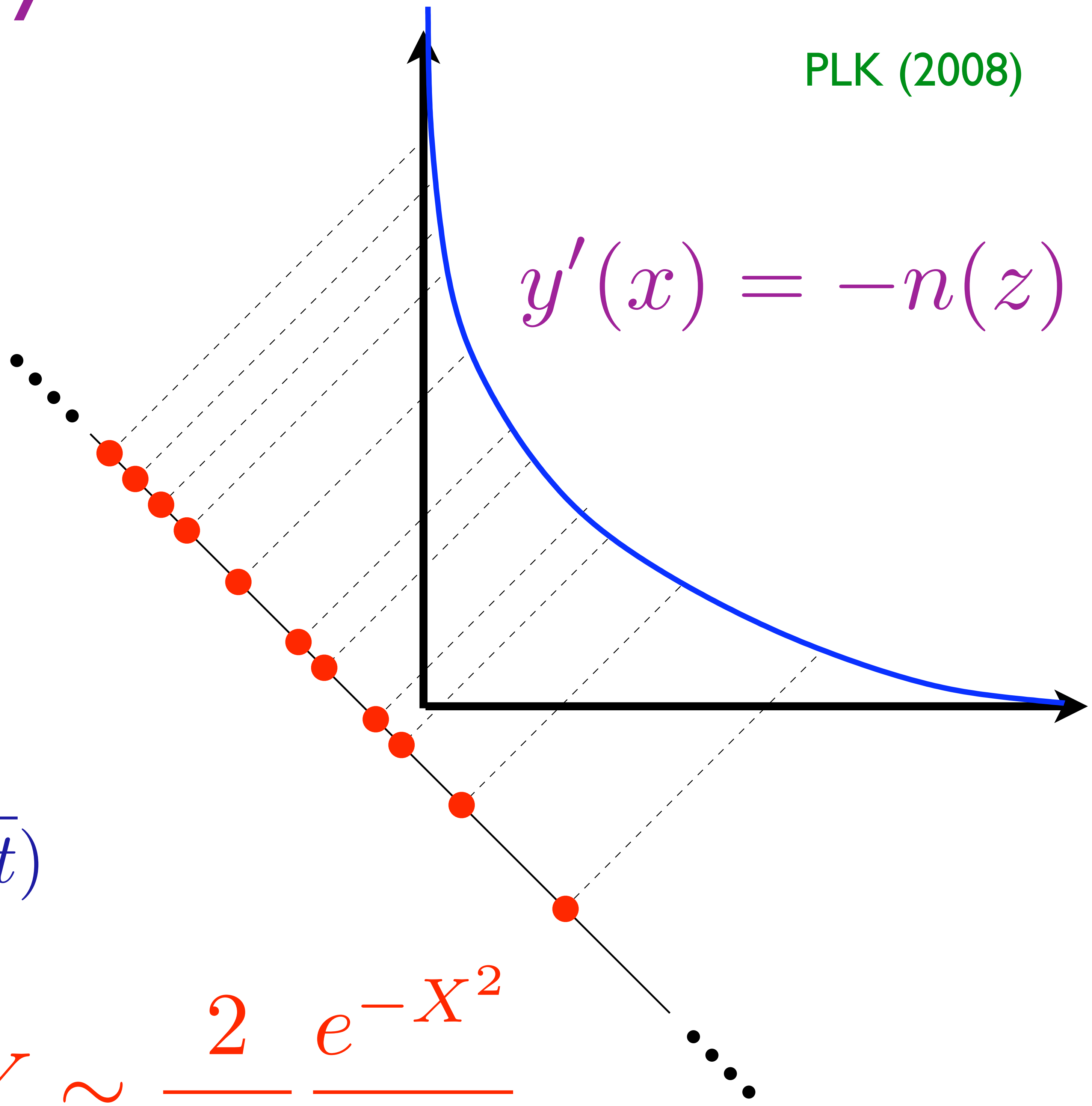
self-similar solution: $y(x,t) = \sqrt{Dt} \, Y(X) \quad X = x/\sqrt{Dt}$

AC equation: $\rightarrow \dfrac{Y - XY'}{2} = \dfrac{Y''}{1+(Y')^2}$ $\lim_{X\to\infty} Y(X) = 0$ $\lim_{X\to+0} Y(X) = \infty$

asymptotic solution: $Y \sim \dfrac{A}{X^2} \, e^{-X^2/4}$ $A \approx 2.74404\ldots$

Equivalent to *Symmetric* Exclusion Process

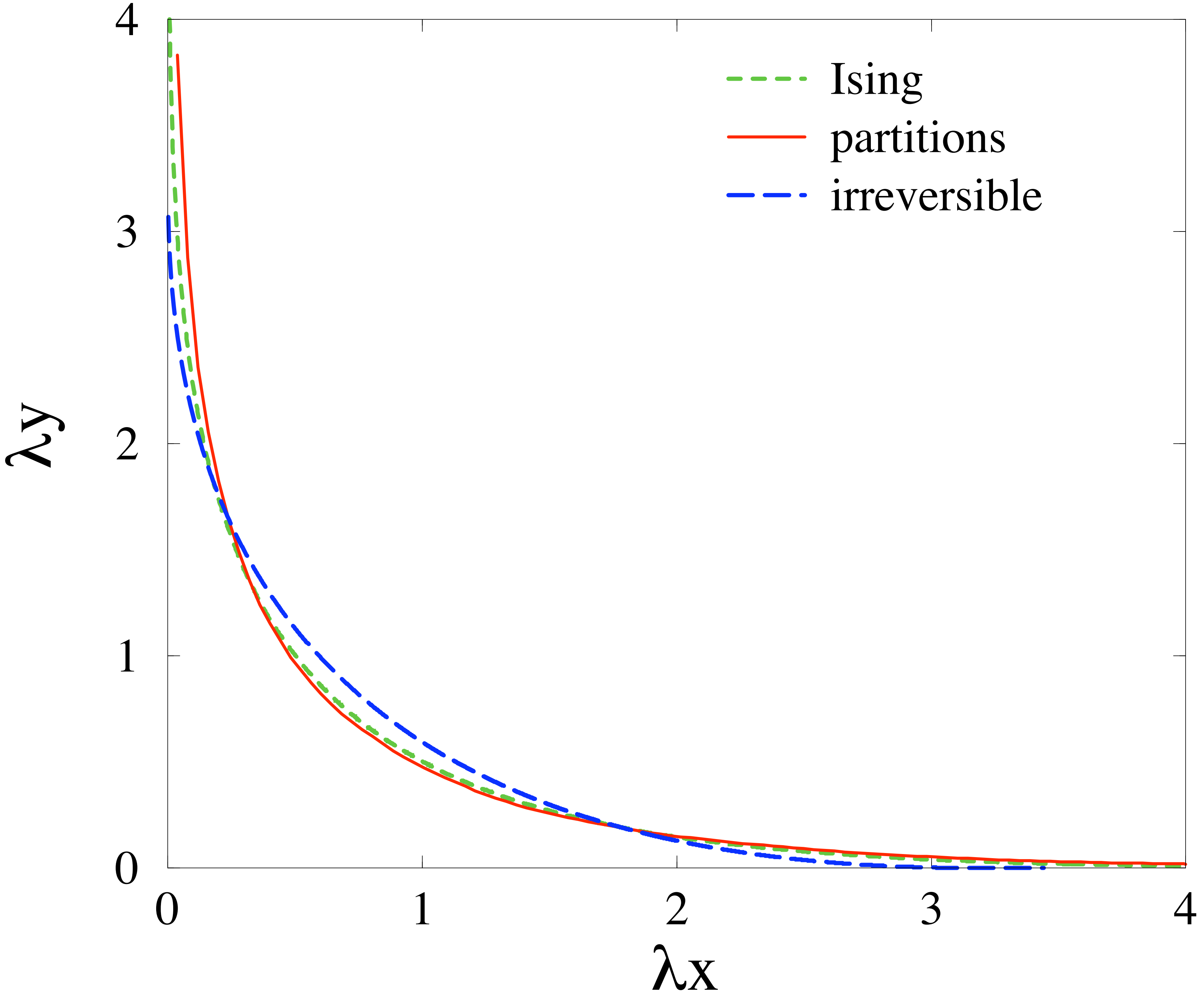
PLK (2008)

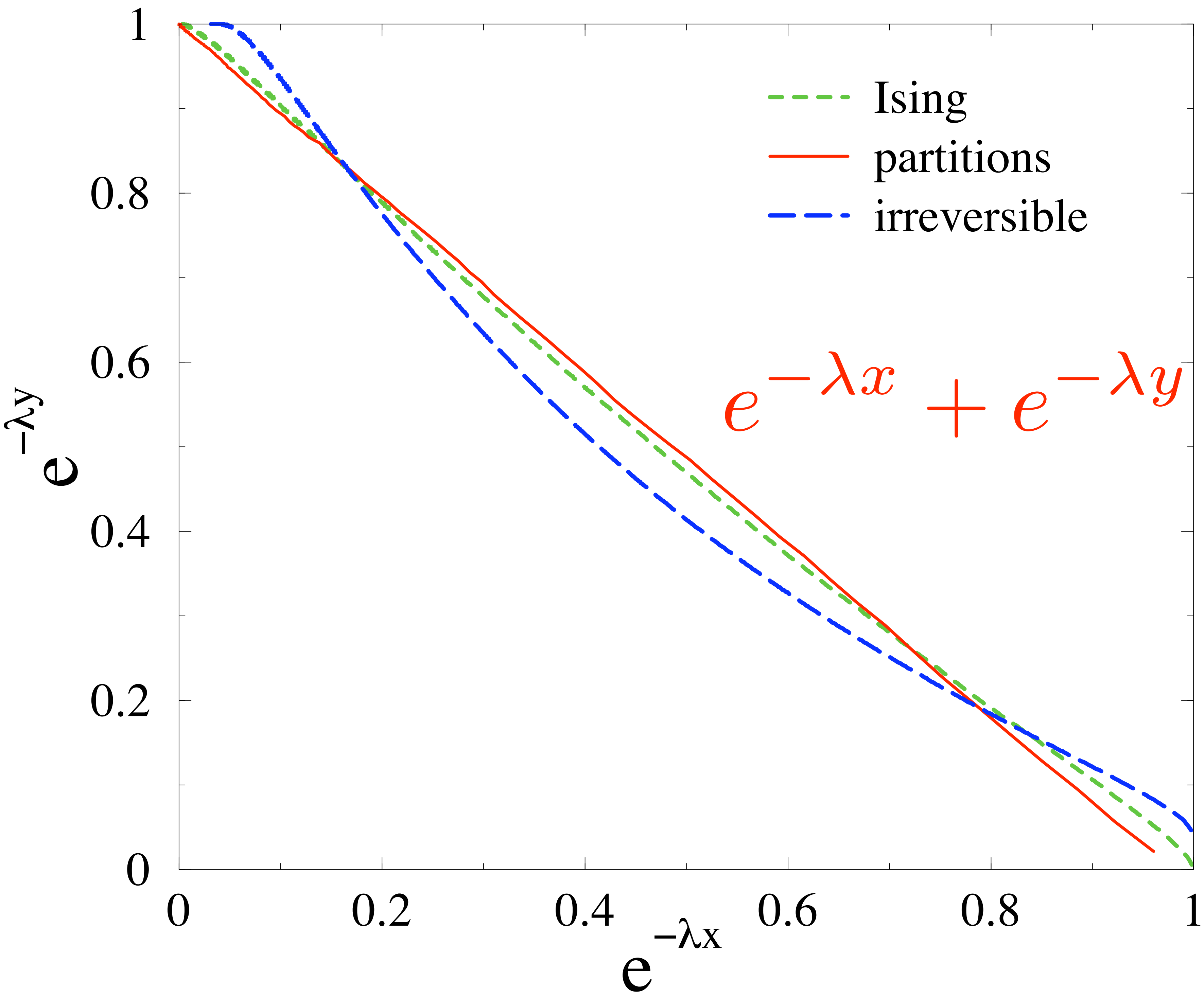


$$n_t = n_{zz}$$

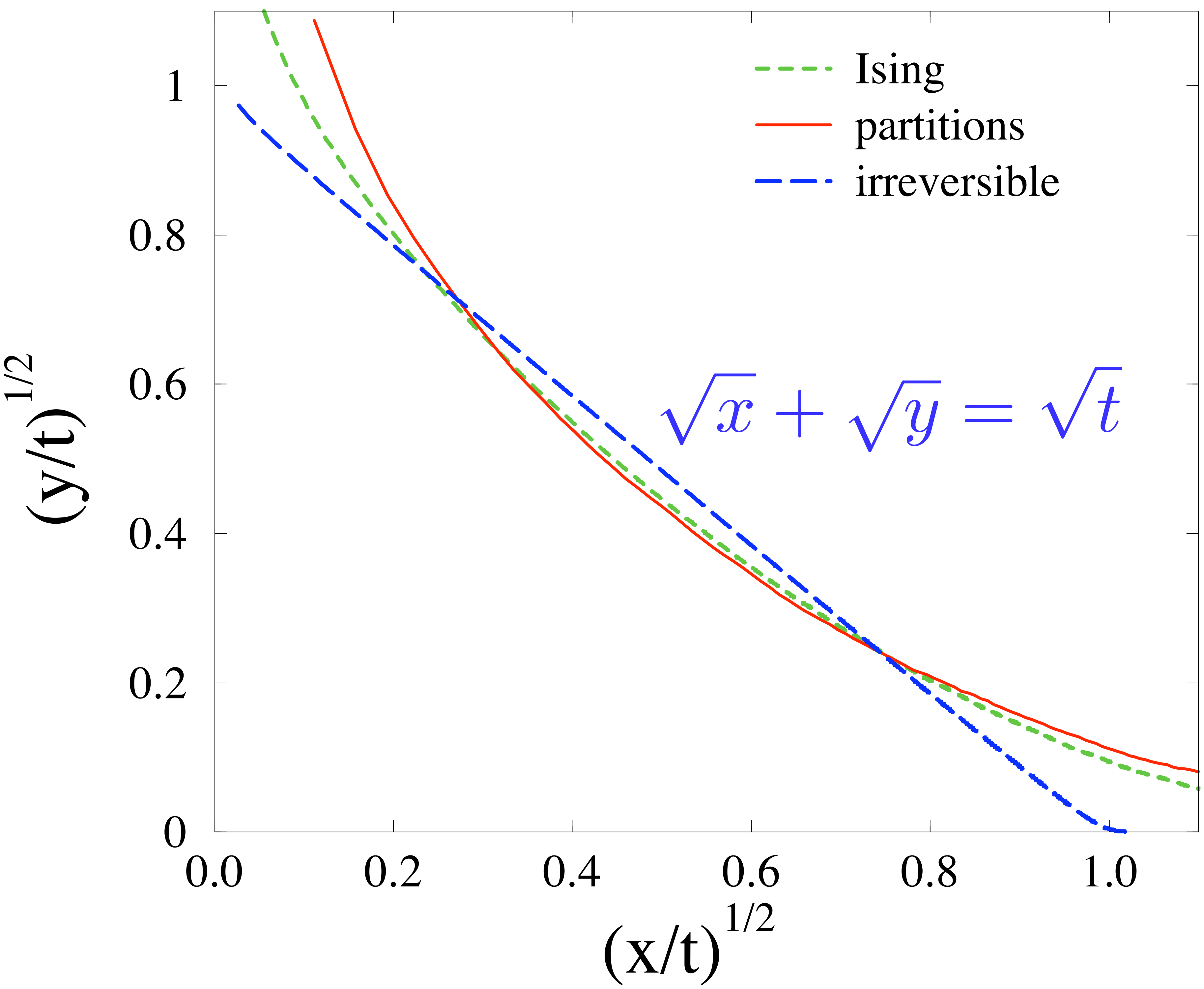
$$n(z,t) = \frac{1}{2}\mathrm{erfc}(z/\sqrt{4t})$$

asymptotic solution: $Y \sim \frac{2}{\sqrt{\pi}}\frac{e^{-X^2}}{X^2}$


Ising
partitions
irreversible
λy
λx
0
1
2
3
4
0
1
2
3
4



$$e^{-\lambda x} + e^{-\lambda y} = 1$$


Ising
partitions
irreversible
$\sqrt{x}+\sqrt{y}=\sqrt{t}$
$(y/t)^{1/2}$
1
0.8
0.6
0.4
0.2
0
0.0
0.2
0.4
0.6
0.8
1.0
$(x/t)^{1/2}$

Outlook

chipping model:

not quite round shapes (in d=2)

large fluctuations between realizations

robust with respect to extensions

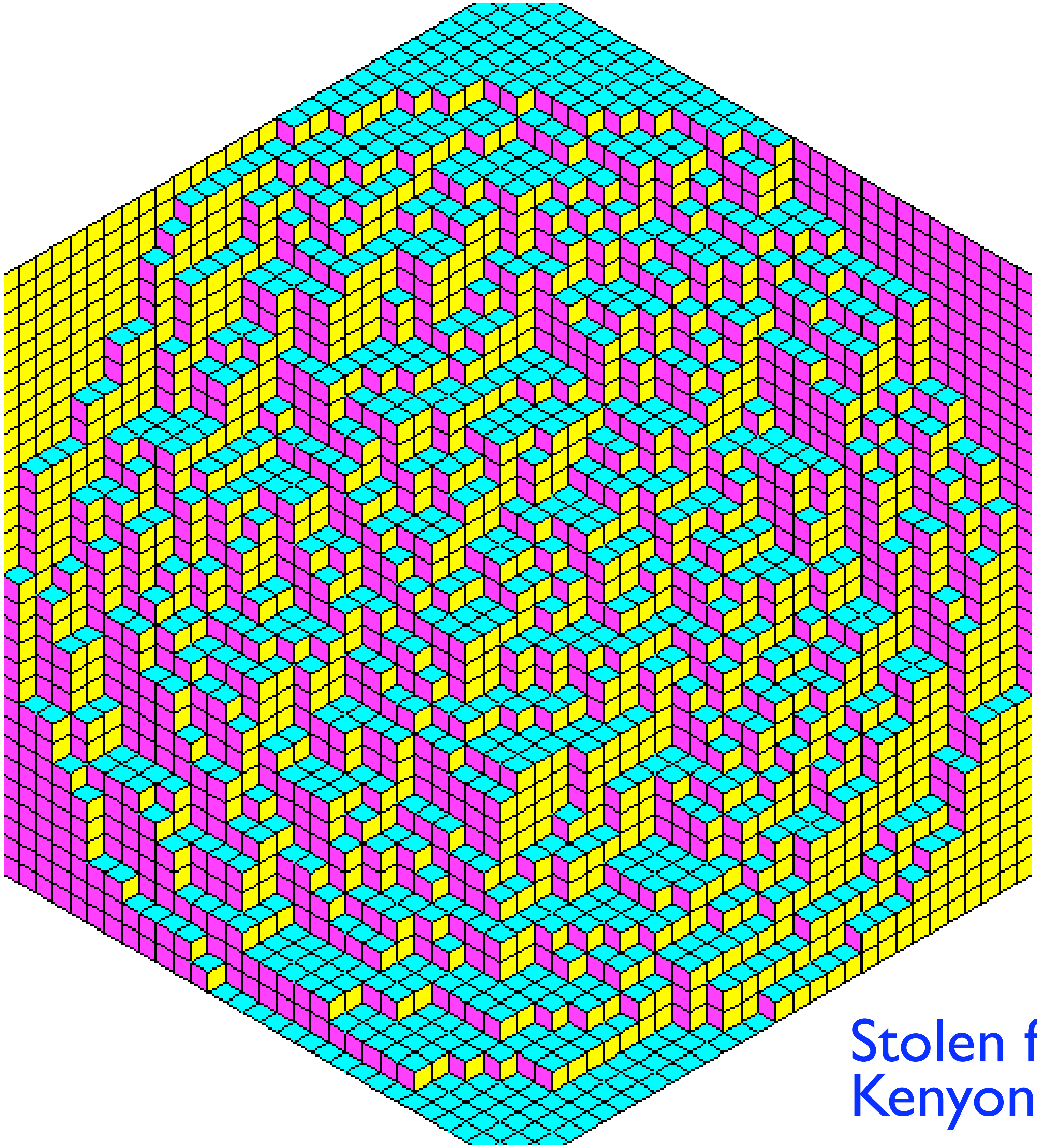
preferentially chip prominent corners; chip more than 1 corner

Ising interfaces:

relation with partitions; single interface solved

challenges: corner interface in d=3

Single Corner in d=3



Stolen from Richard Kenyon's website

Outlook

chipping model:

- not quite round shapes (in d=2)
- large fluctuations between realizations
- robust with respect to extensions
  - *preferentially chip prominent corners; chip more than 1 corner*

Ising interfaces:

- relation with partitions; single interface solved
- challenges: corner interface in d=3

  ¿final fate with random IC?

What is the final state?

d=1: ground state is *always* reached

d=2: ground state is *sometimes* reached

d>2: ground state is *never* reached

Please prove for d ≥ 2!

Outlook

chipping model:

not quite round shapes (in d=2)

large fluctuations between realizations

robust with respect to extensions

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