Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

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≥ 3	wrong

Central Dogma of Spin Dynamics

Start at $T \gg T_c$ and suddenly quench to T_f.

1. **Supercritical dynamics** $T_f > T_c$
2. **Critical** $T_f = T_c$
3. **Subcritical** $T_f < T_c$ *universal*, same as $T_f = 0$.

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Dynamic scaling hypothesis:

1. **Single length scale** $L(t) \to$ coarsening
2. **Algebraic scaling** $L(t) \sim t^z$
3. **Universality** z independent of most details.

The System

Ising Hamiltonian $\mathcal{H} = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$

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- periodic boundaries

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- periodic boundaries

Initial state:

- antiferromagnetic
- ↑ with prob. ½, ↓ with prob. ½
- ½ spins ↑, ½ spins ↓

Endowing Spins with a Dynamics

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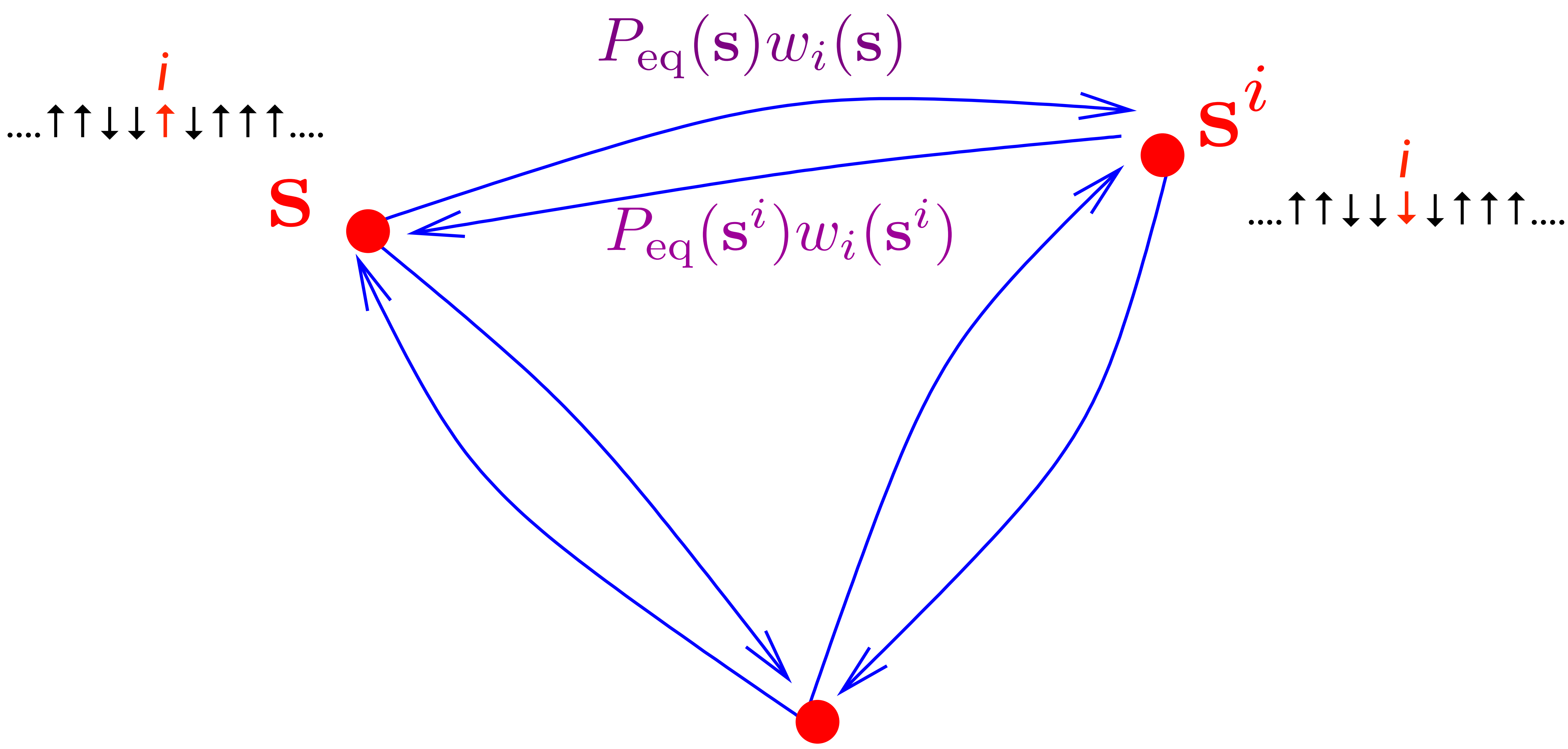
detailed balance condition:

$$\frac{w_i(\mathbf{s})}{w_i(\mathbf{s}^i)} = \frac{P_{\rm eq}(\mathbf{s}^i)}{P_{\rm eq}(\mathbf{s})} = \frac{e^{-\beta s_i \sum J_{ij} s_j}}{e^{\beta s_i \sum J_{ij} s_j}} = \frac{1 - s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}{1 + s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}$$

detailed balance condition in state space:



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flip rate in one dimension:

$$w_i = \tfrac{1}{2}\{1 - s_i \tanh[\beta J\,(s_{i-1} + s_{i+1})]\}$$

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Glauber dynamics at T=0: Pick a random spin and consider outcome of reversing it

if	$\Delta E < 0$	do it
if	$\Delta E > 0$	don't do it
if	$\Delta E = 0$	do it with prob. 1/2

Equations of Motion

Correlation functions: $S_i = \langle s_i \rangle, \;\; S_{i,j} = \langle s_i s_j \rangle$ .....

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mean spin:

$$\frac{dS_j}{dt} = -2\left\langle s_j \left\{ \tfrac{1}{2}\left[1 - s_j \frac{s_{j-1} + s_{j+1}}{2}\right] \right\} \right\rangle$$
$$= -S_j + \tfrac{1}{2}(S_{j-1} + S_{j+1})$$

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kth-neighbor correlation function: $G_k \equiv S_{i,i+k}$

$$\frac{dG_k}{dt} = -2G_k + \left(G_{k-1} + G_{k+1}\right) \qquad G_0(t) = \langle s_i^2 \rangle = 1$$

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$$\simeq (4\pi t)^{-1/2}$$

$$
\begin{aligned}
G_1 &= \langle s_i \, s_{i+1} \rangle \\
&= \text{Prob(aligned)} - \text{Prob(anti-aligned)} \\
&= 1 - 2 \ \text{Prob(anti-aligned)} \\
&= 1 - 2 \ \text{Prob(domain wall particle exists)} \\
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$$= 1 - 2\rho$$

ρ = density of domain-wall “particles”

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→ *domain wall picture*

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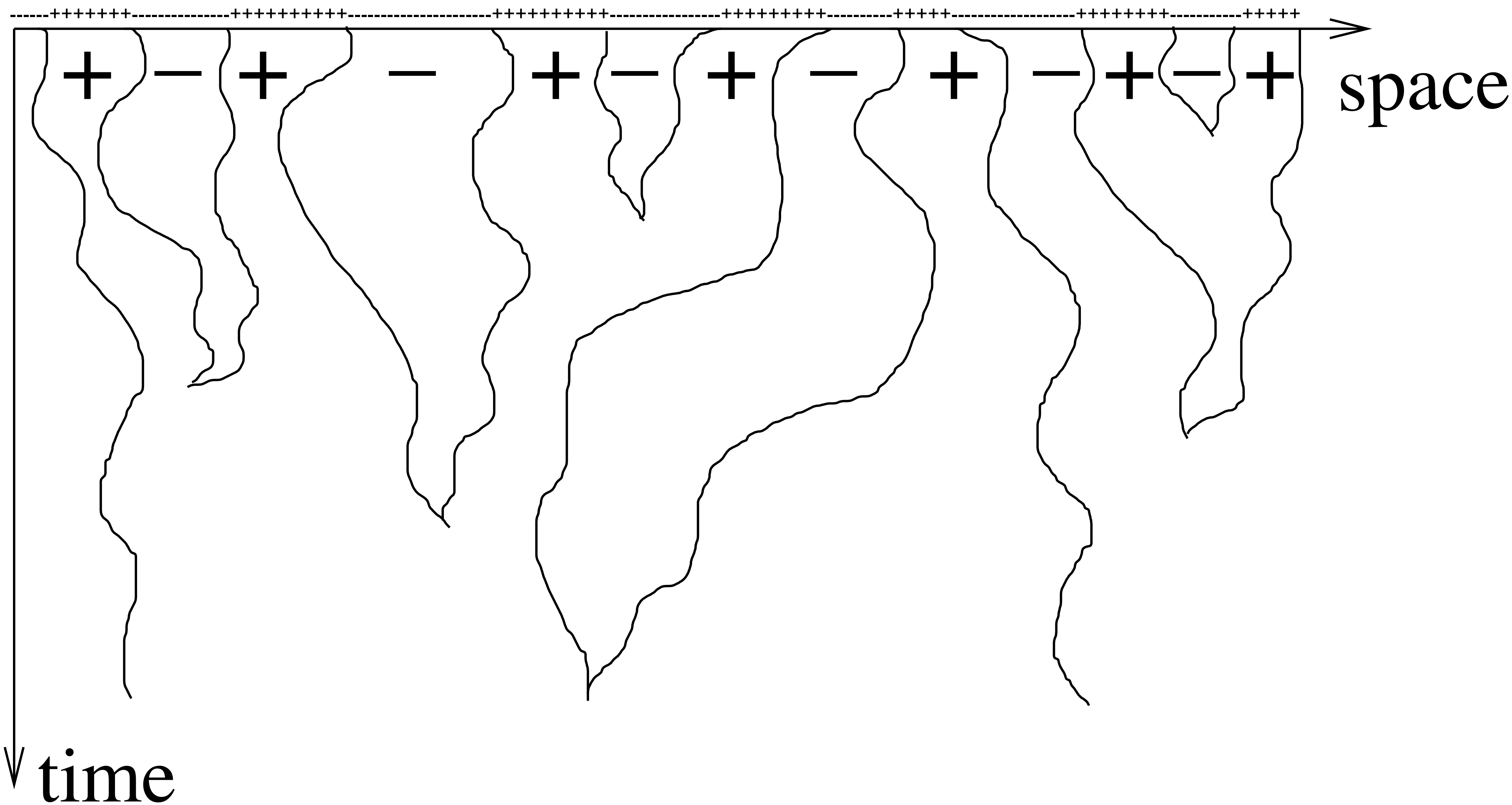

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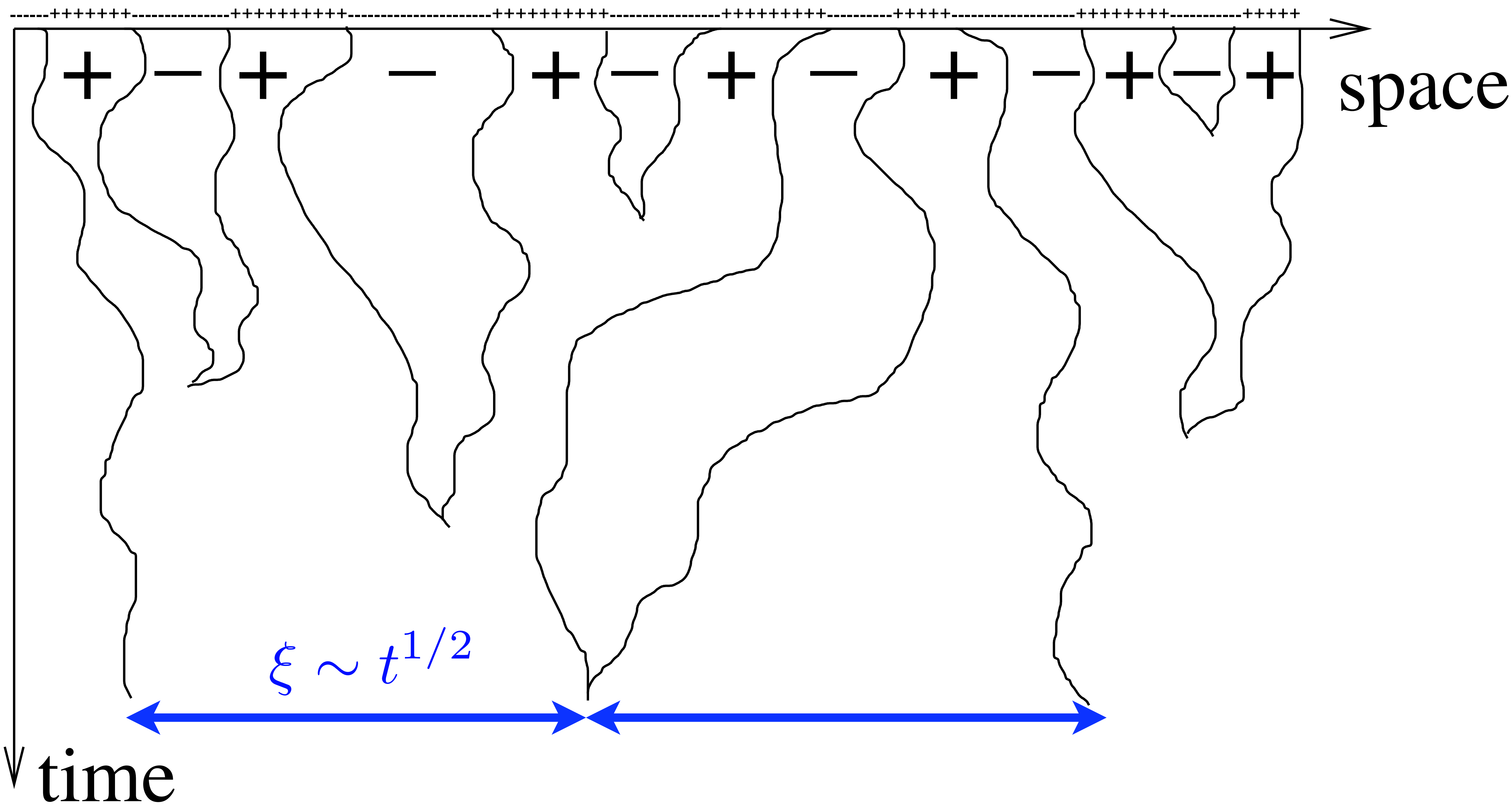

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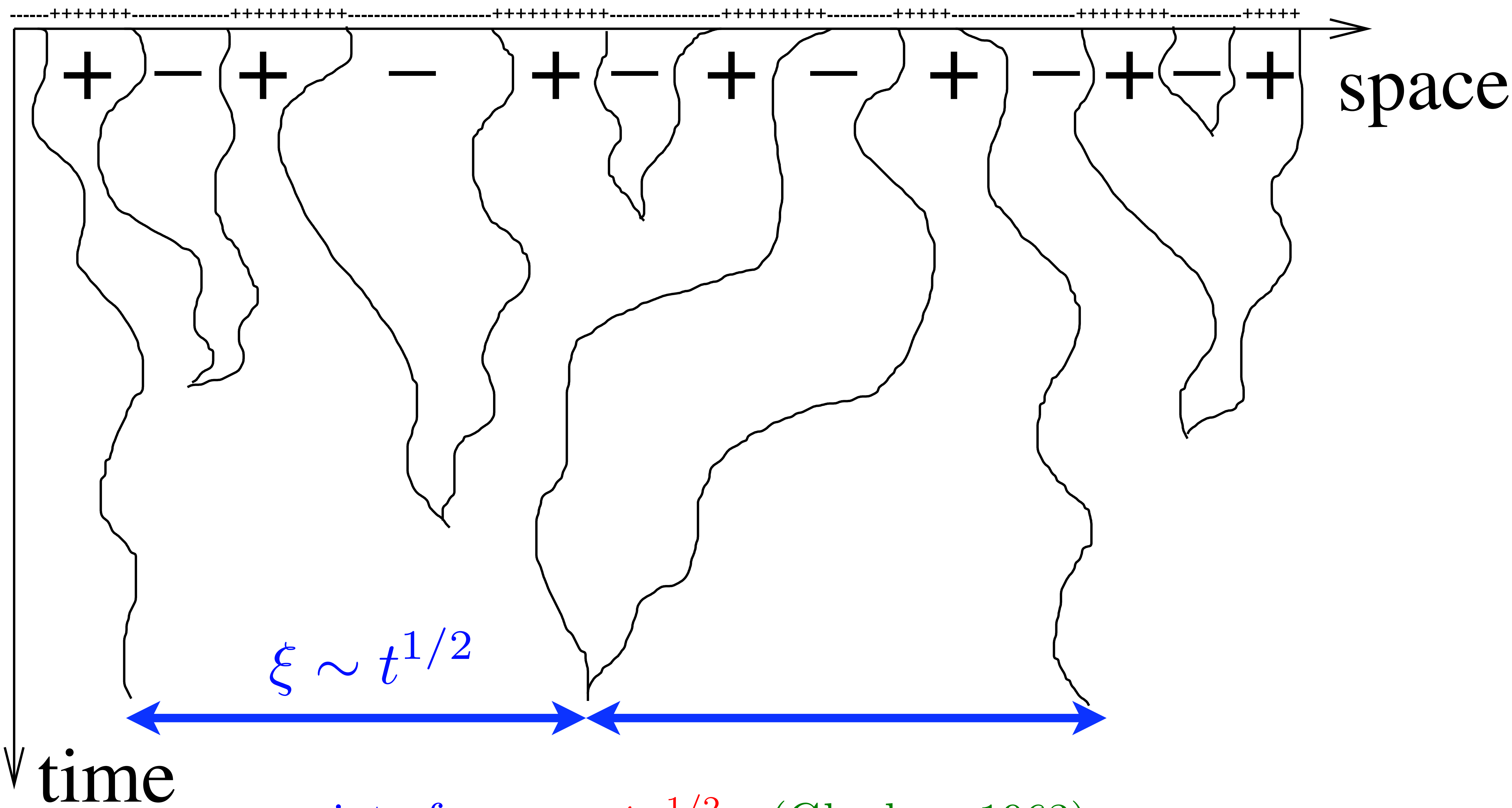

- no. interfaces: $\propto t^{-1/2}$ (Glauber, 1963)

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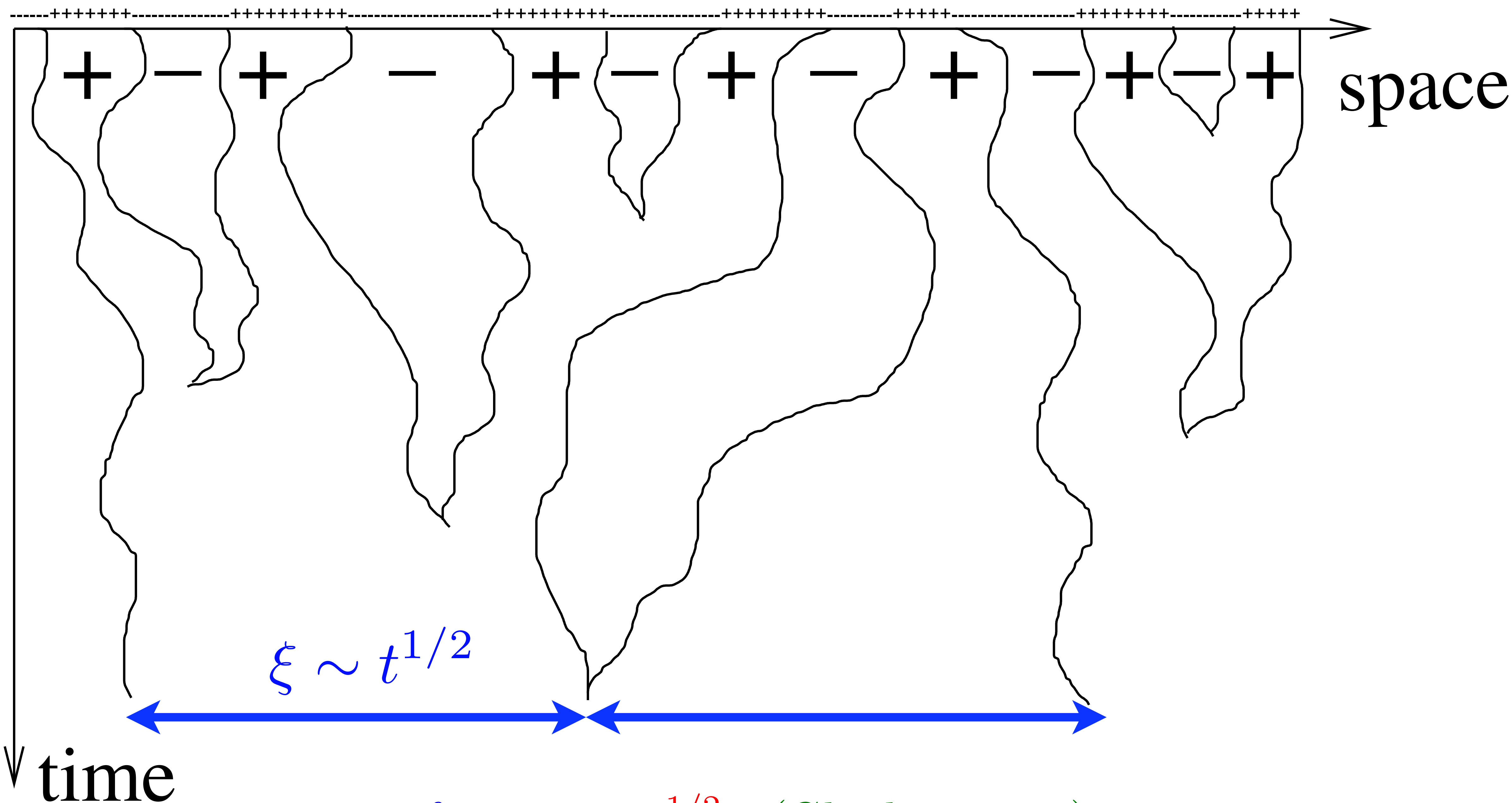

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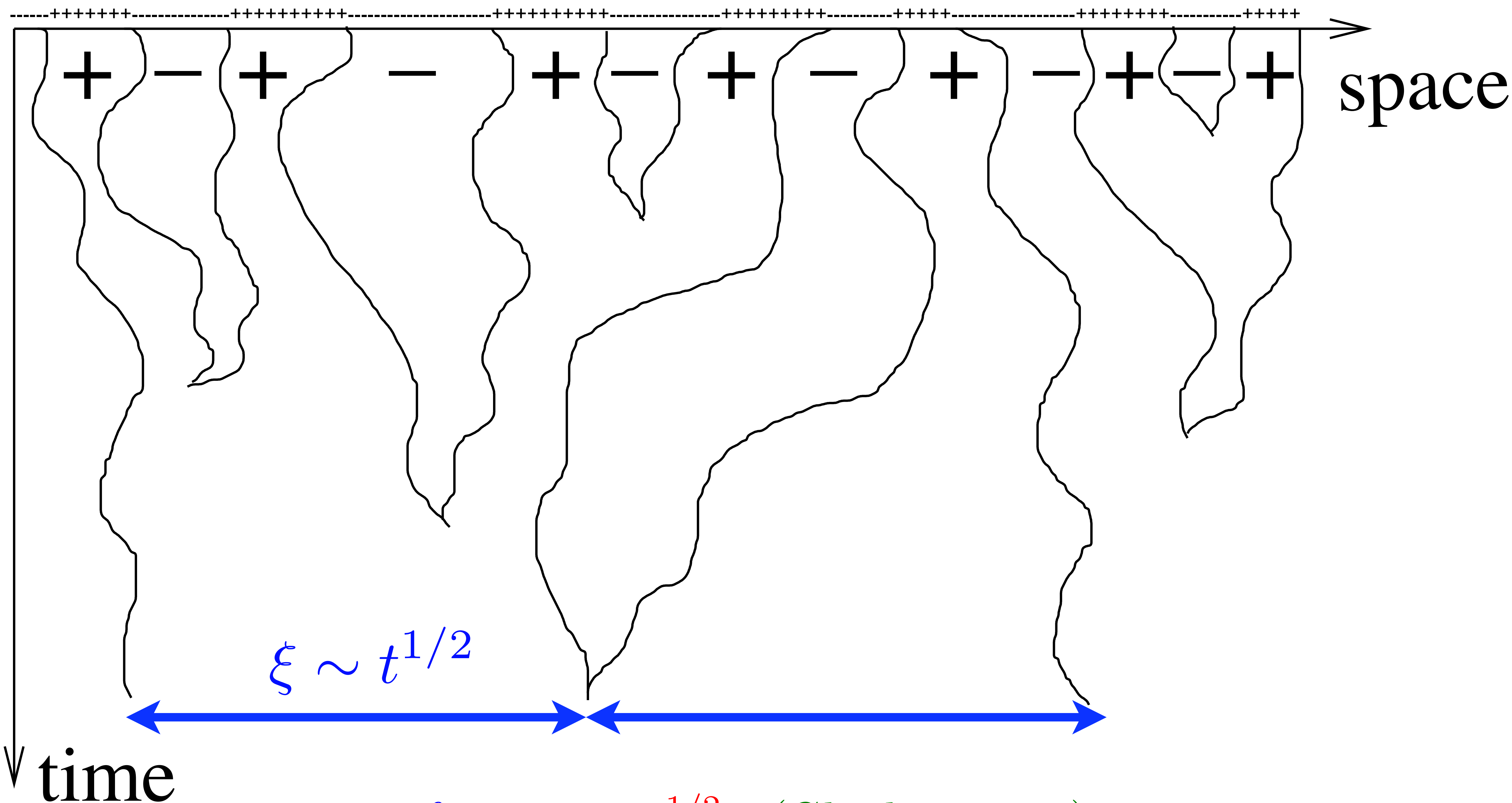

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- time to ground state: $T \propto L^2$

Domain Length Distribution (Ben-Naim & Krapivsky 97)

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scaling ansatz: $P_k(t) \simeq C\, t^{-1}\, \Phi(k t^{-1/2})$ $\quad \int_0^\infty x\Phi(x)\, dx = 1$

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$$\rho = \sum_{k\geq 1} P_k \simeq (4\pi t)^{-1/2} \to \int_0^\infty \Phi(x)\, dx = (4\pi)^{-1/2} \equiv C$$

$$\frac{d\rho}{dt} = -2P_1 \to P_1 \simeq \frac{1}{4}\, C\, t^{-3/2}$$

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master equation:

$$\frac{dP_k}{dt} = -2P_k + P_{k+1} + P_{k-1}\left(1 - \frac{P_1}{\rho}\right) + \frac{P_1}{\rho^2}\sum_{i+j=k-1} P_i\, P_j - \frac{P_1}{\rho} P_k$$

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2}\frac{d(x\Phi)}{dx} + \frac{1}{4C}\int_0^x \Phi(y)\,\Phi(x-y)\,dy = 0$$

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Laplace transform: $\Phi(s) = \frac{1}{C}\int_0^\infty \Phi(x)\, e^{-sx}\, dx$

$$\frac{d\Phi}{ds} = \frac{\Phi^2}{2s} + 2s\Phi - \frac{1}{2s}$$

$$\Phi = 1 - 2s^2 - 2s\frac{d}{ds}\ln D_{1/2}(\sqrt{2}\, s) \simeq Ae^{-\lambda x} \qquad x \to \infty$$

Two Dimensions

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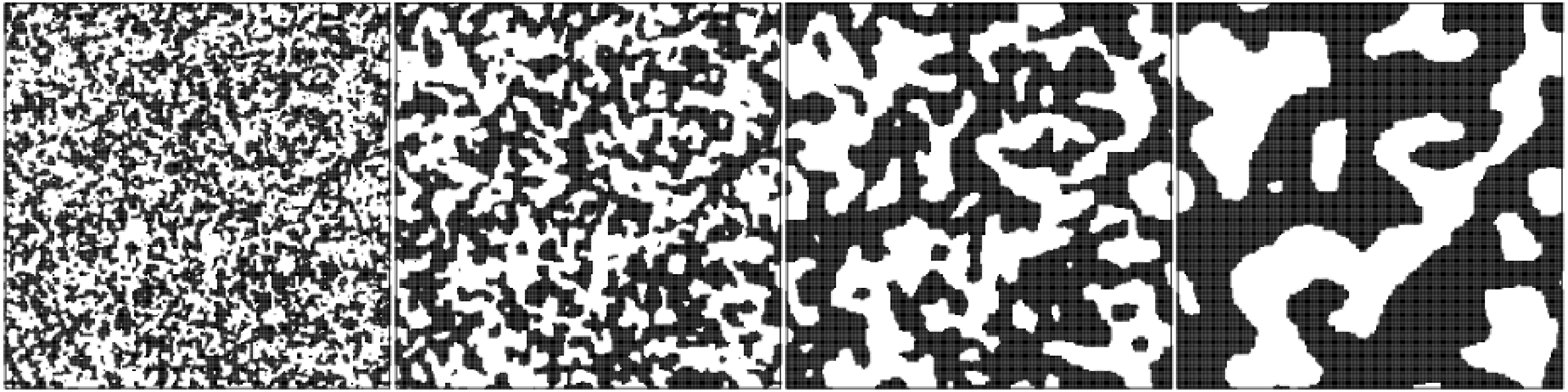
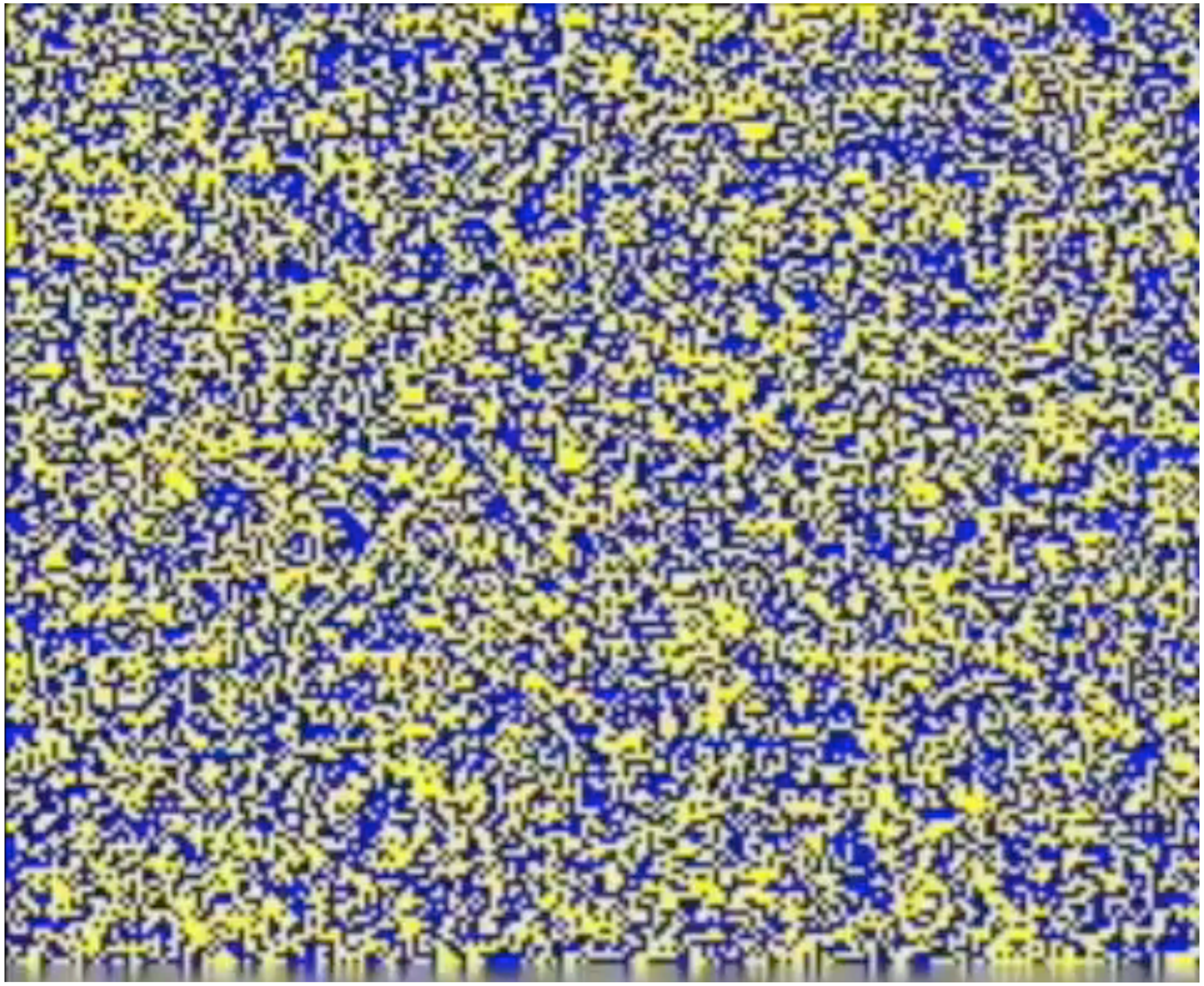
coarsening of 256x256 system (courtesy of V. Spirin)



t=4 t=16 t=64 t=256

Two Dimensions:
Evolution to Ground State



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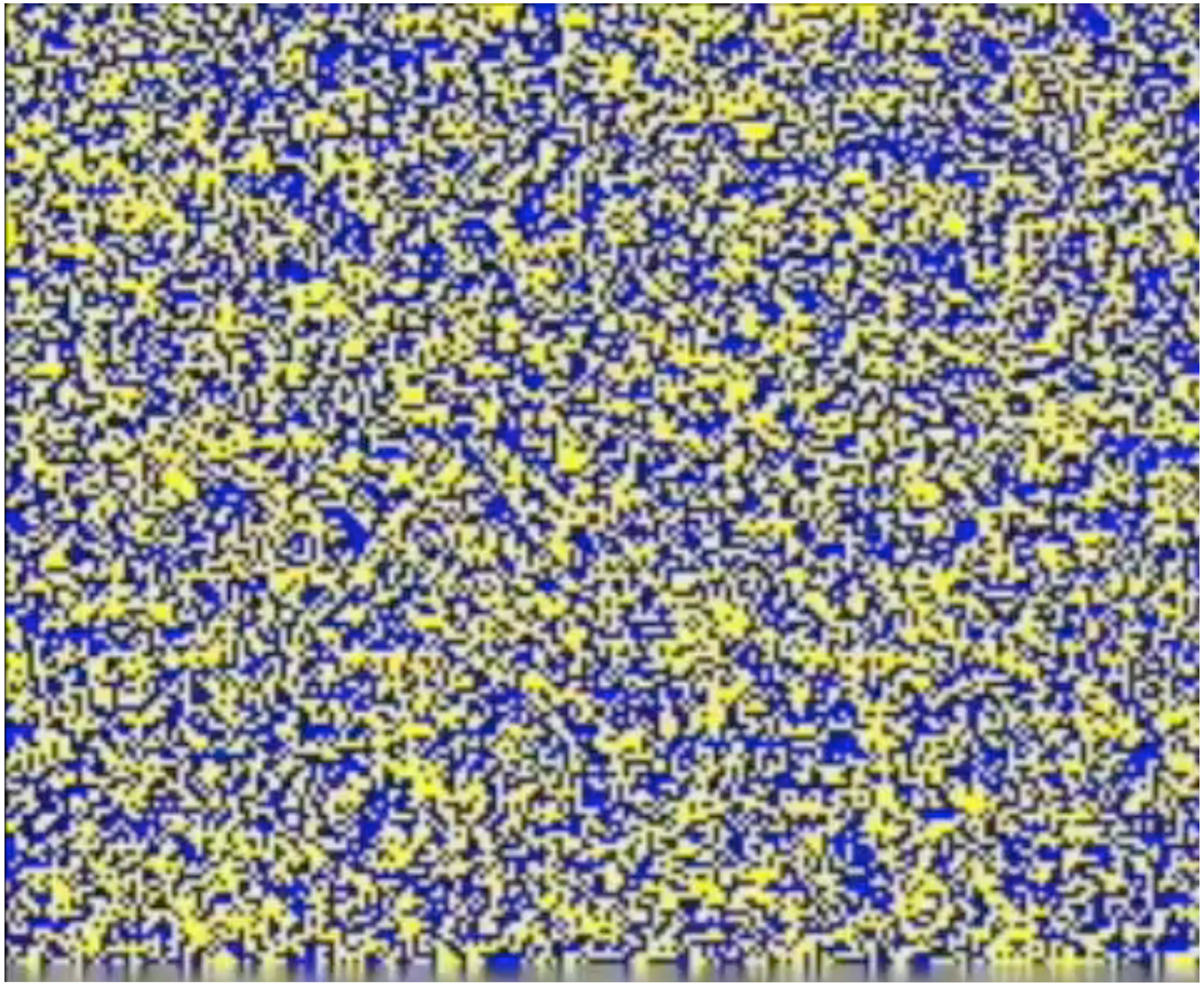
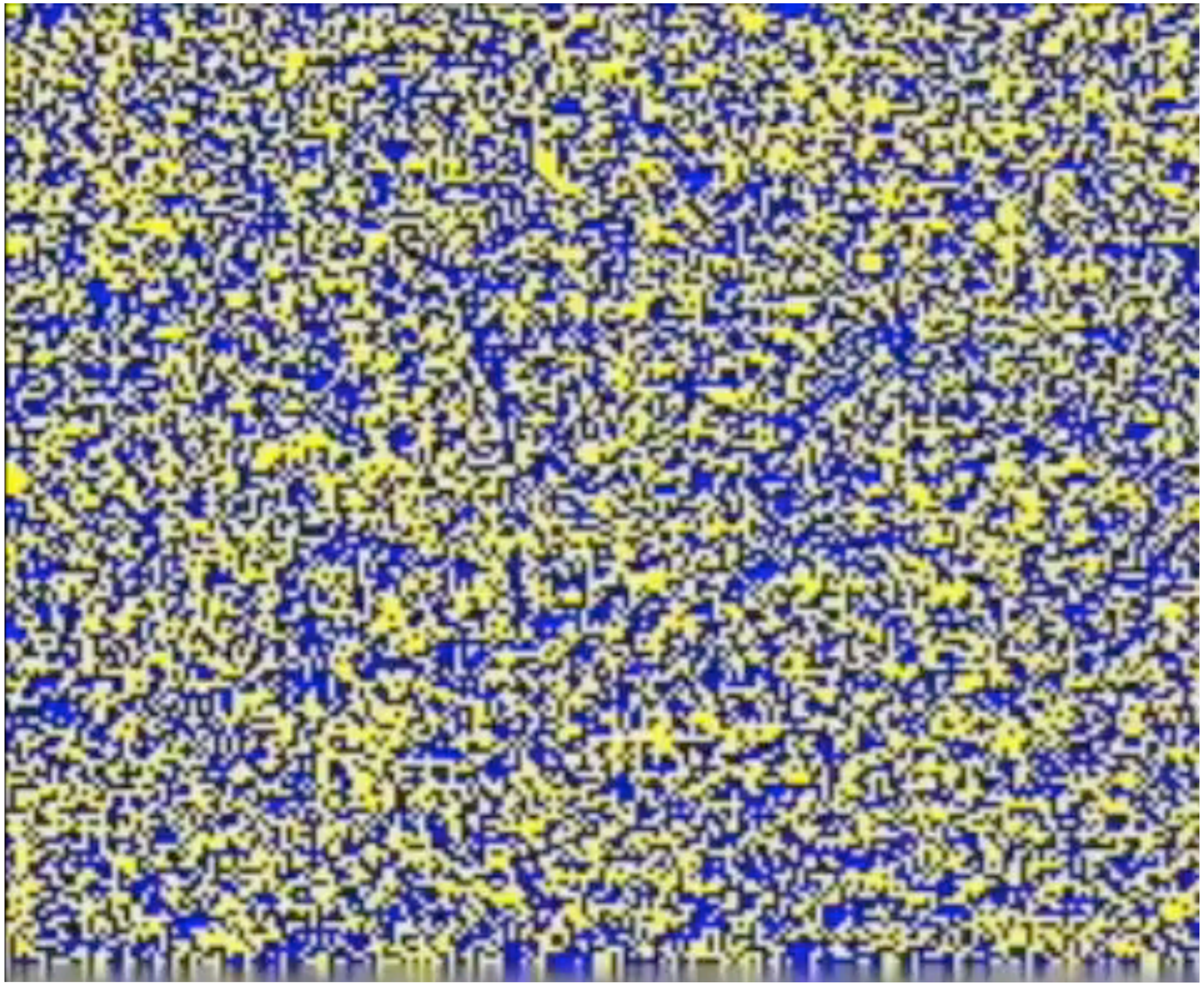


Two Dimensions:
Evolution to Stripe State



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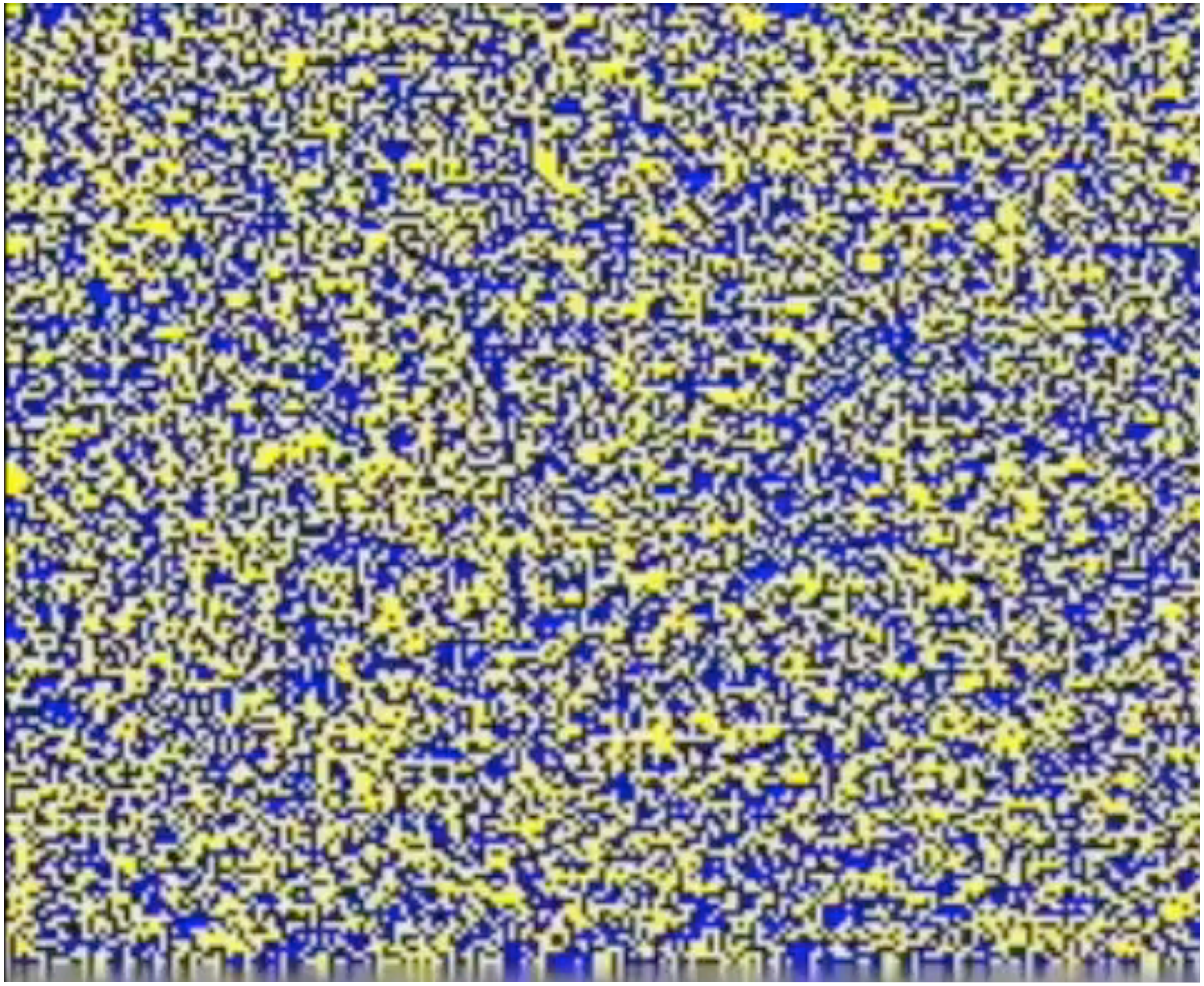
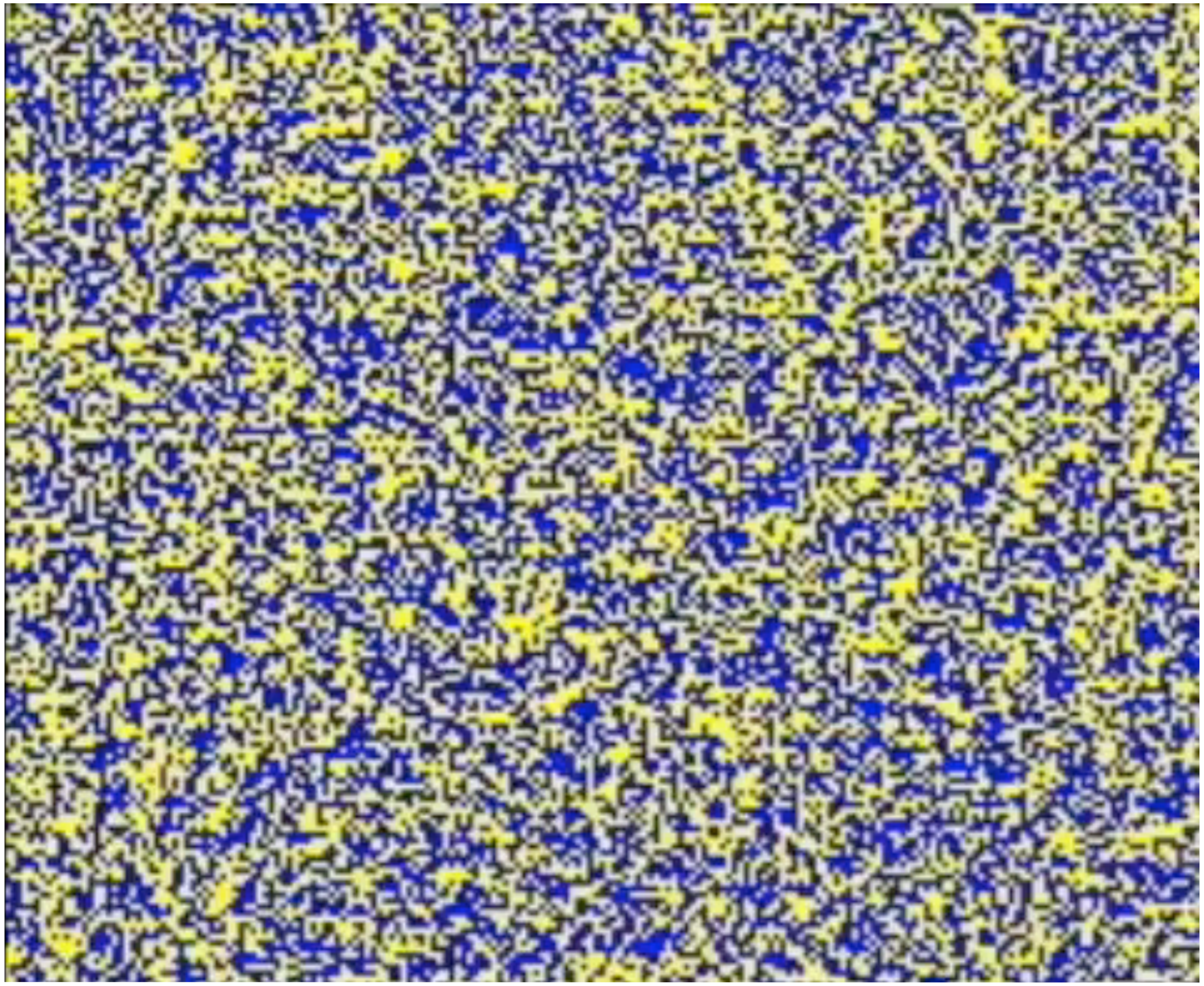


Two Dimensions:
Evolution to Diagonal Stripe State



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Question: what ***is*** the final state?

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Answer from simulations: (Spirin, Krapivsky, & SR 01, 02)

ground state with probability ≈ 2/3

stripe state with probability ≈ 1/3

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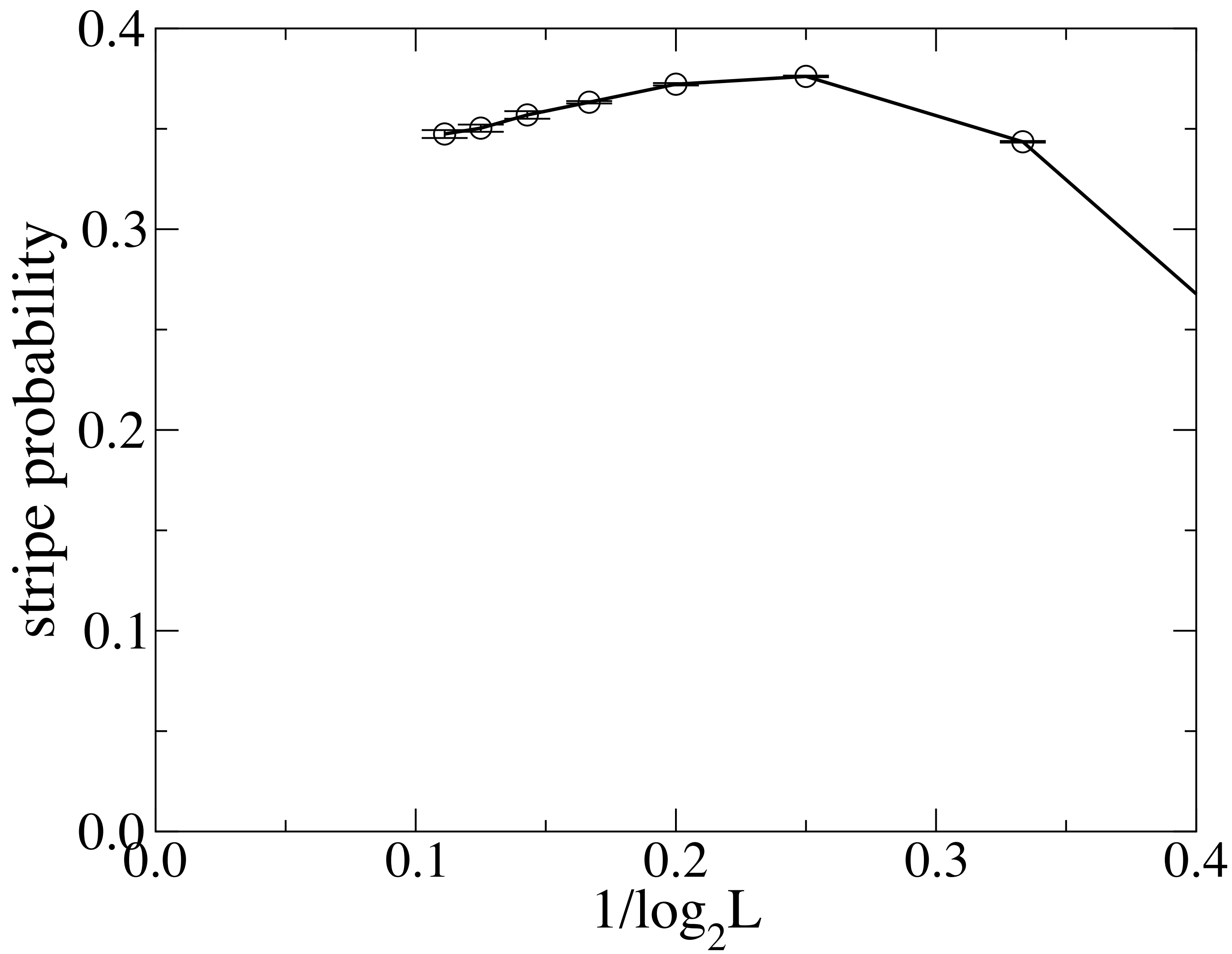

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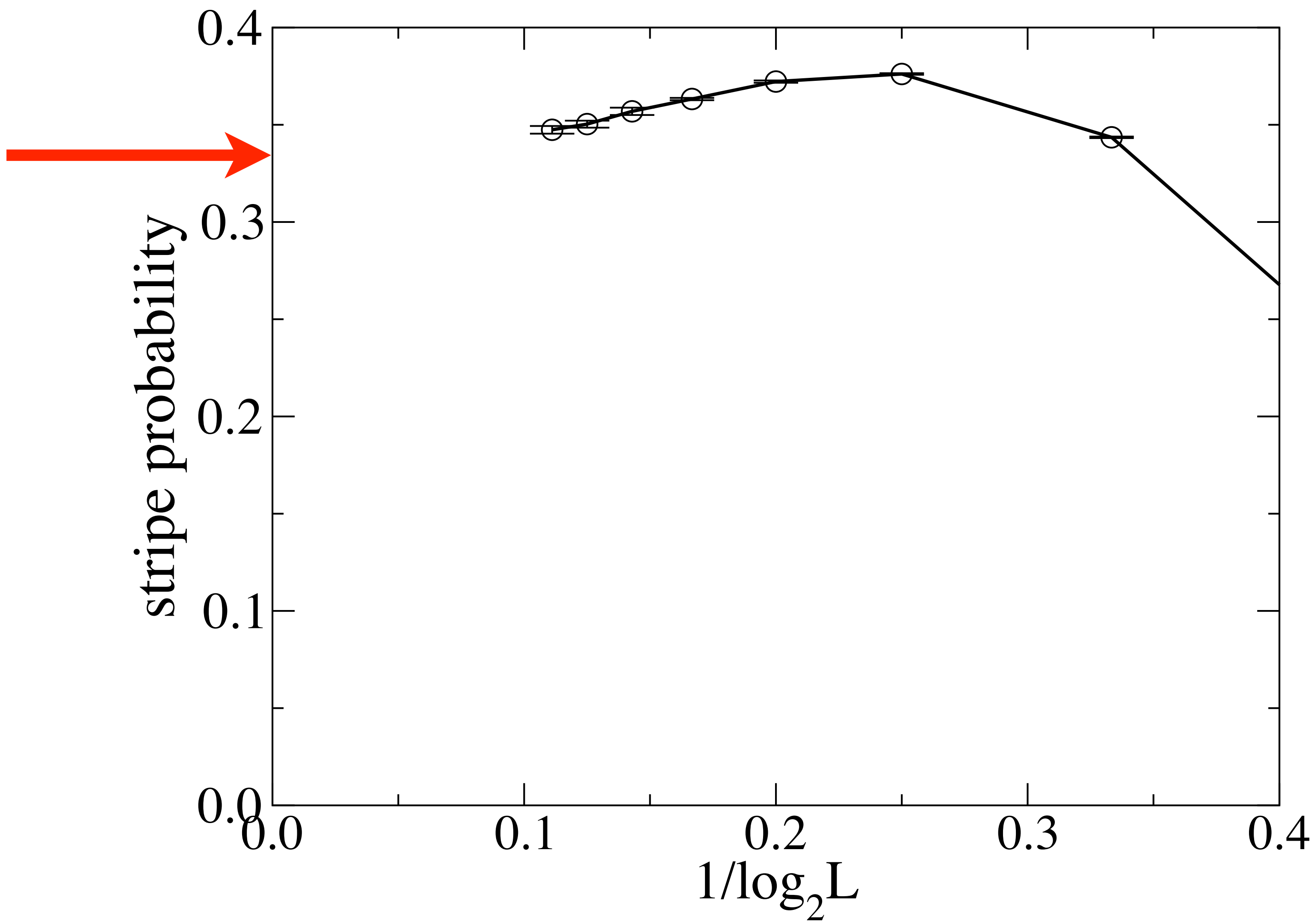
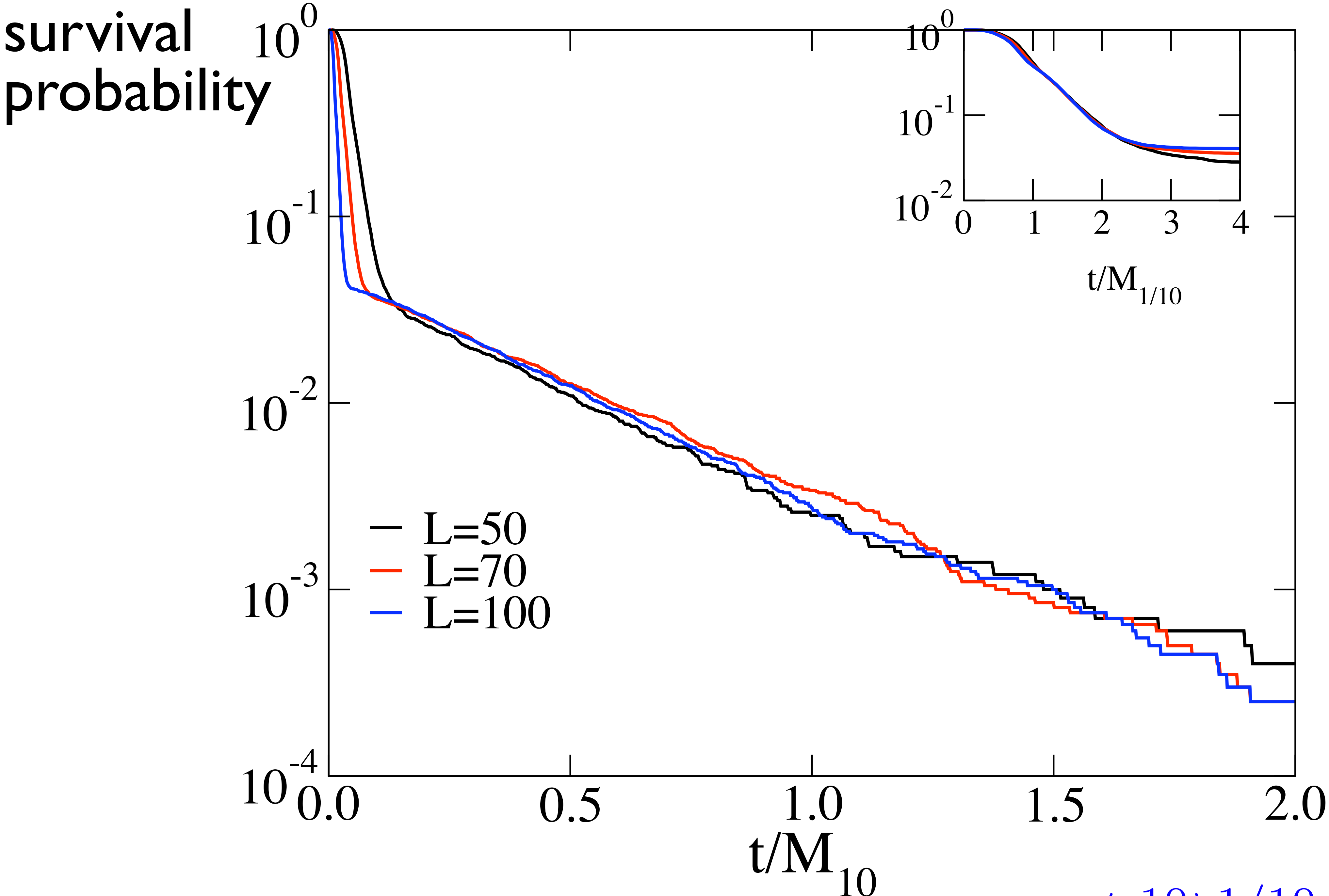


Two Dimensions

How long to reach the final state?



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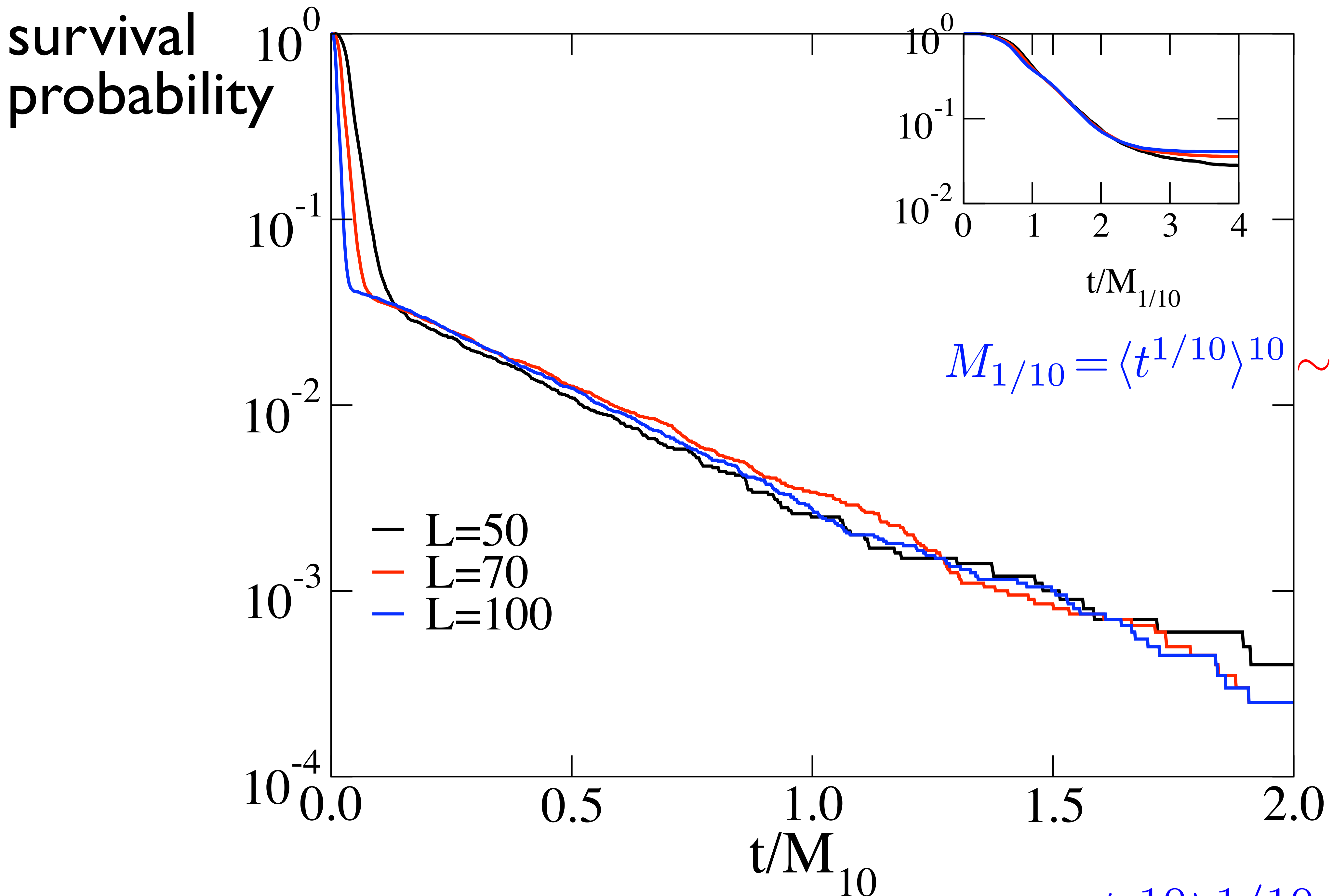

$M_{1/10} = \langle t^{1/10} \rangle^{10} \sim L^2$

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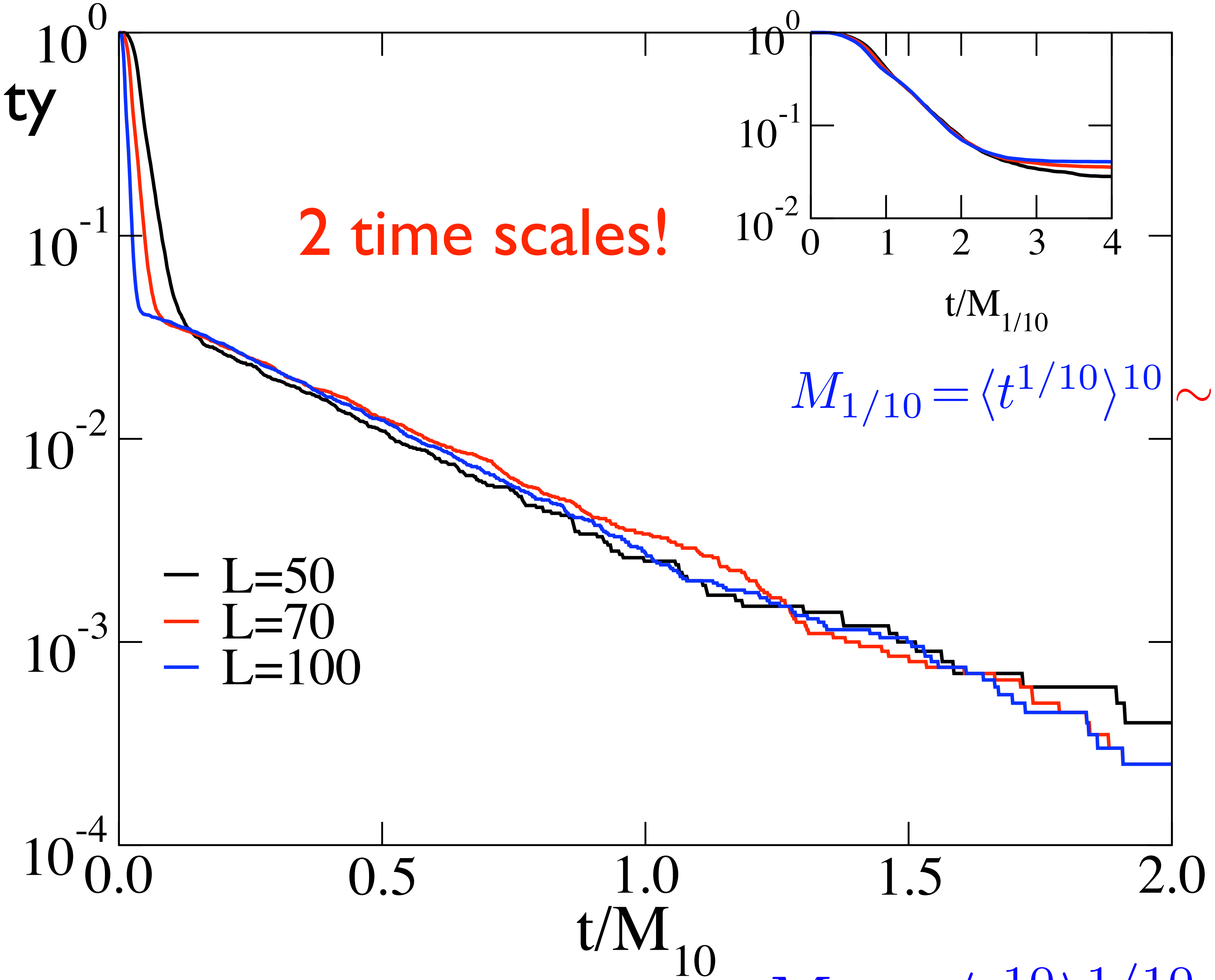
Two Dimensions

How long to reach the final state?

survival probability



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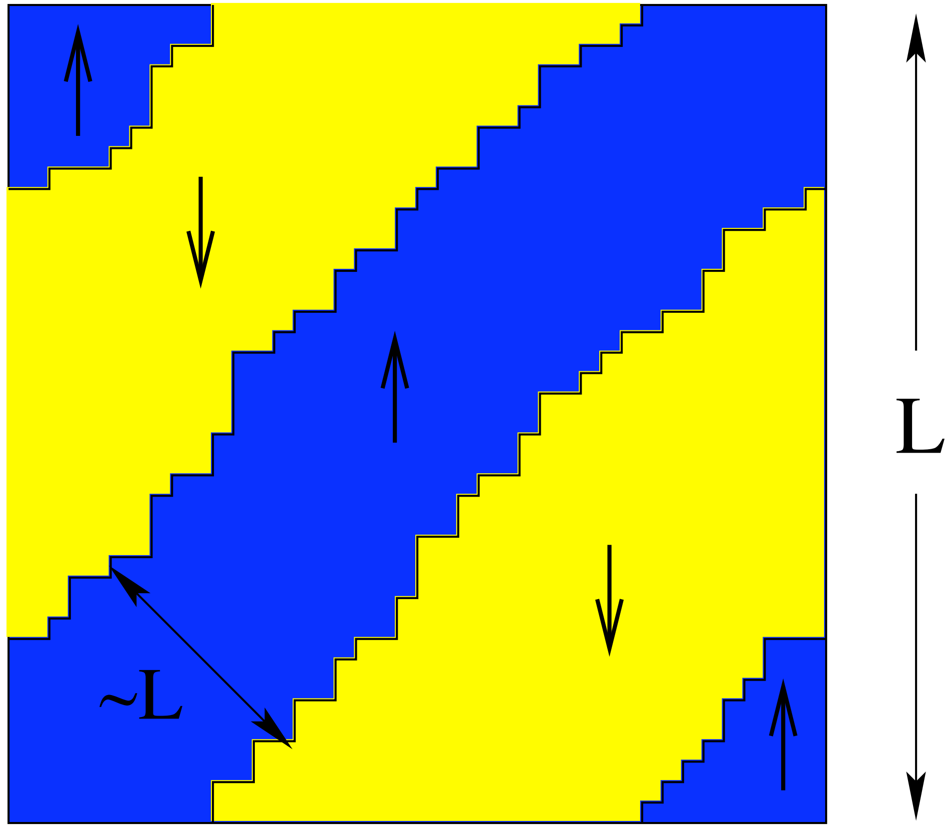
Two time-scale relaxation

95% short-lived, but 5% long lived because of diagonal stripes!



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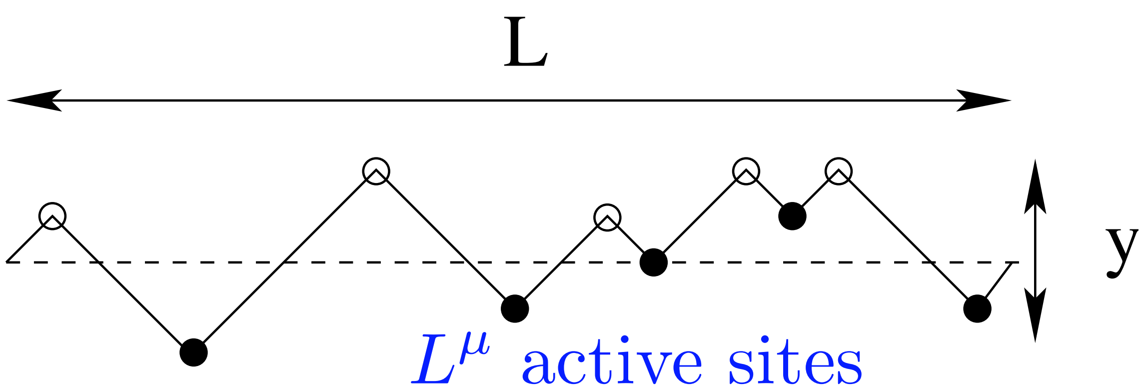
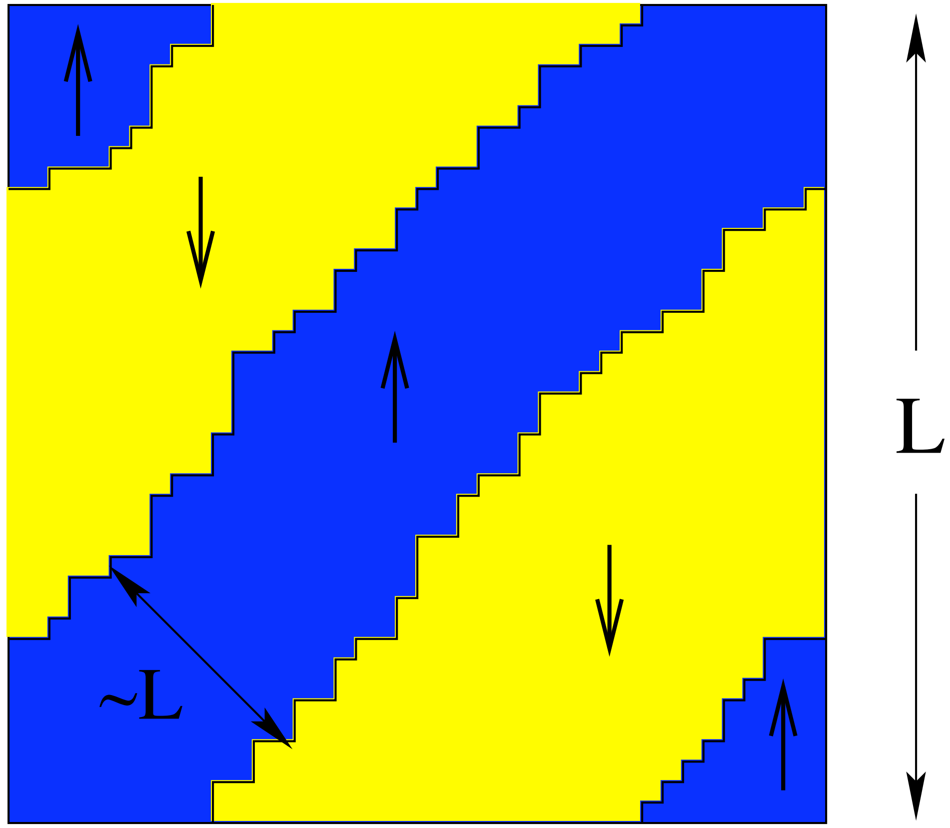
Diagonal stripe dynamics: (Plischke et al 87)





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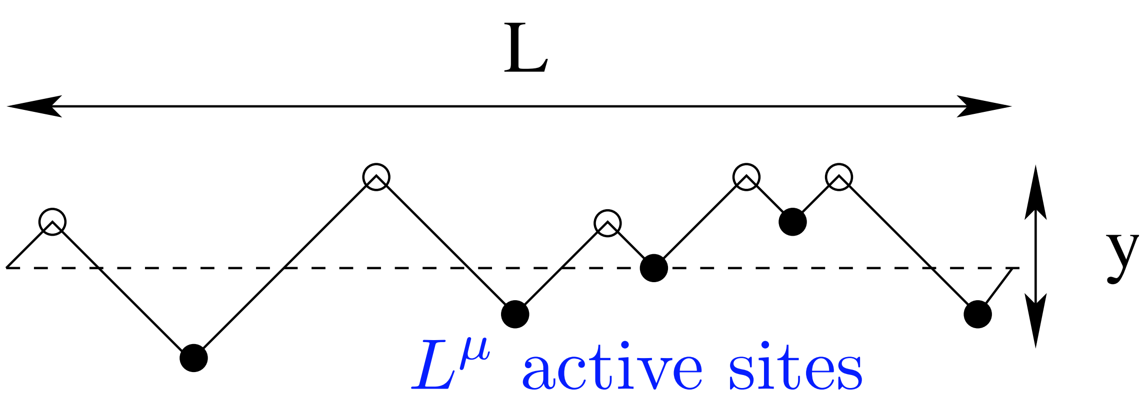

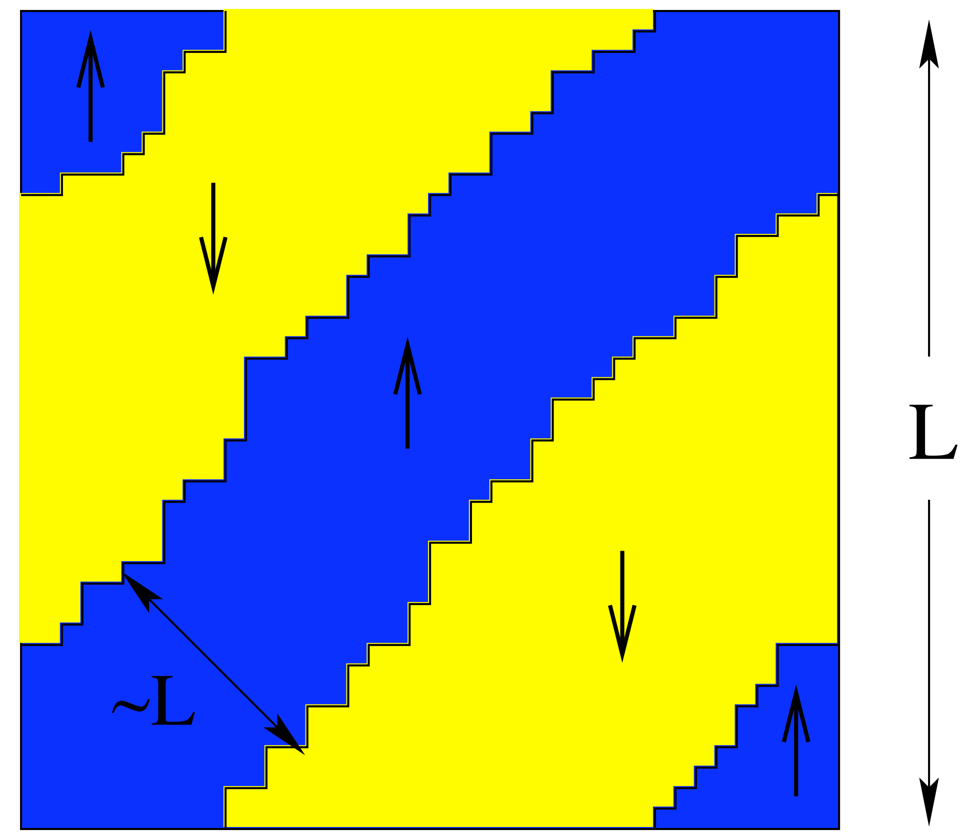

$$\Delta t = 1, \quad L^\mu \text{ events} \quad \rightarrow \quad \Delta y_{\rm cm} \sim L^{\mu/2}/L$$

$$\rightarrow \quad D(L) \sim L^{\mu-2}$$

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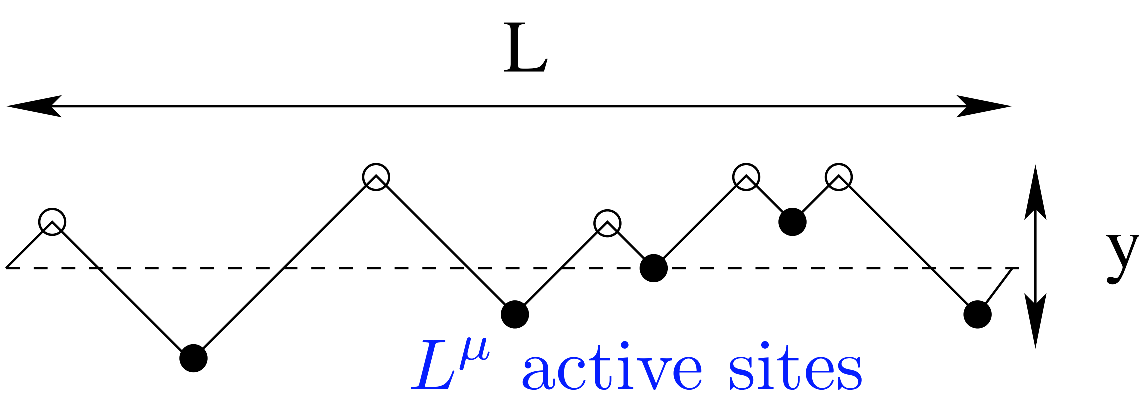
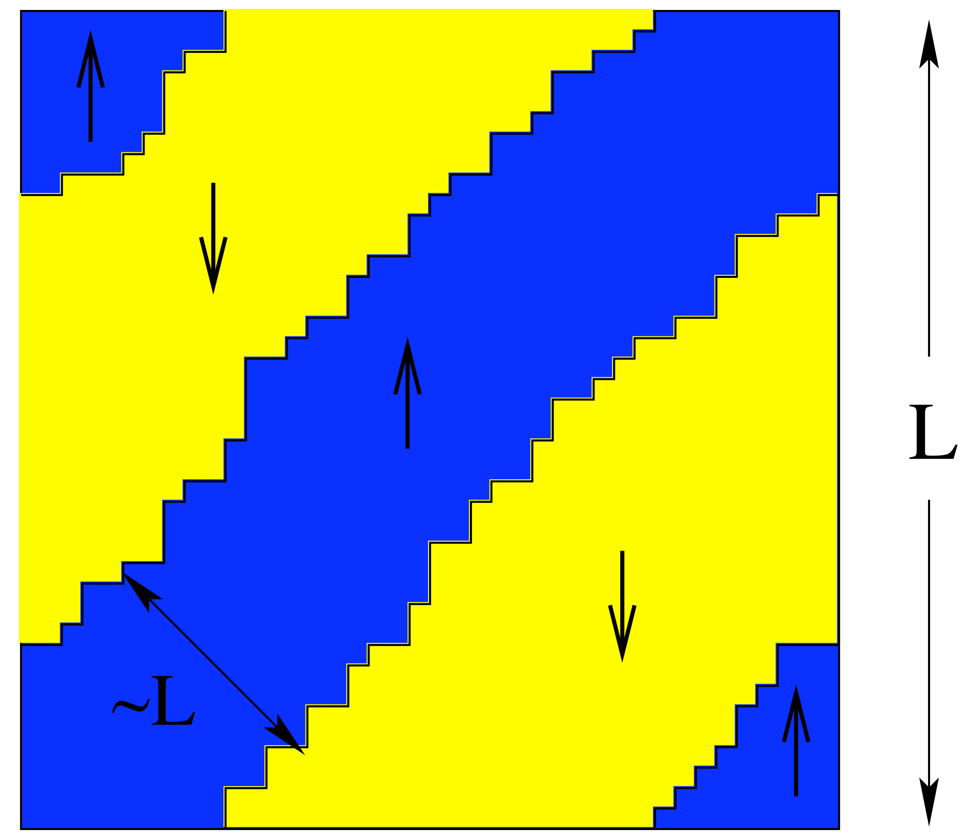




$$\Delta t = 1, \quad L^\mu \text{ events} \quad \rightarrow \quad \Delta y_{\rm cm} \sim L^{\mu/2}/L$$

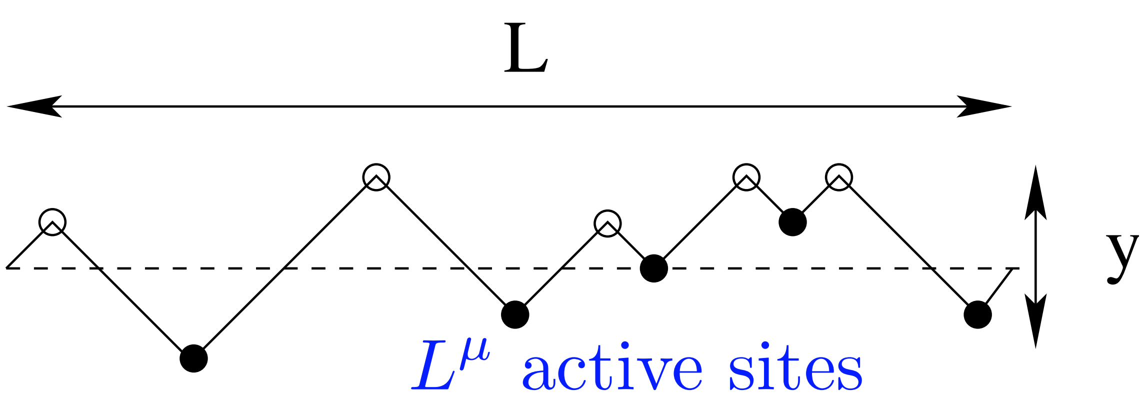
$$\rightarrow \quad D(L) \sim L^{\mu-2}$$

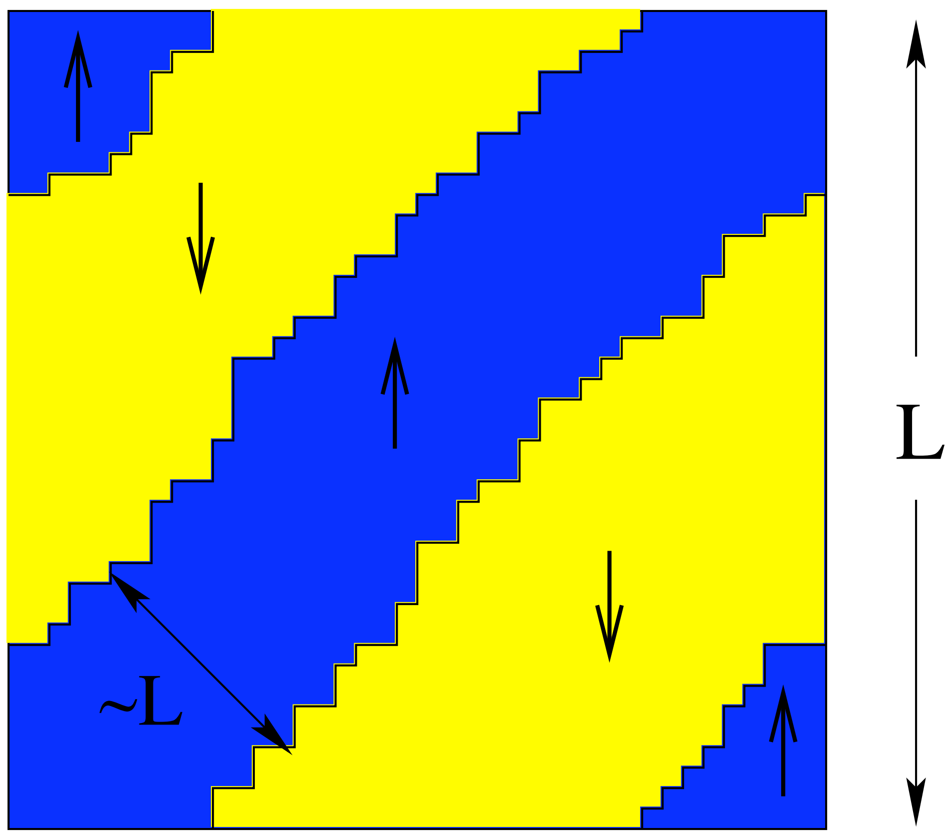
survival time $\tau \sim L^2/D \quad \sim \quad L^{4-\mu}$

Two time-scale relaxation

95% short-lived, but 5% long lived because of diagonal stripes!

Diagonal stripe dynamics: (Plischke et al 87)





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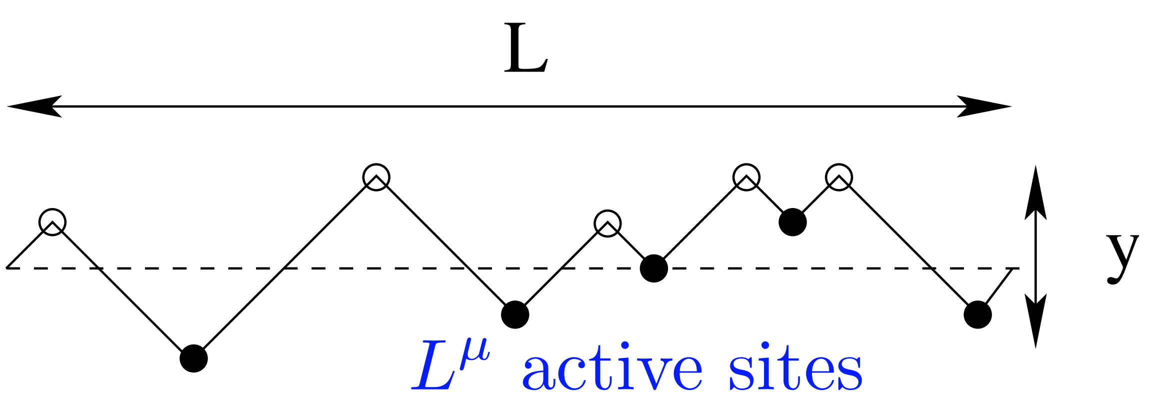

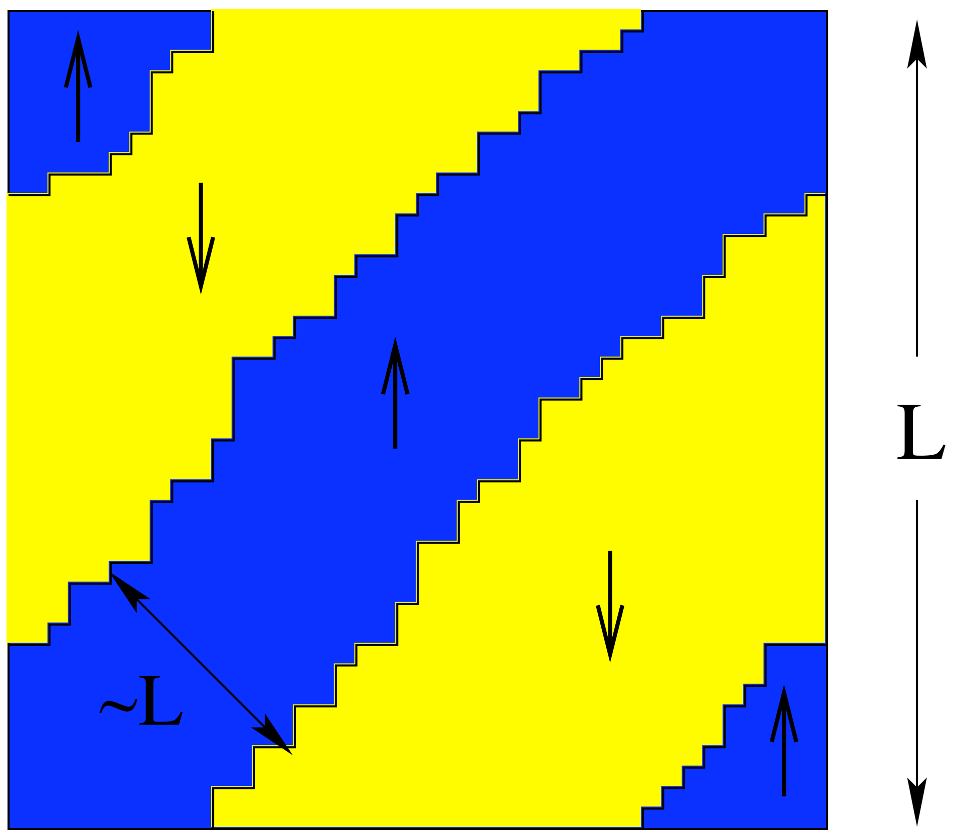

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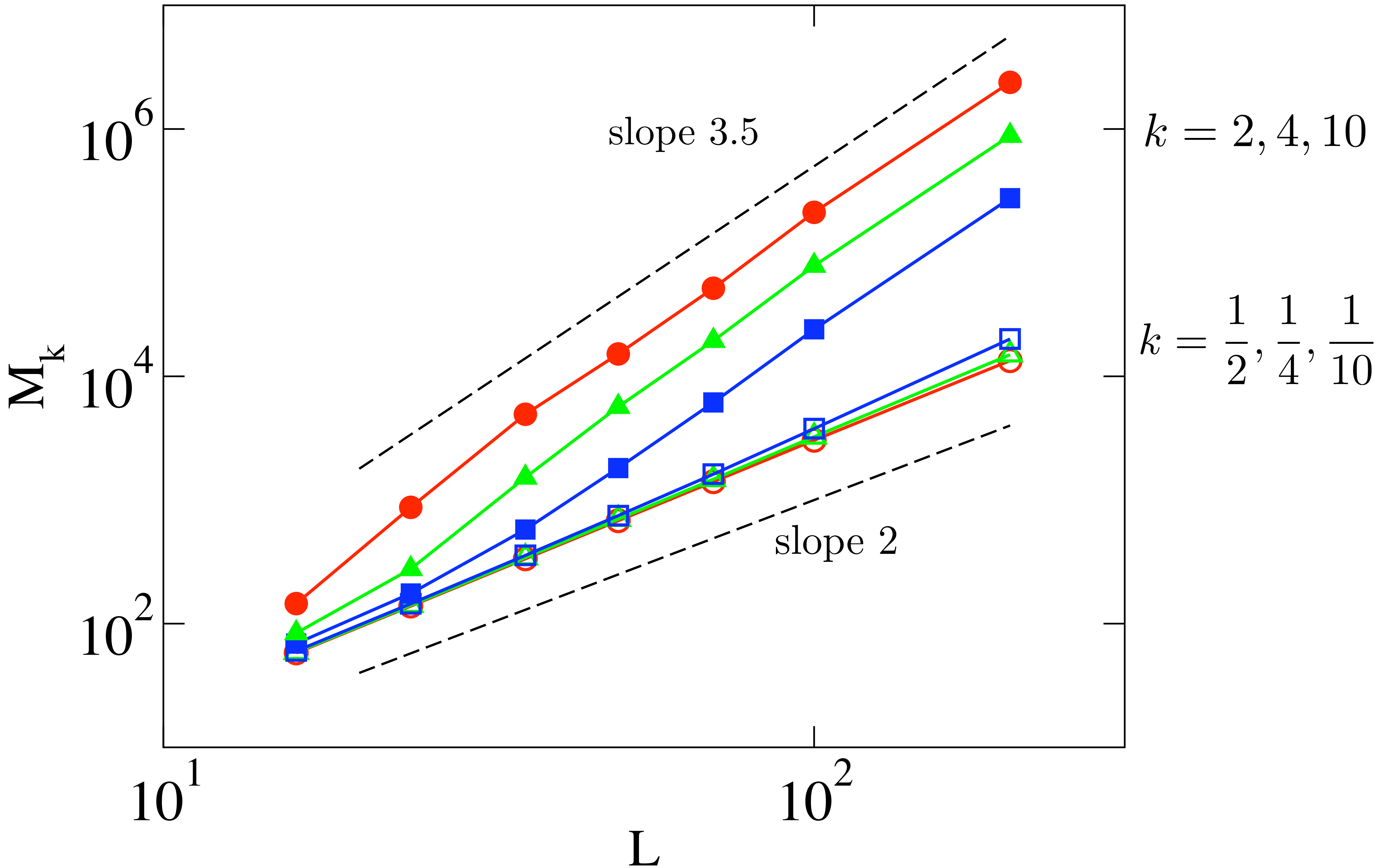
$$\sim \quad L^{3.5}$$

Multiscaling in moments of the stopping time



Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

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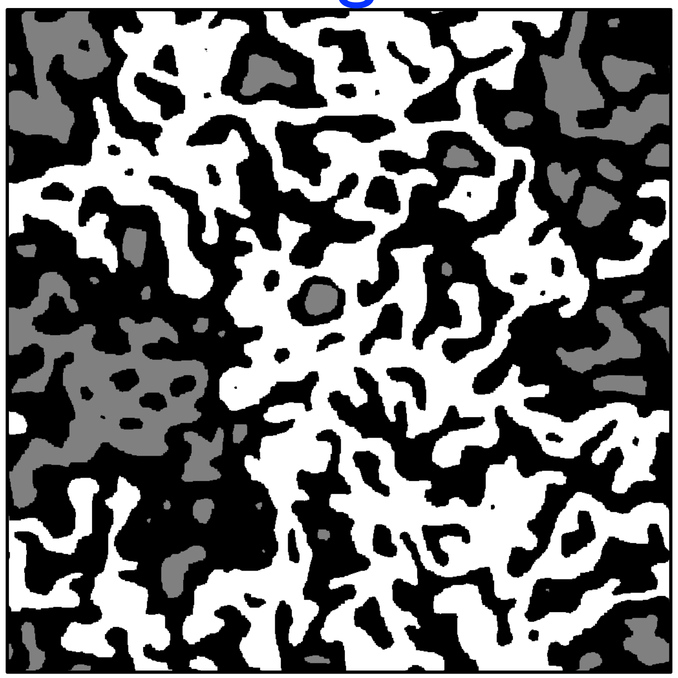
coarsening of 1024x1024 system



t=200

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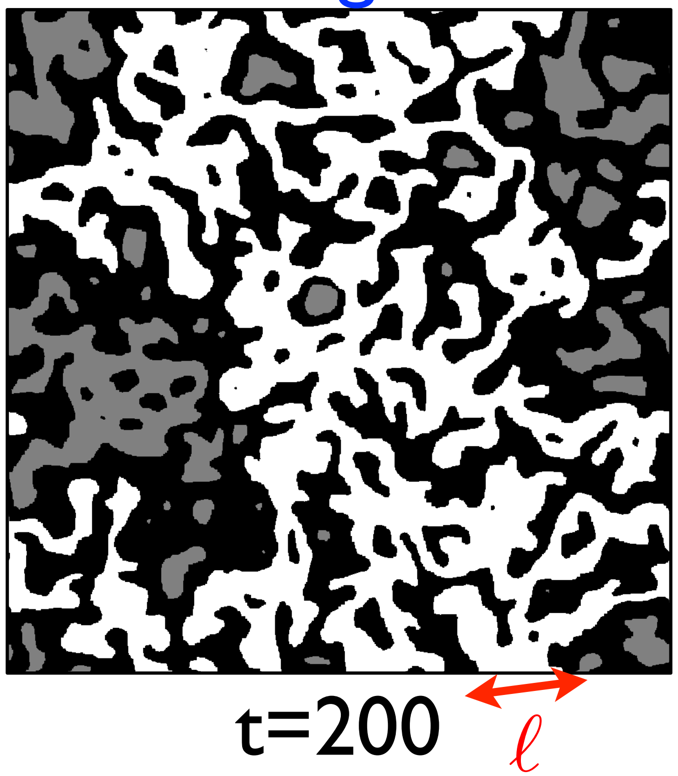

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critical pt of continuum percolation $a \ll \ell(t) \ll L$

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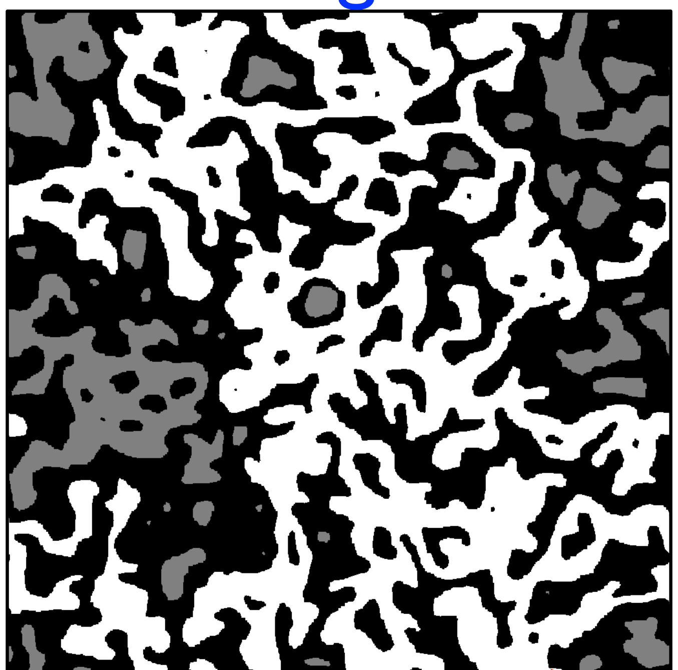
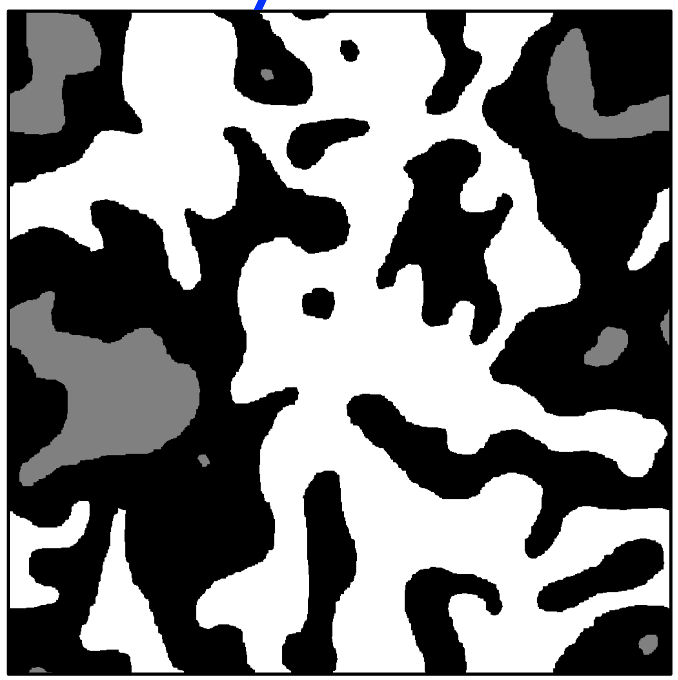


t=200 ℓ



t=1000

critical pt of continuum
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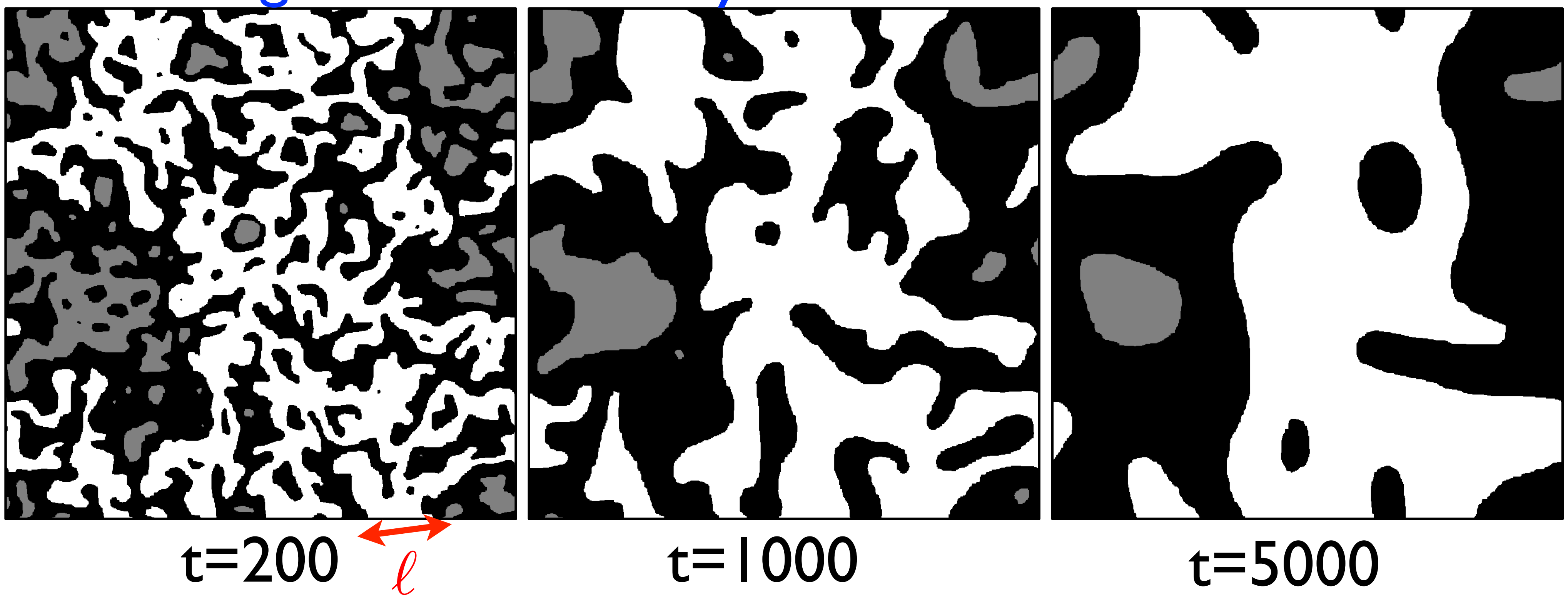

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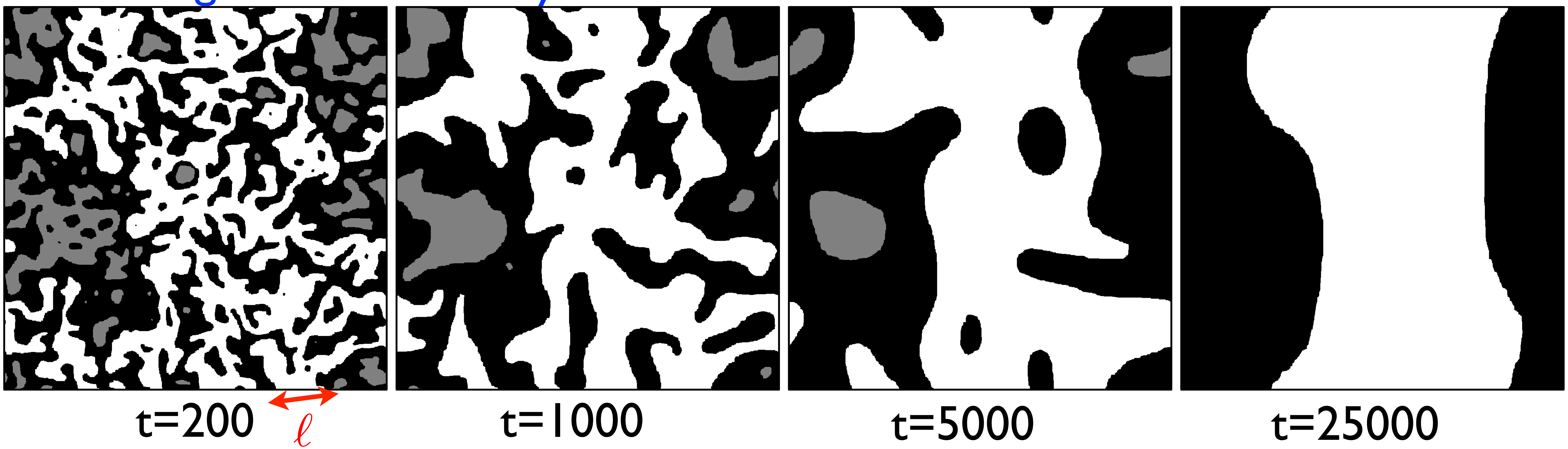

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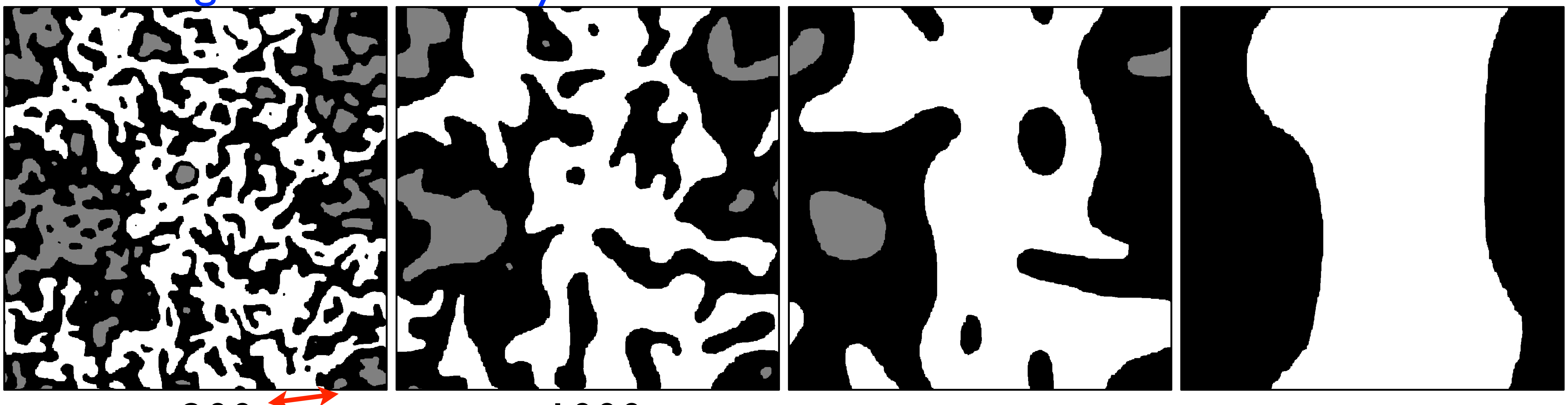

t=200 ℓ t=1000 t=5000 t=25000

critical pt of continuum percolation $a \ll \ell(t) \ll L$

deterministic, curvature-driven evolution for $\ell(t) \gg a$
→ Invariant topology in the coarsening regime

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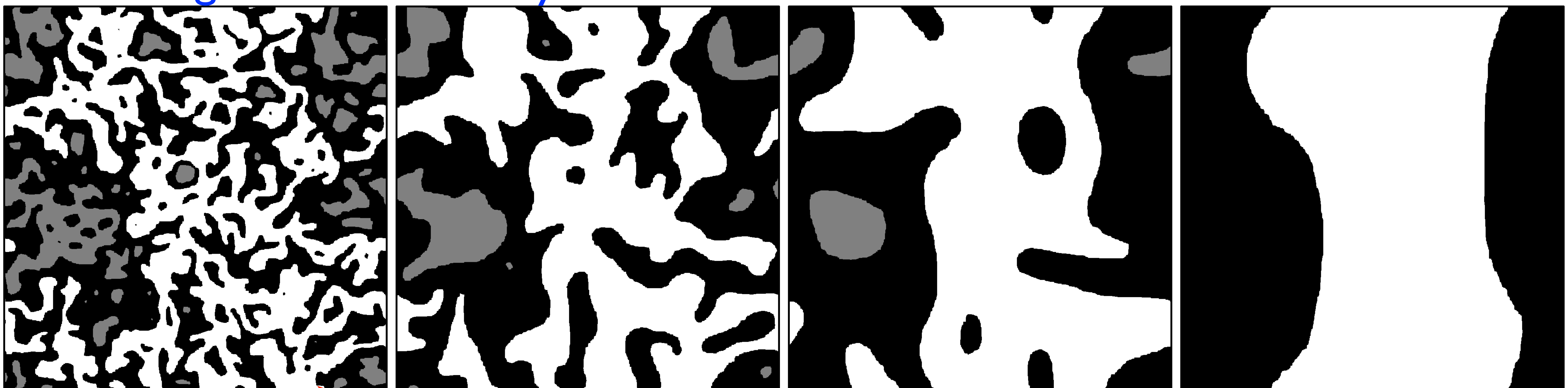

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FBC: $$P_{\text{stripe}} = \frac{\sqrt{3}}{2\pi}\,\lambda(r)\;\; {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right)$$

$$\lambda(r) = \left(\frac{1-k}{1+k}\right)^2$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2\pi}\ln\frac{27}{16} = 0.3558\ldots$$

$$r = \text{aspect ratio} = \frac{2K(k^2)}{K(1-k^2)}$$

PBC: $P_{\text{stripe}} \approx 0.3388$

(Cardy 1992, Watts 1996, Simmons et al. 2007)

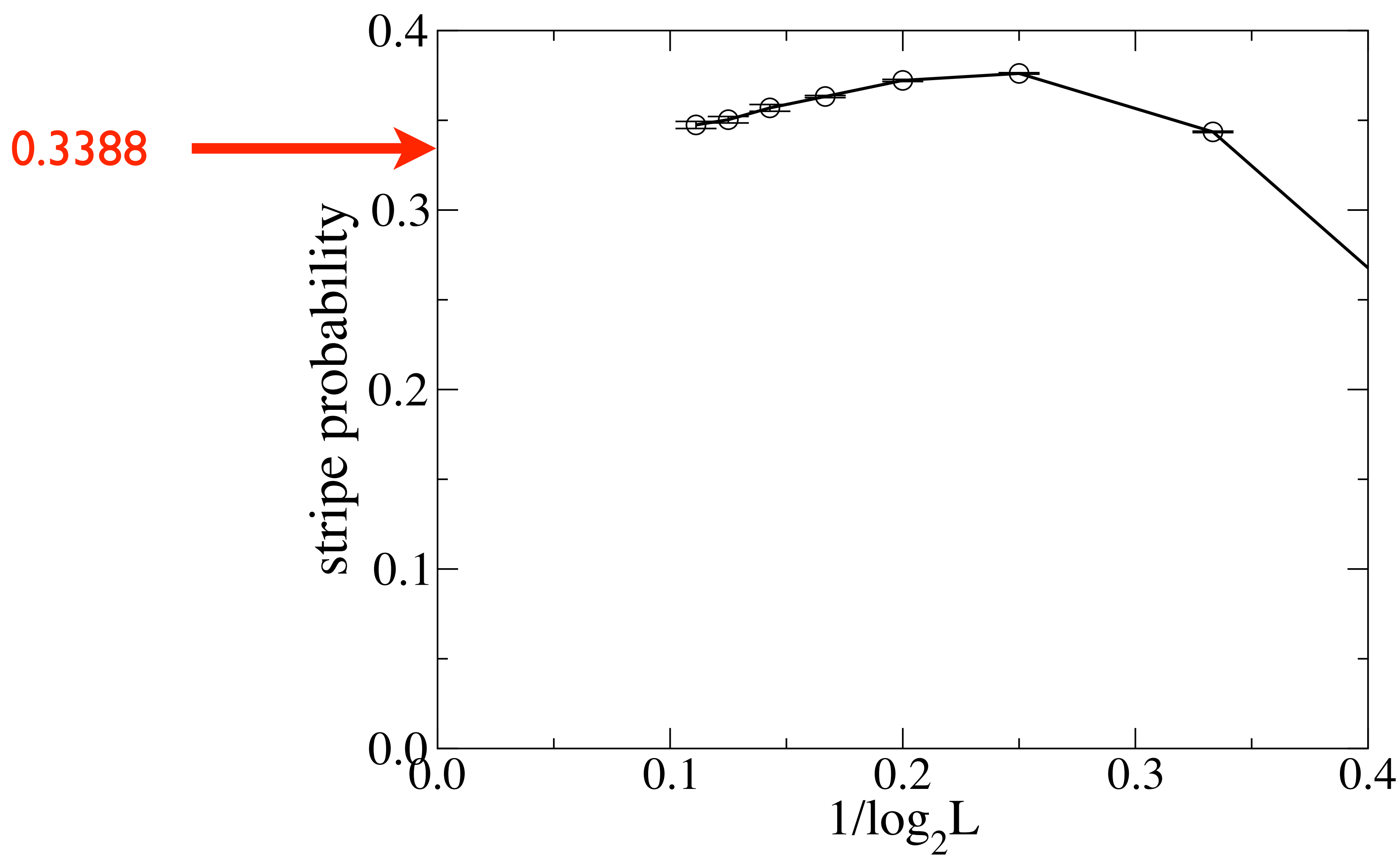
Two Dimensions

Question: what ***is*** the final state?

Answer from simulations: (Spirin, Krapivsky, & SR 01, 02)

ground state with probability ≈ 2/3

stripe state with probability ≈ 1/3



Three Dimensions (Olejarz, Krapvisky, & SR)

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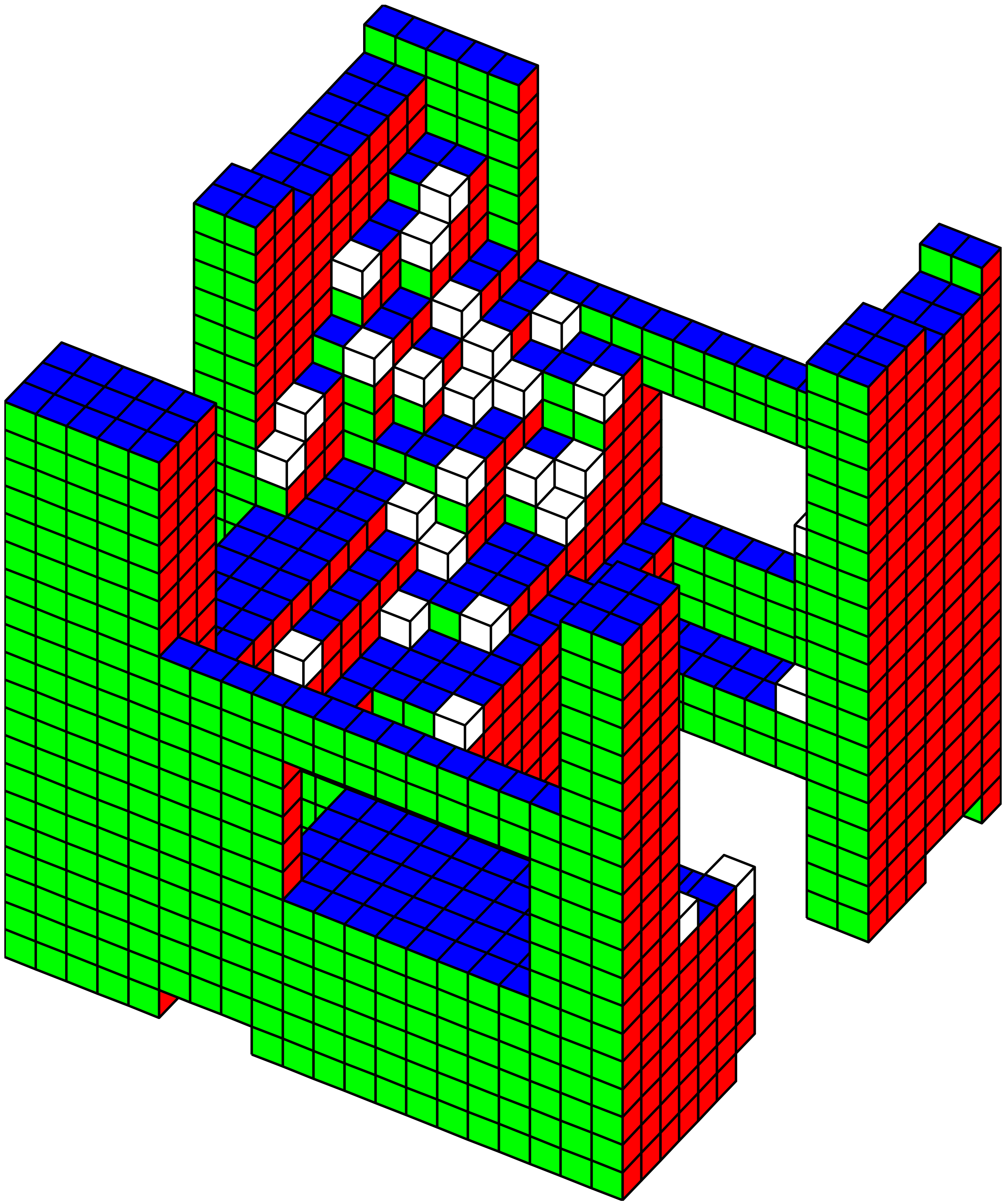
typical 20x20x20 system



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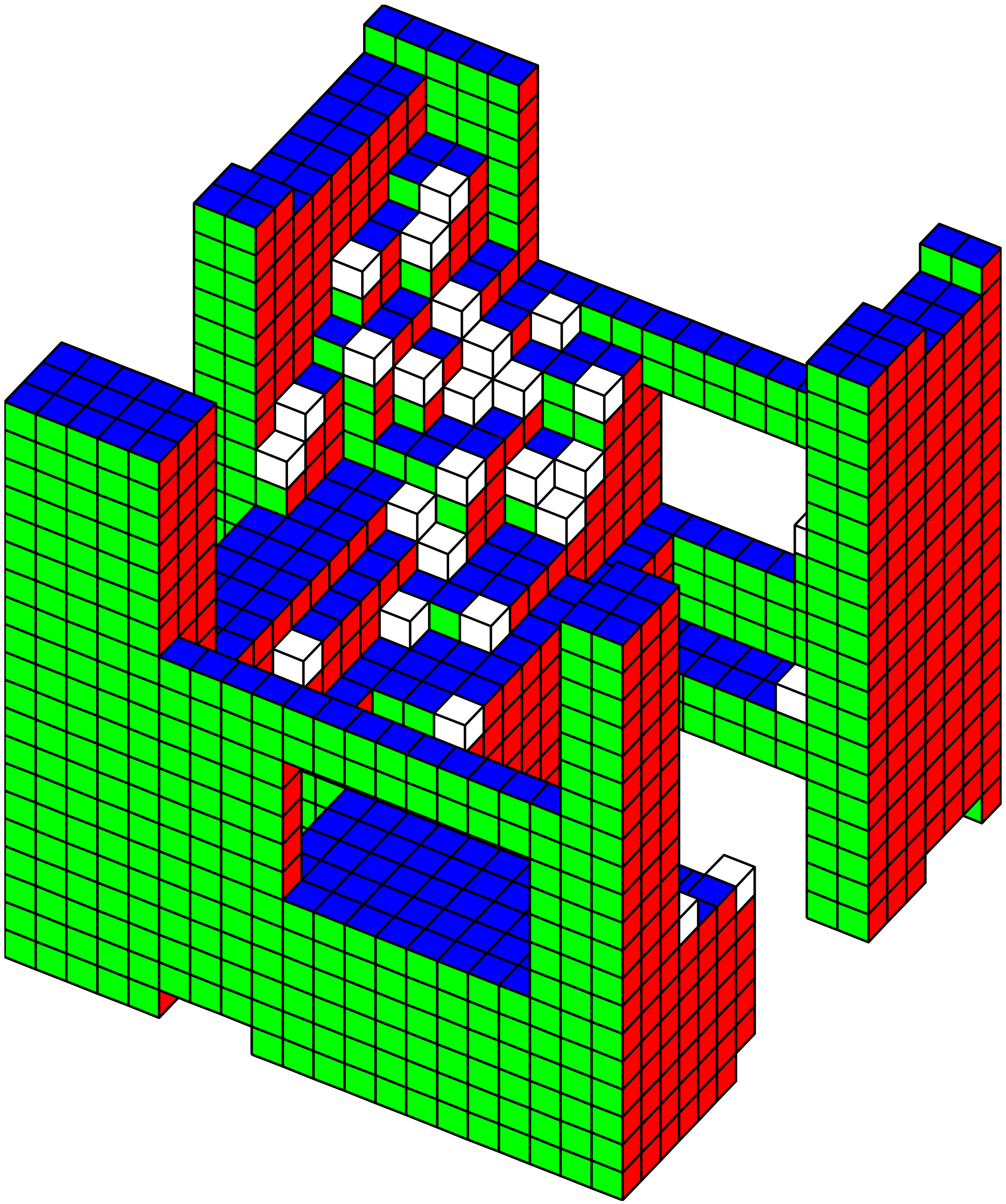

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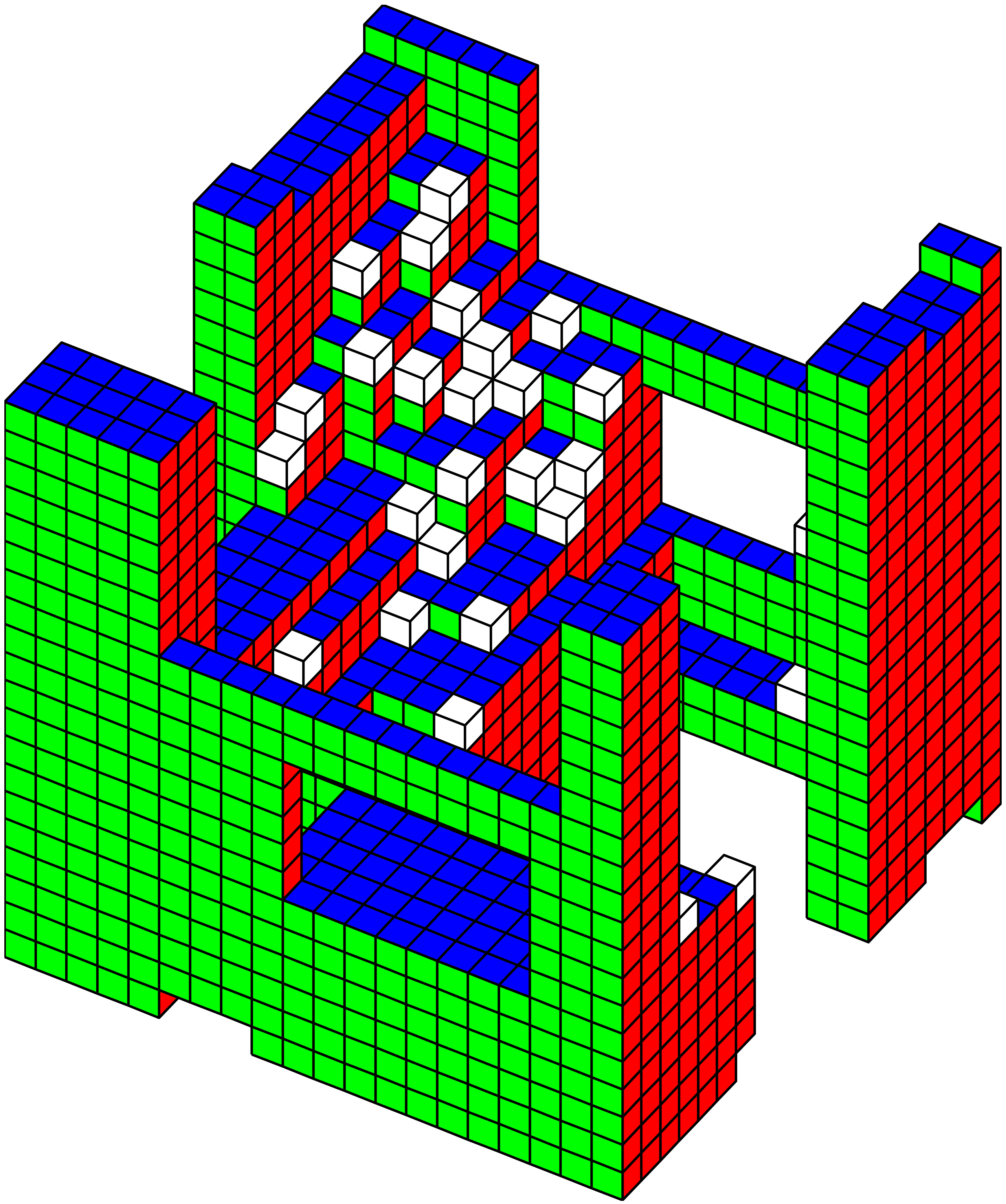

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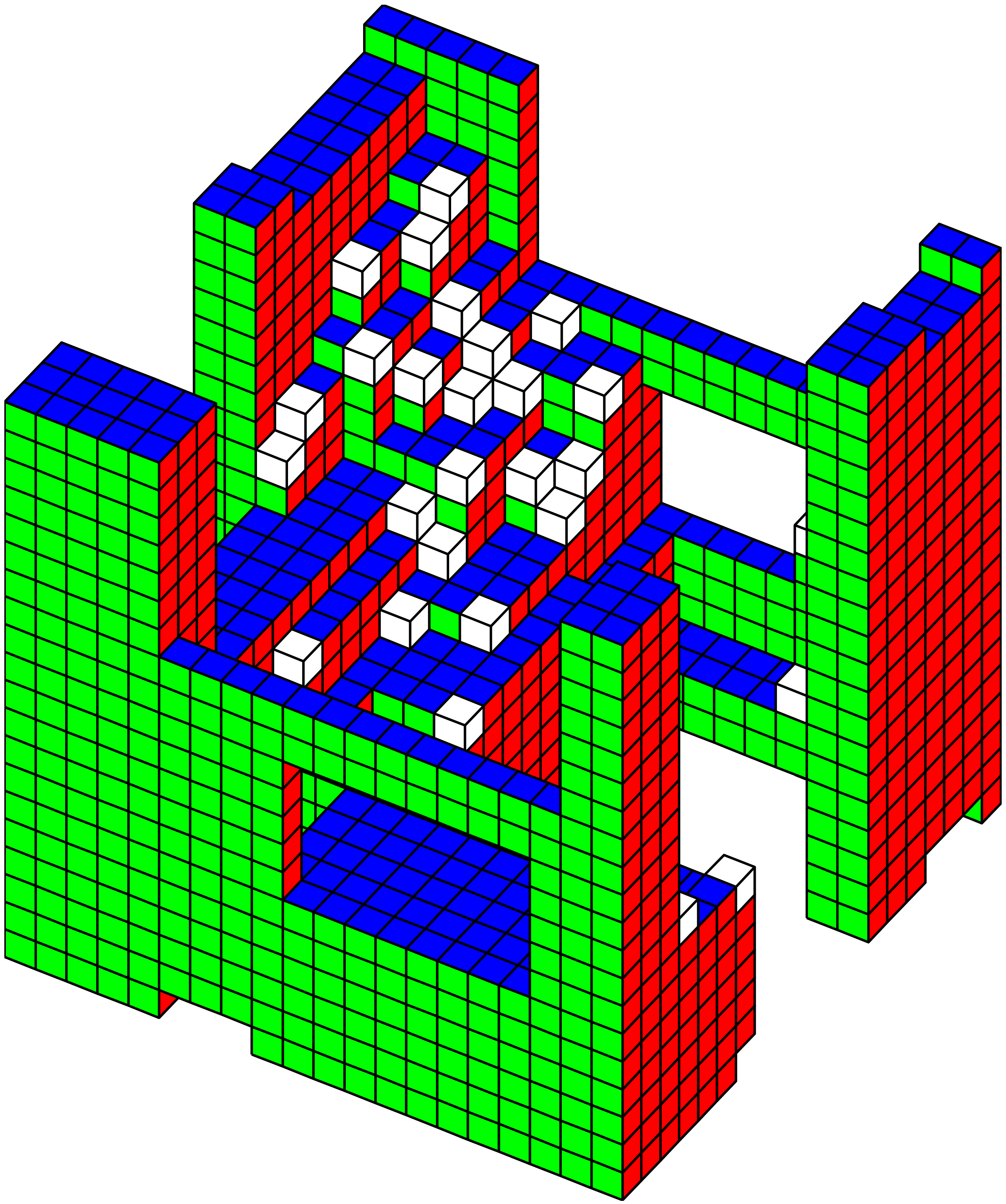

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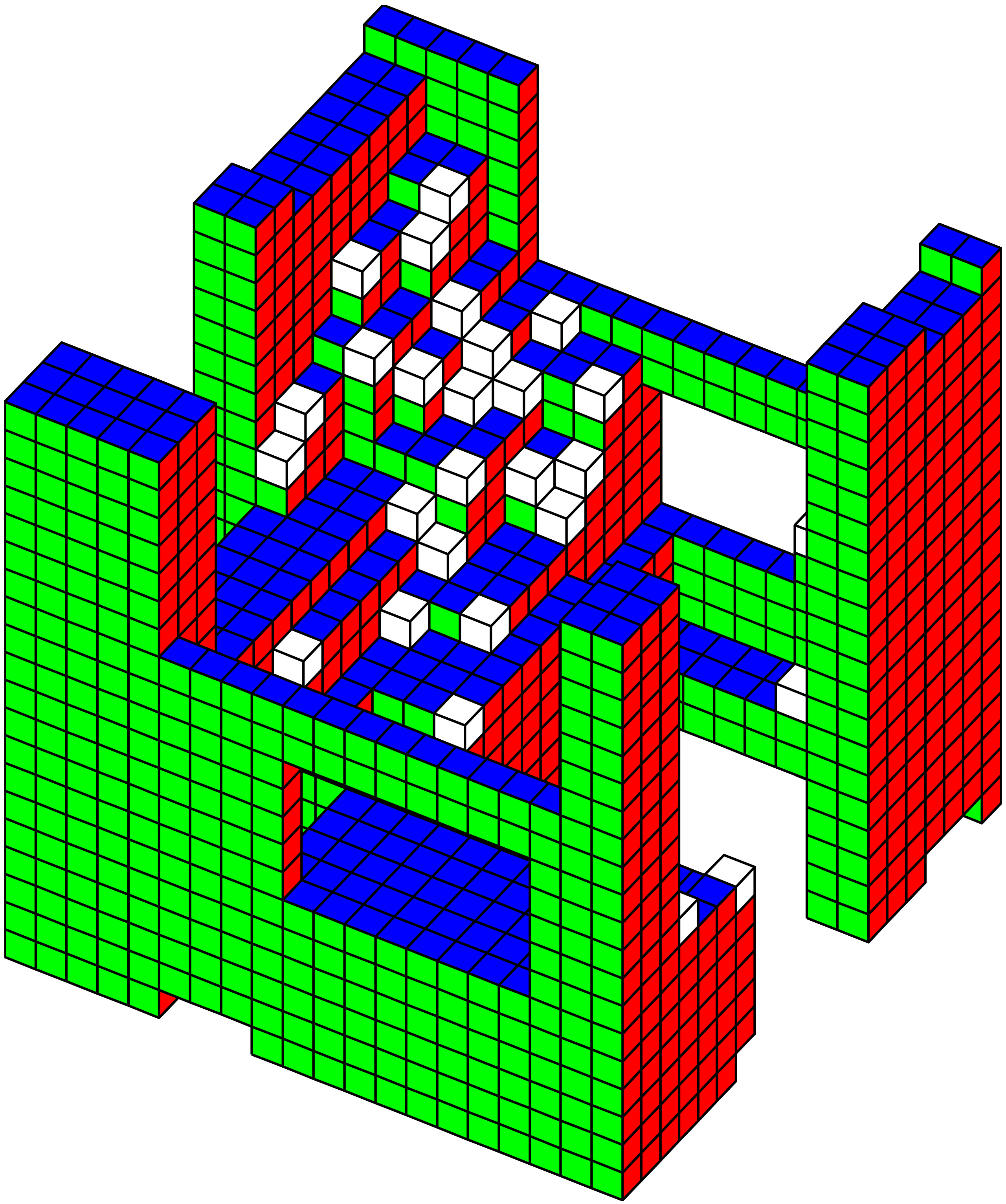

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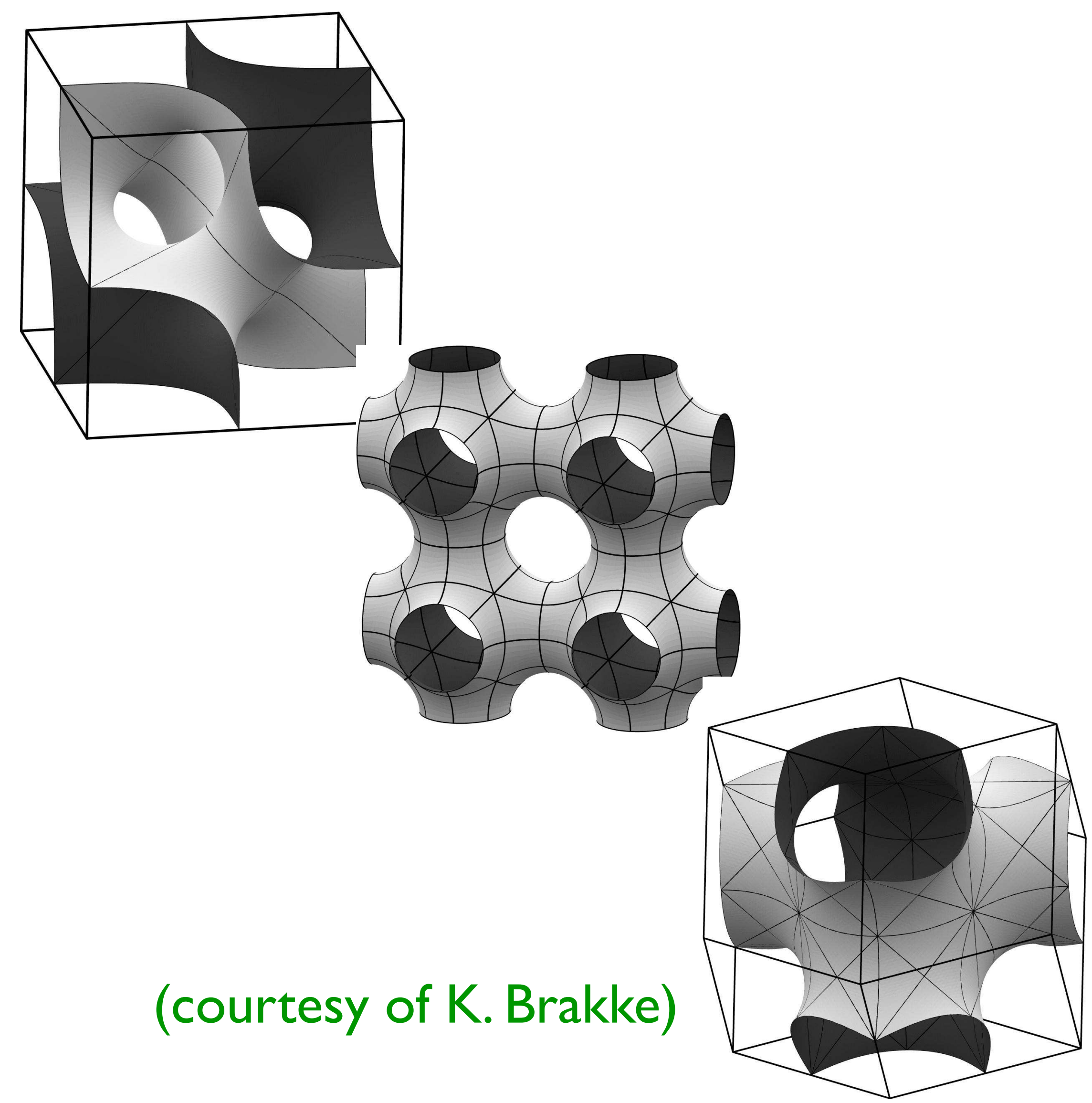


(courtesy of K. Brakke)

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

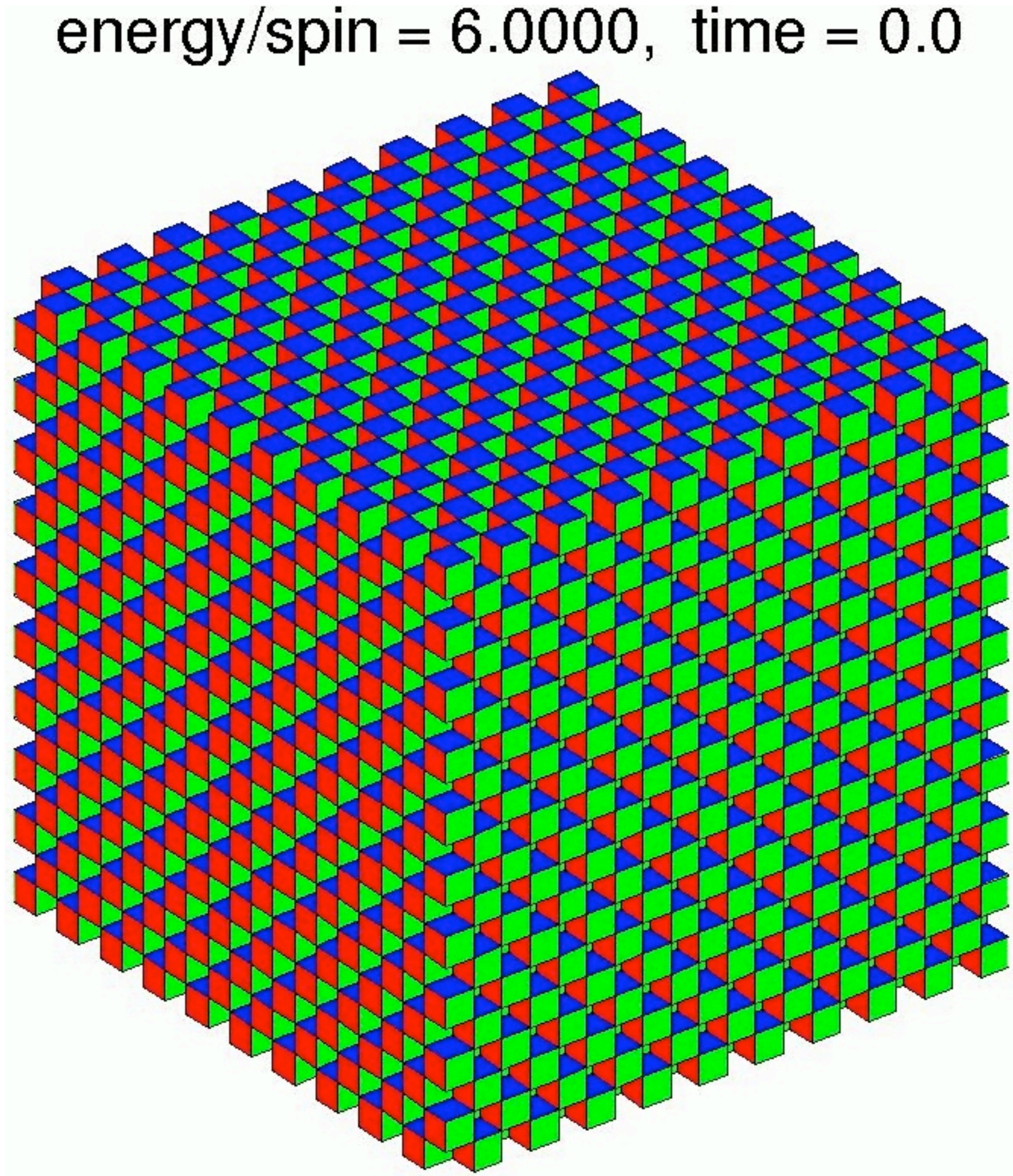
Basic question: What is the final state of the Ising-Glauber model @ T=0 with symmetric initial conditions?

Expectation:
- Ground state is approached as $t \to \infty$
- Power-law coarsening

The result:

dimension	expectation
1	absolutely correct
2	“sort of” correct
≥ 3	wrong

Evolution from Antiferromagnetic State



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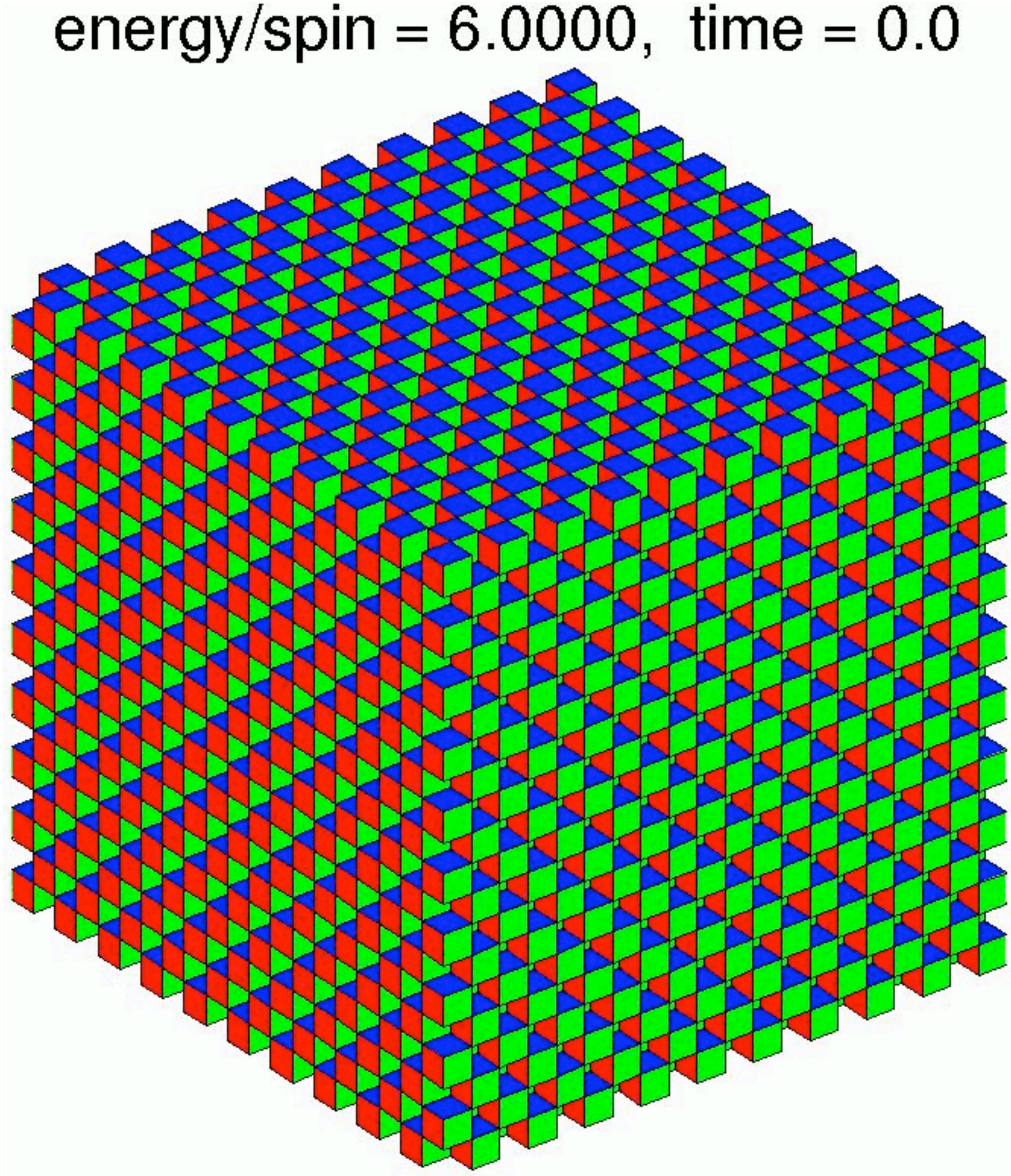
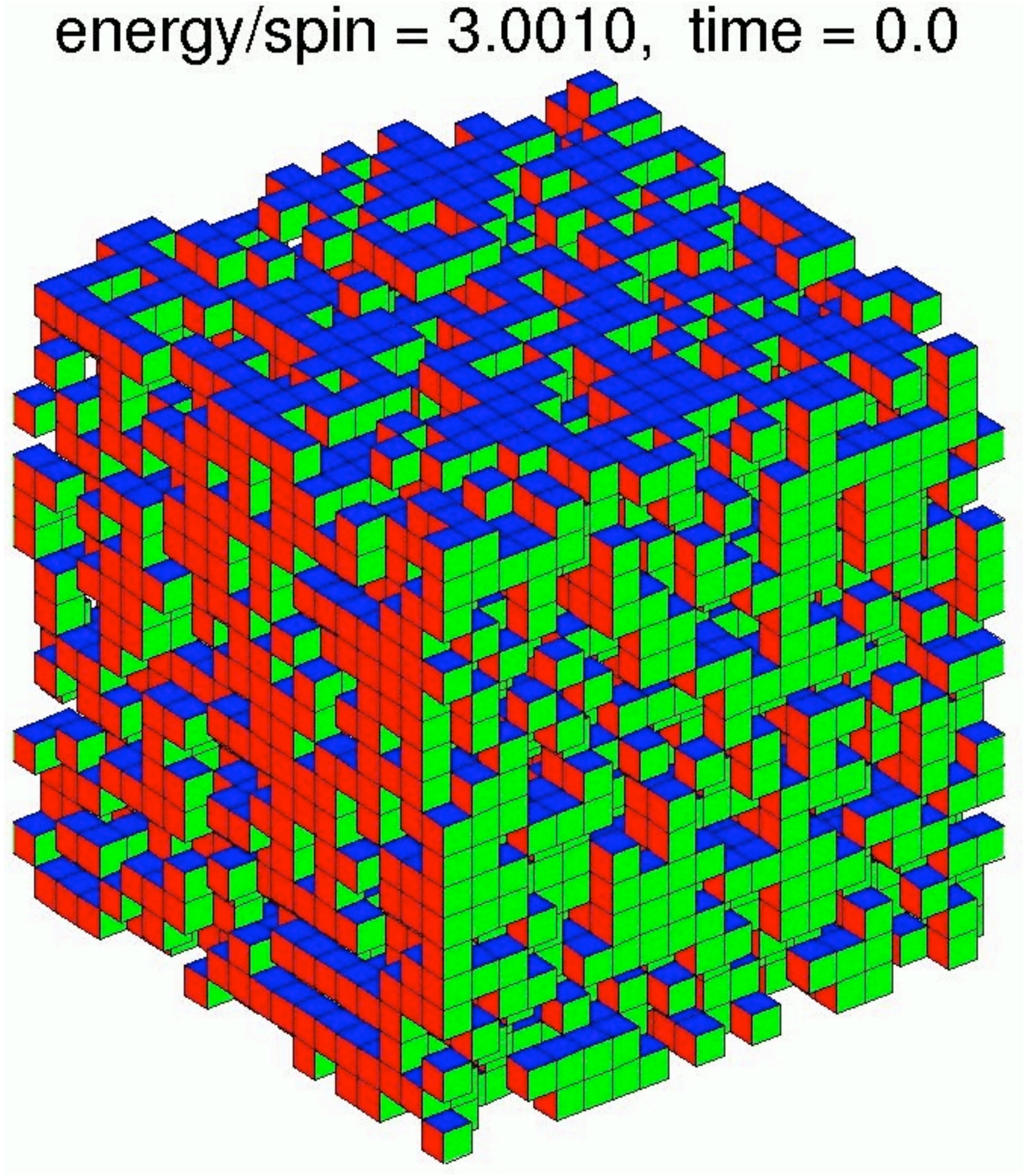


Evolution from Random* State



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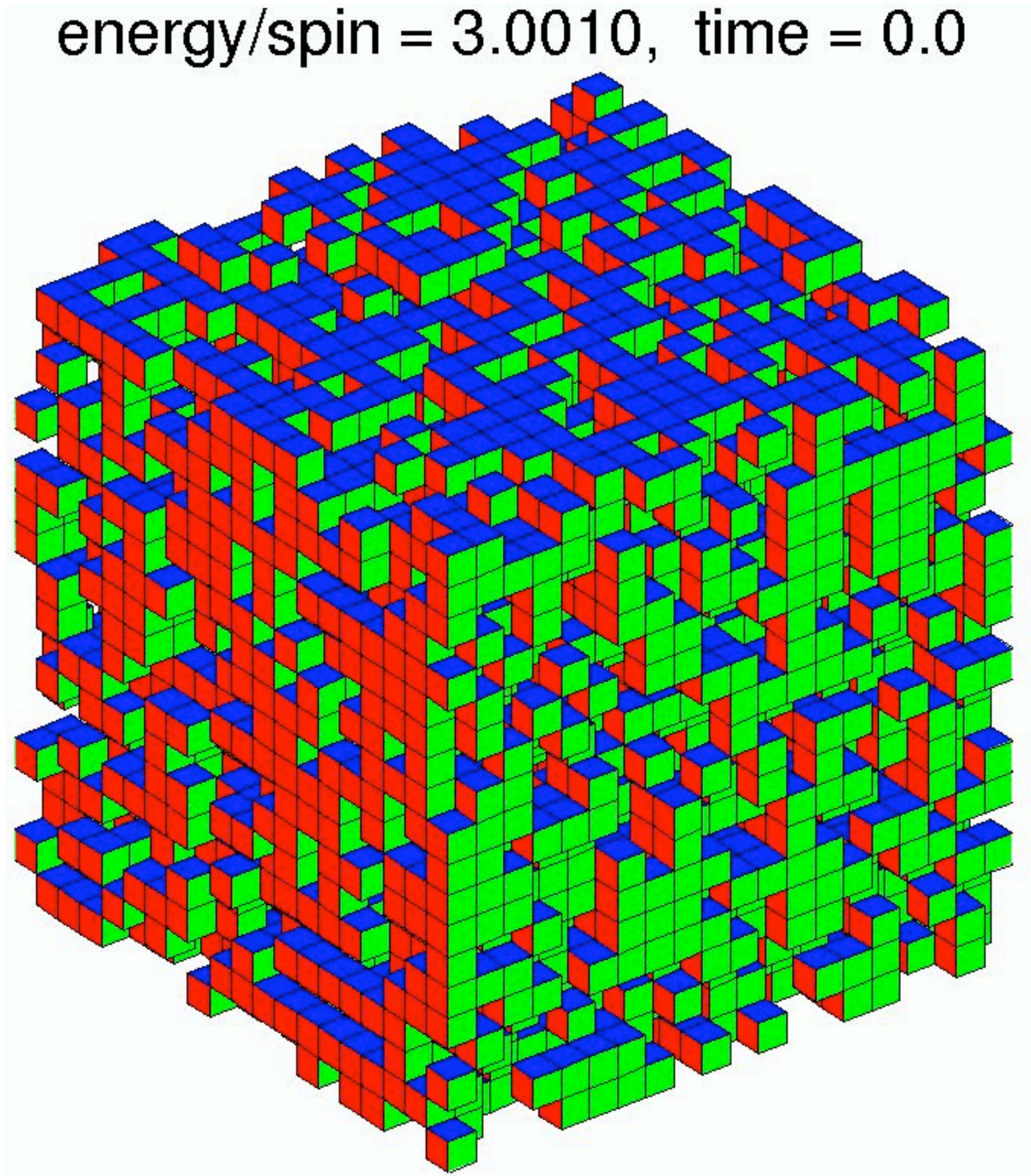

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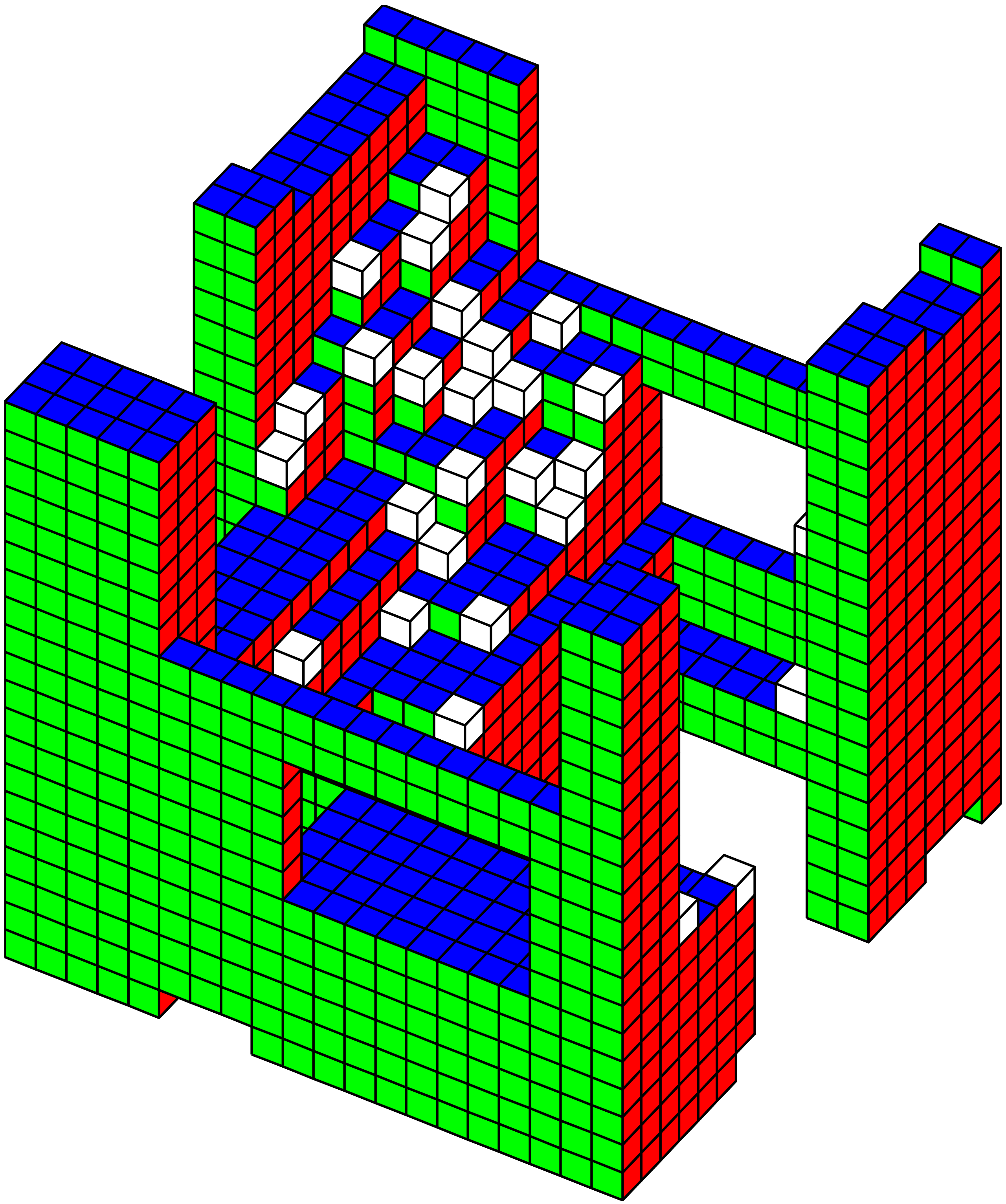

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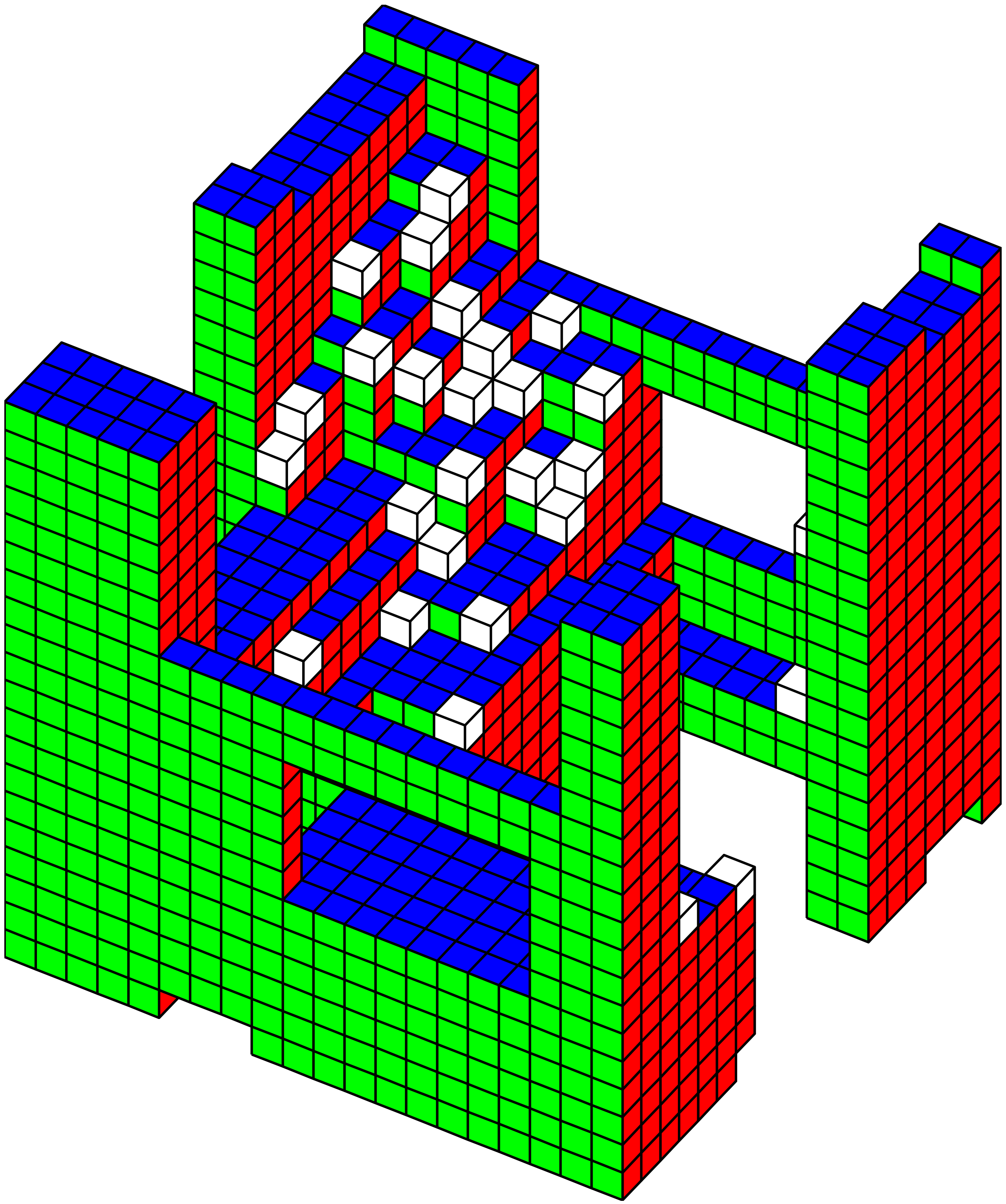

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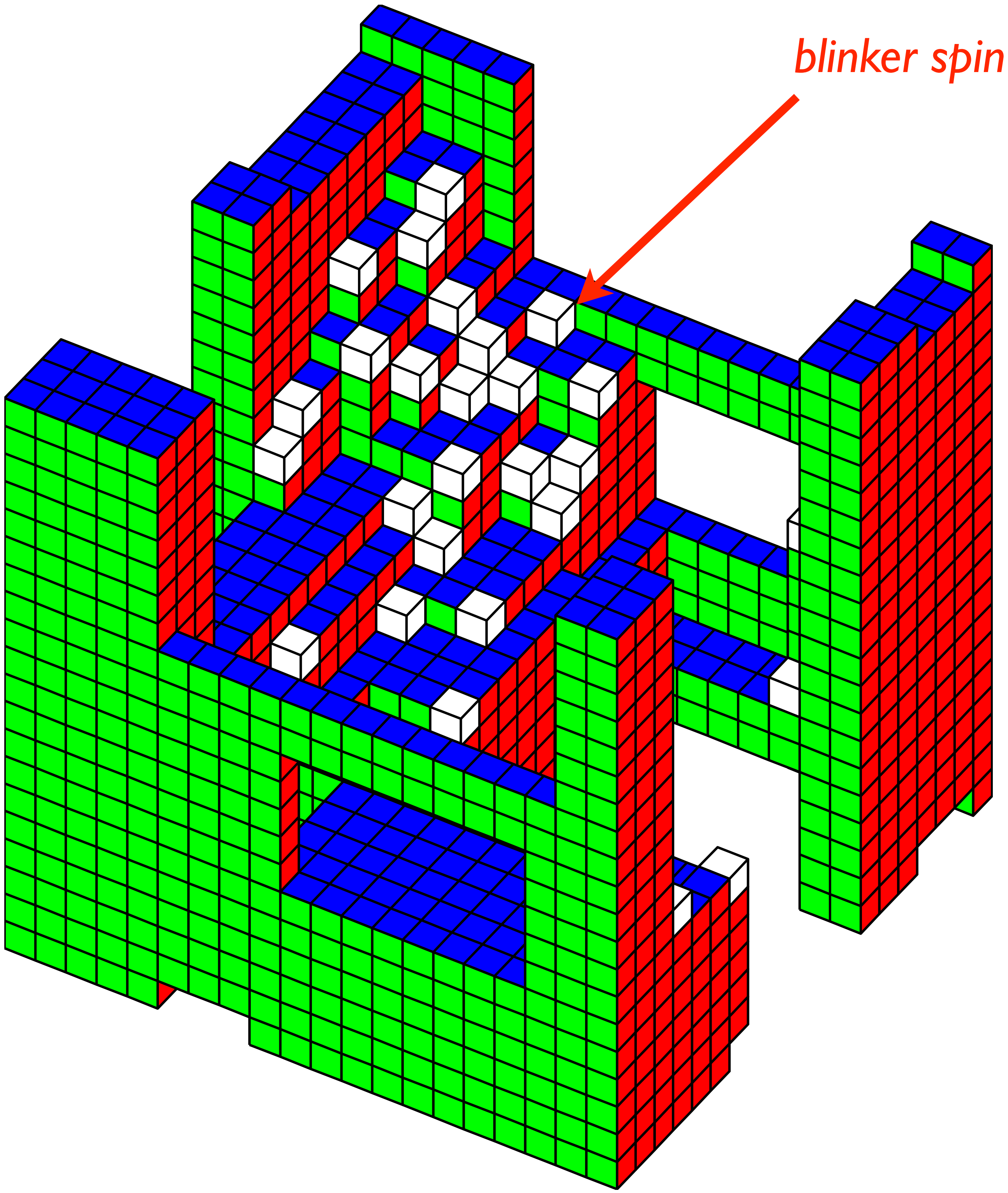

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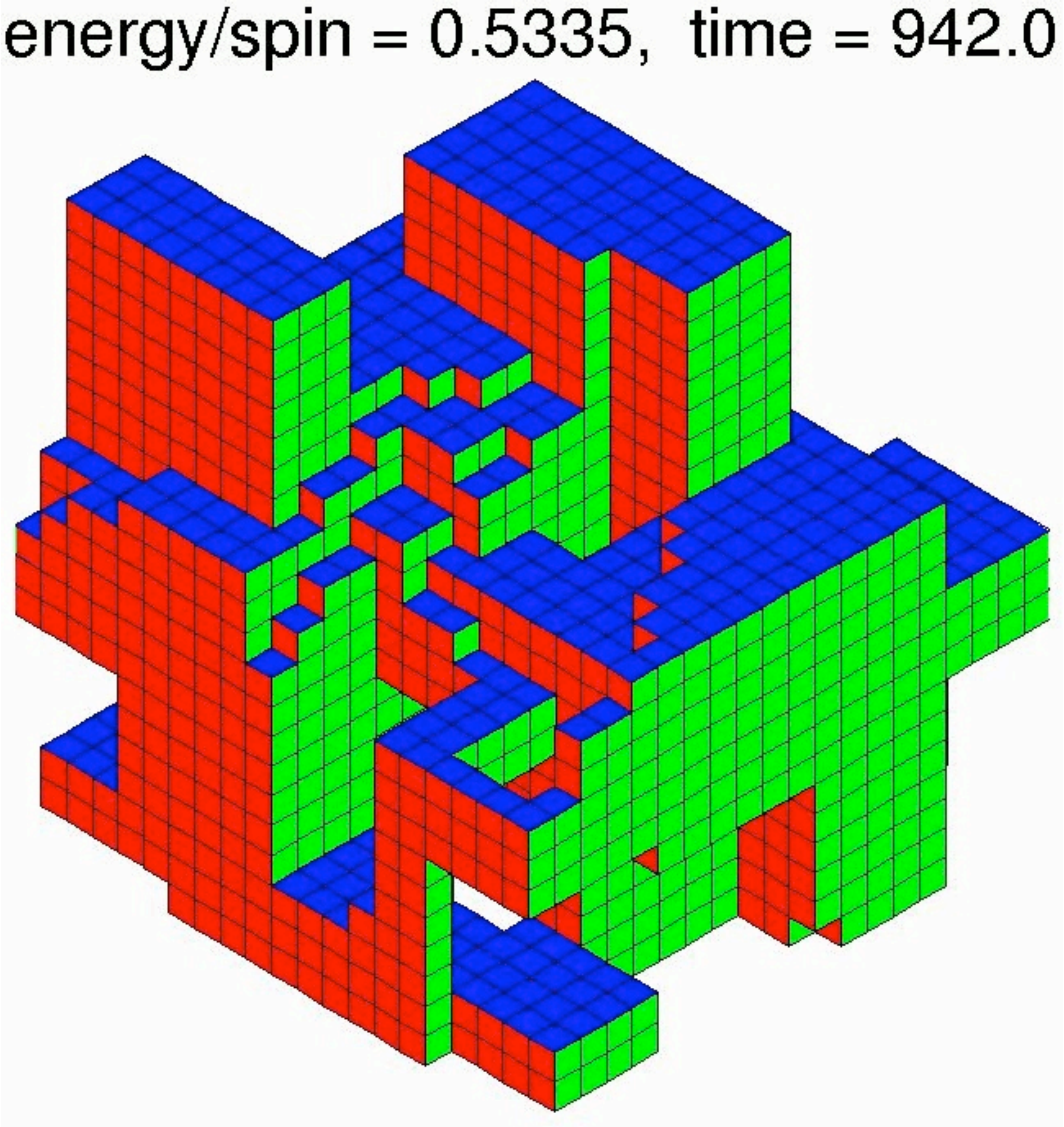
Blinker Evolution in Three Dimensions



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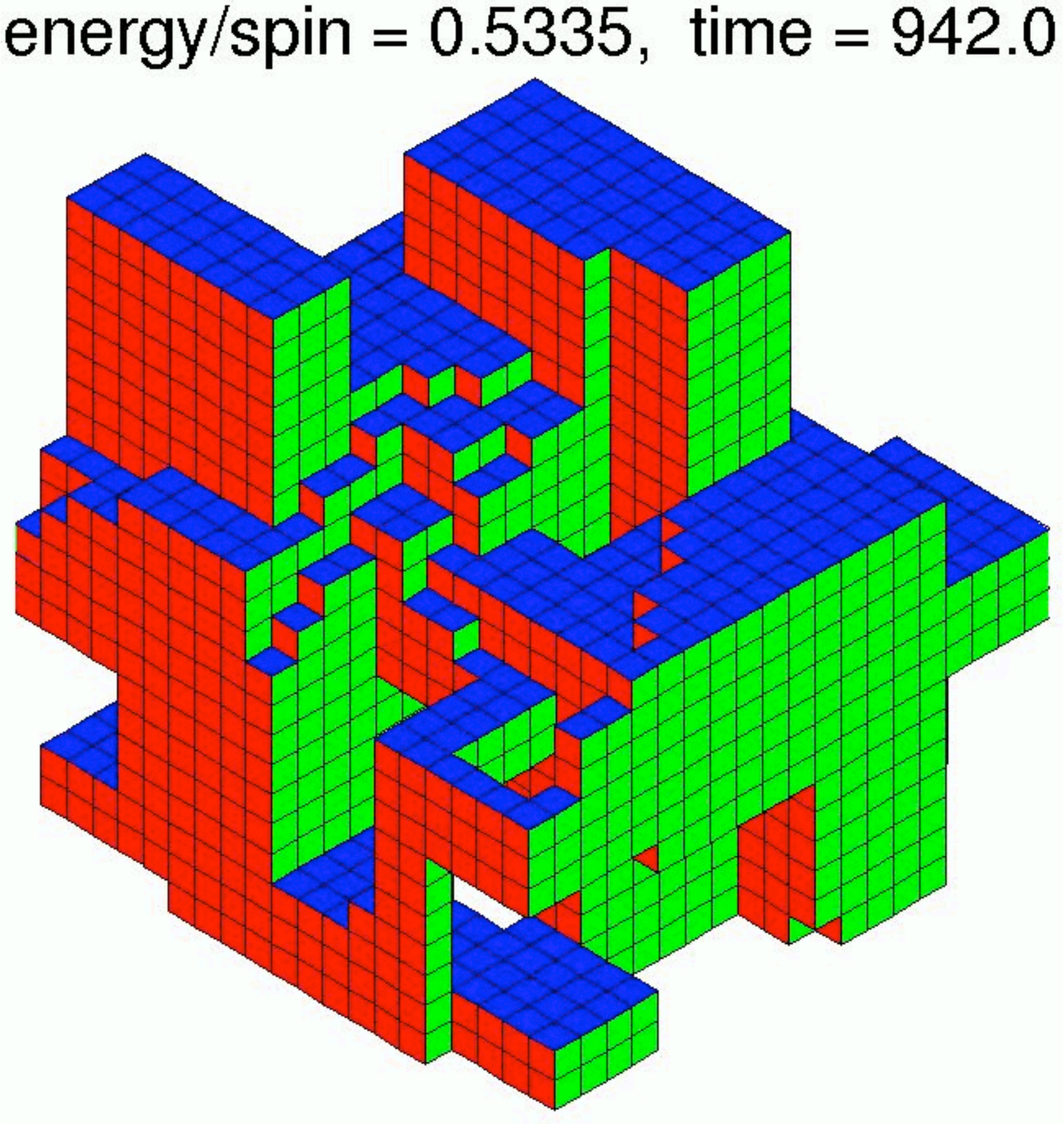
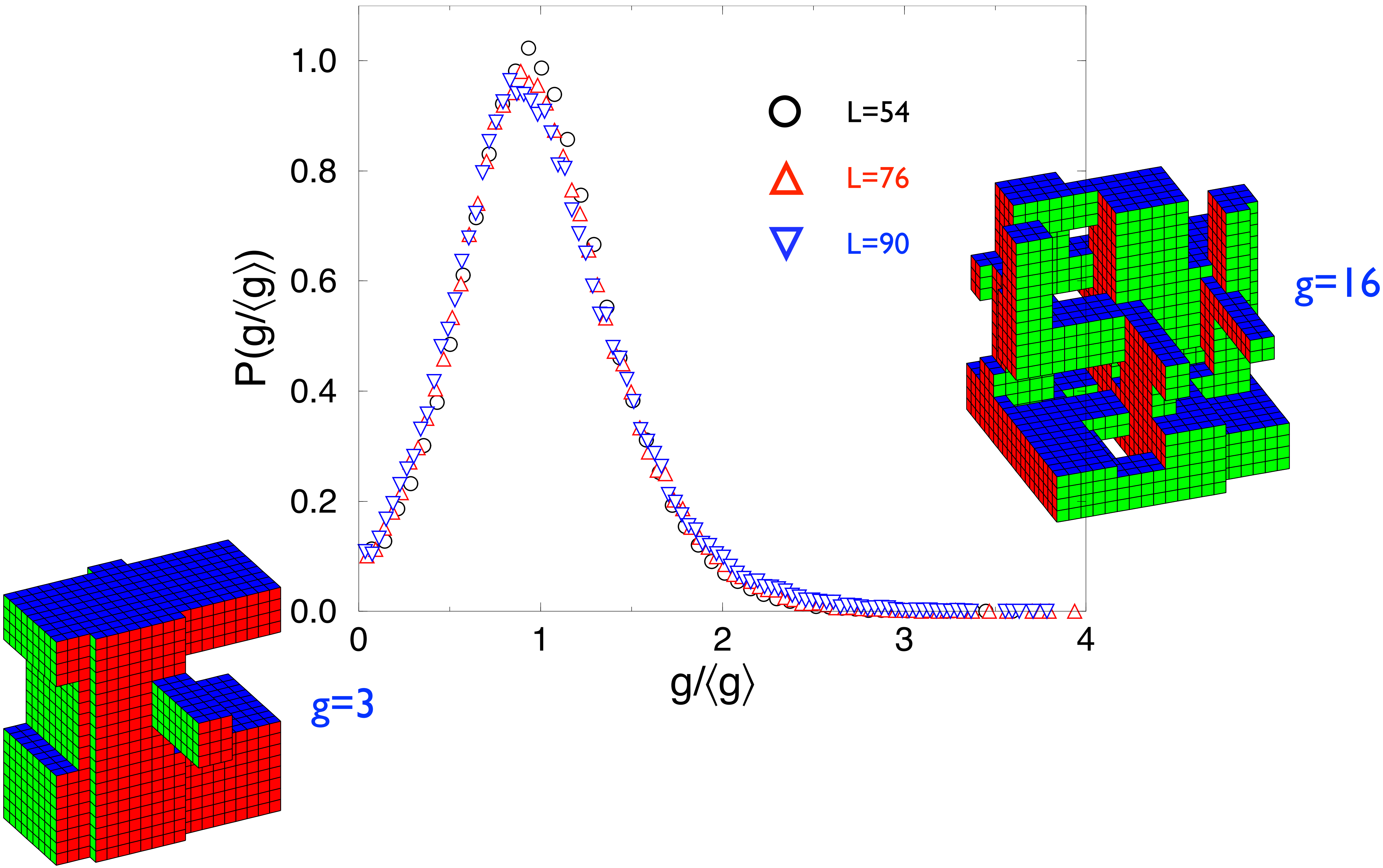


Genus Distribution



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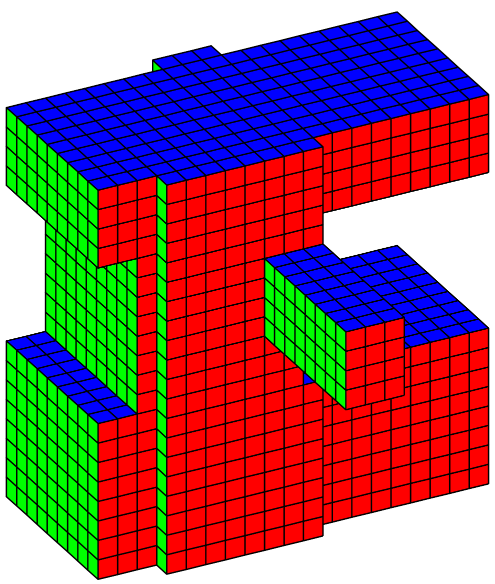
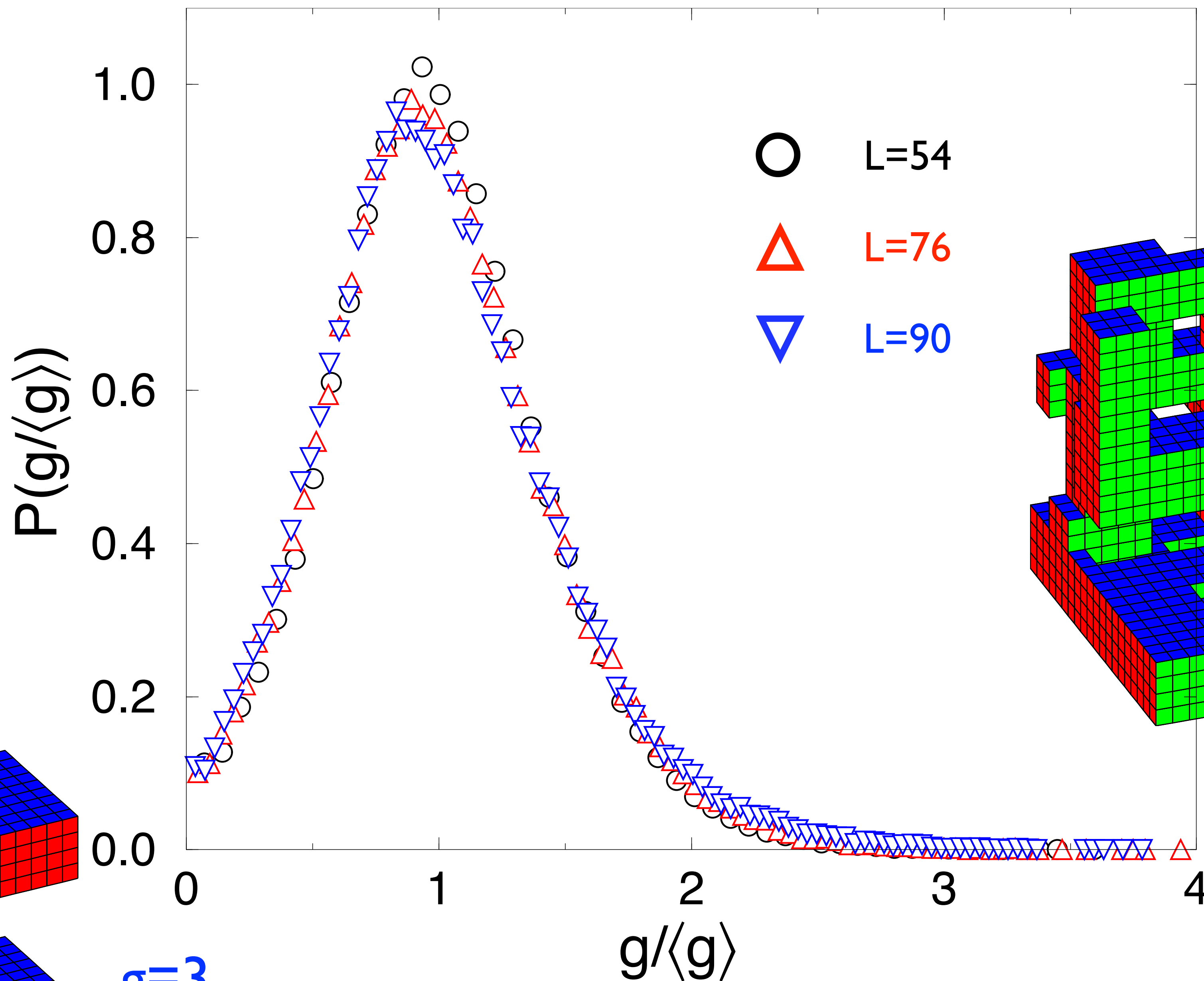
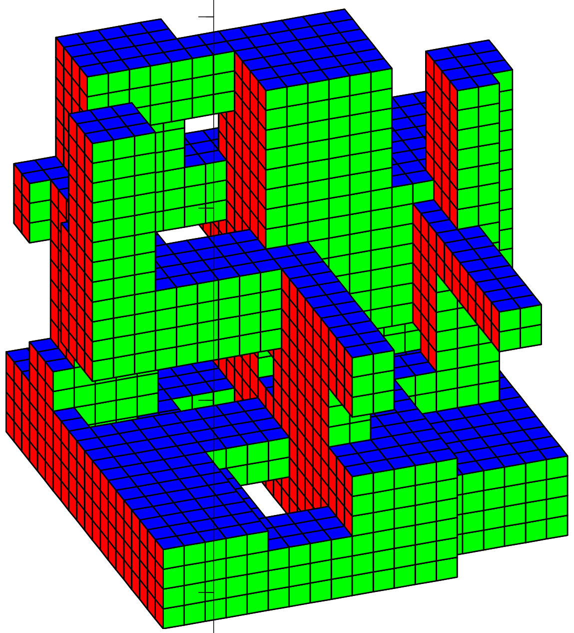
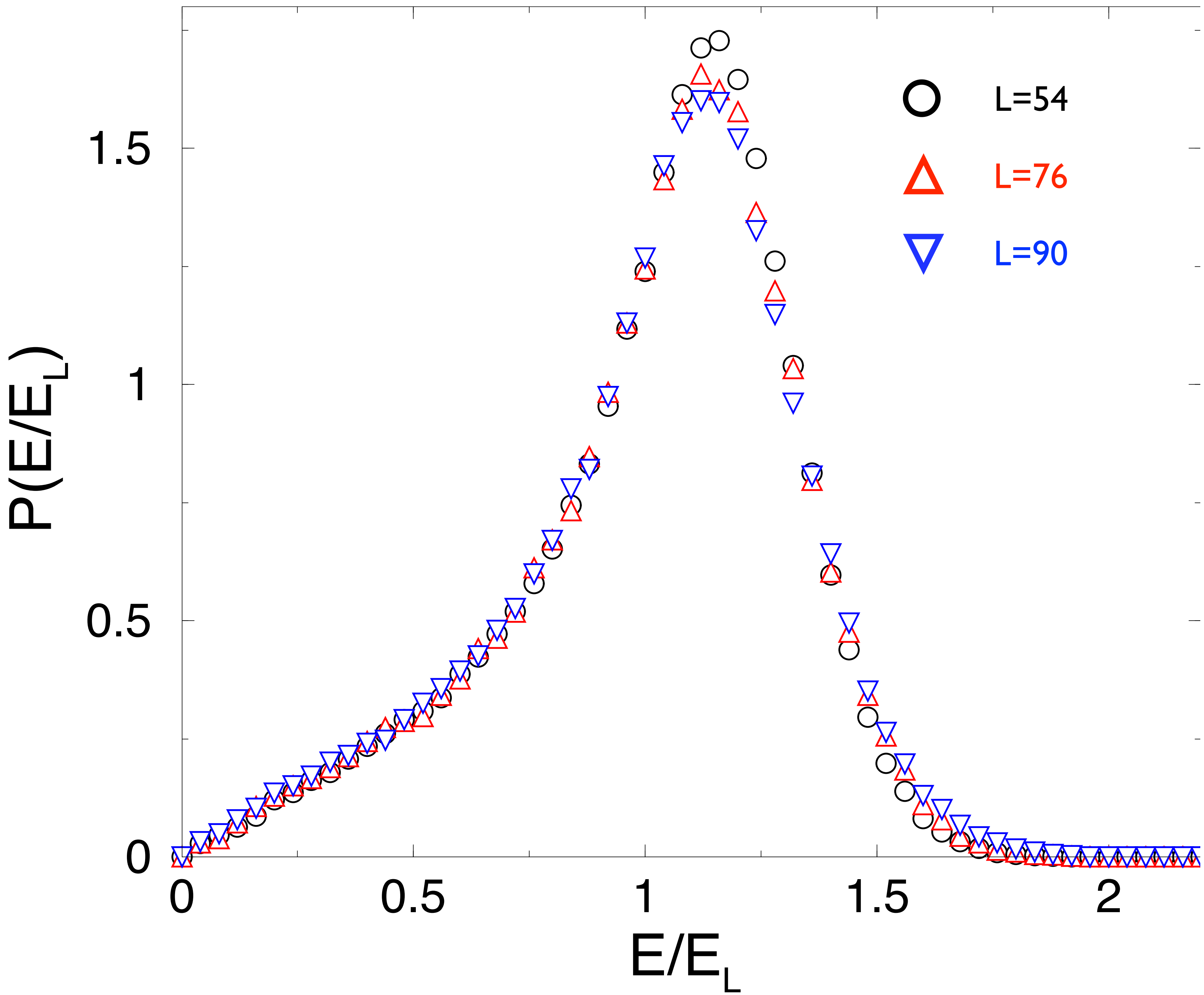






$$\langle g \rangle \sim L^{1.7}$$

Energy Distribution



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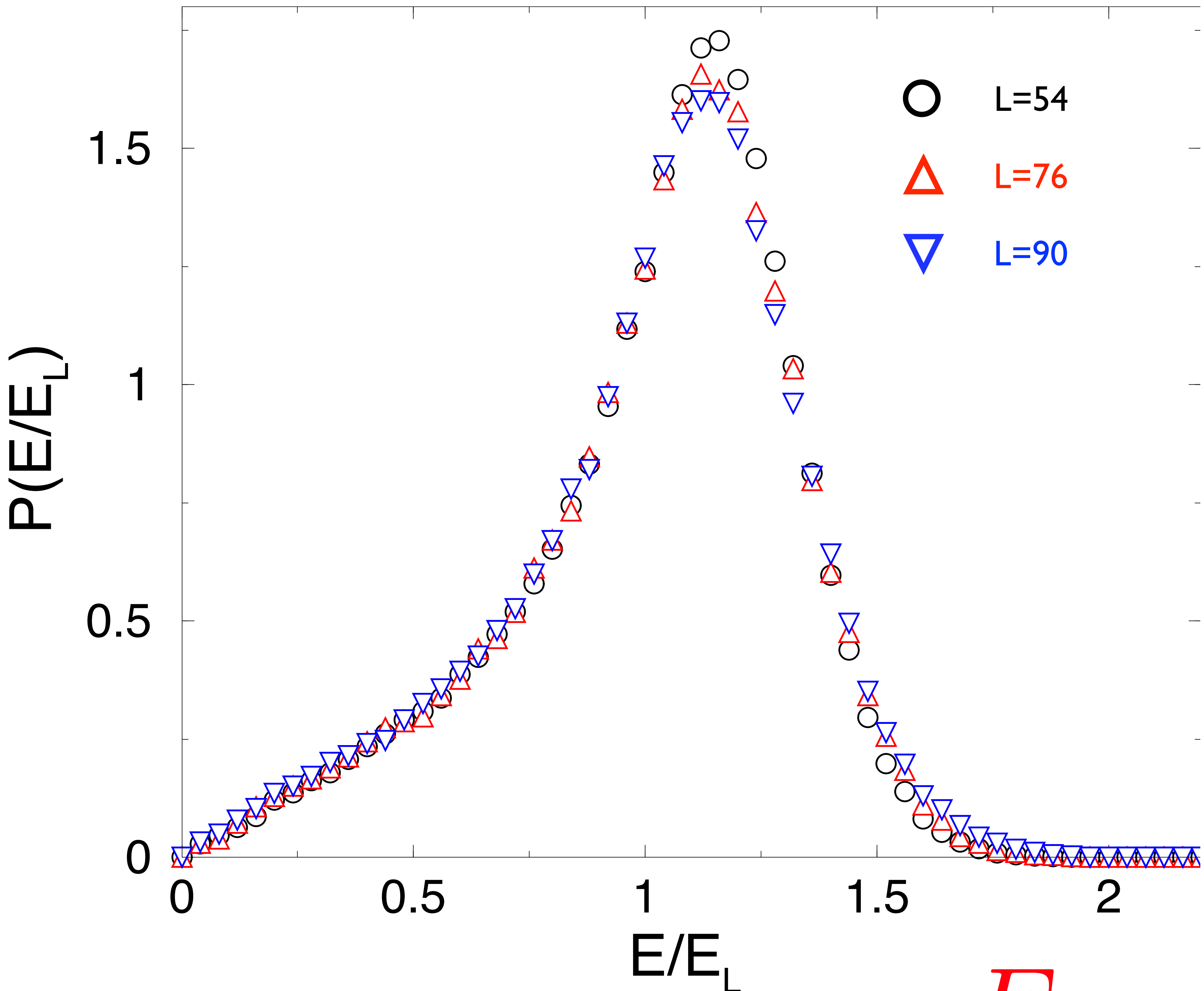

$$E_L \sim L^{-1}$$

Relate Genus and Energy by Topology

$$\chi = 2(1-g) = \mathcal{V} - \mathcal{E} + \mathcal{F}$$

Euler characteristic — genus — vertices — edges — faces

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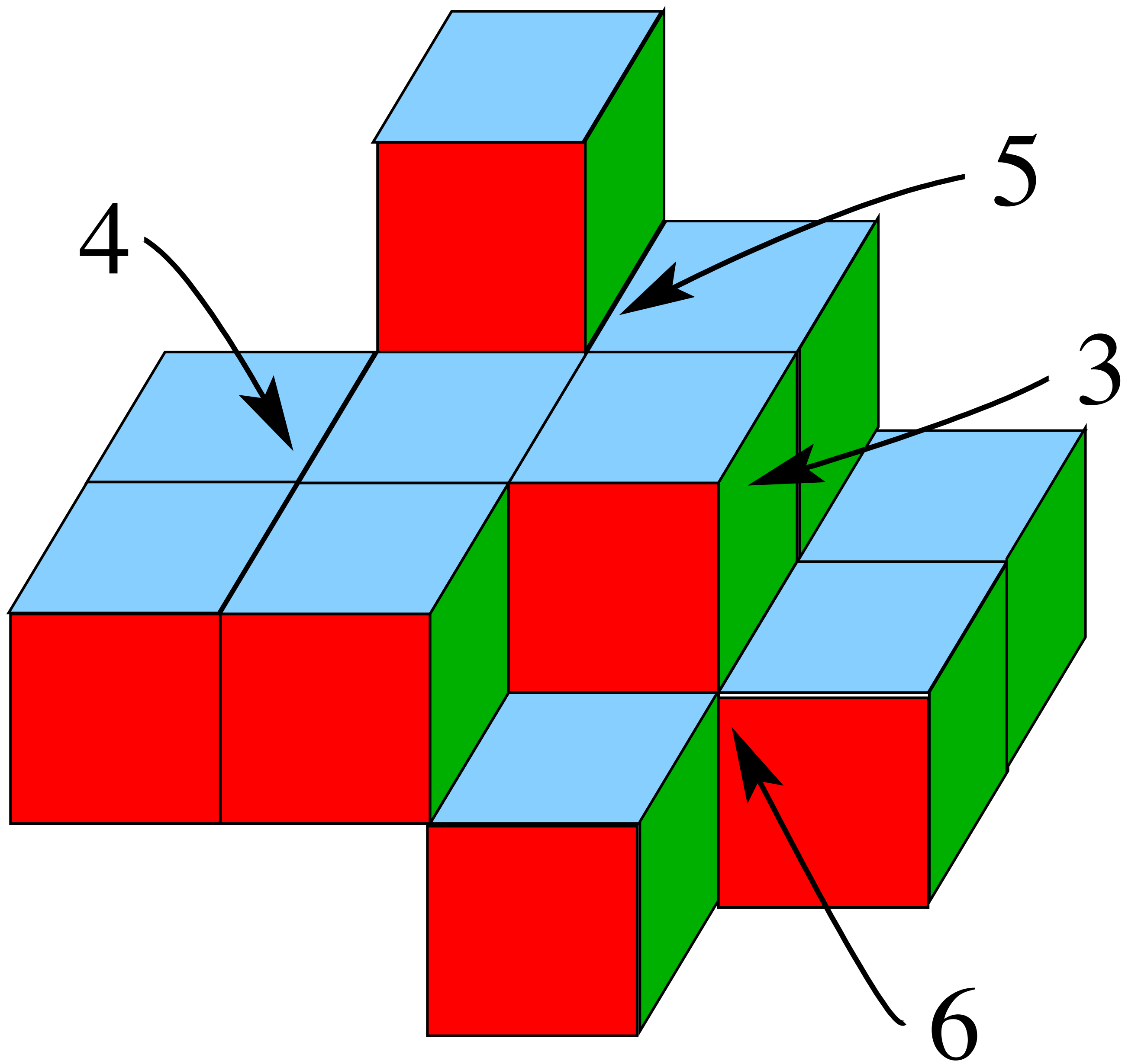

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$$\longrightarrow 0 \leq g \leq 1 + \frac{\mathcal{F}}{6}$$

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Euler characteristic   genus   vertices   edges   faces

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$$\to \epsilon + \gamma \leq 3$$

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Euler characteristic genus vertices edges faces

constraints between $\mathcal{V}$, $\mathcal{E}$, $\mathcal{F}$: $\mathcal{E} = 2\mathcal{F}$ $\frac{\mathcal{E}}{3} \leq \mathcal{V} \leq \frac{2\mathcal{E}}{3}$

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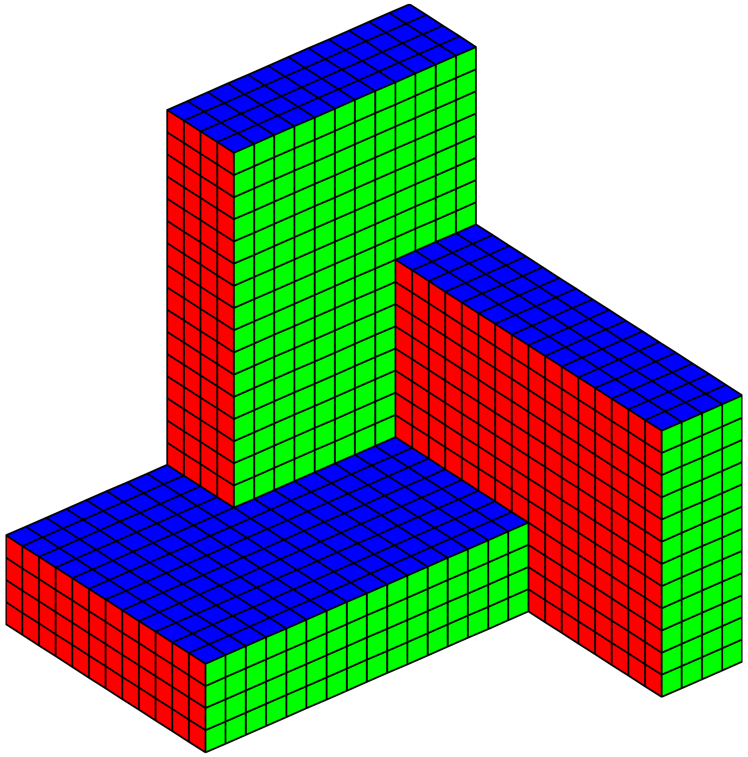
1 1.7 simulations

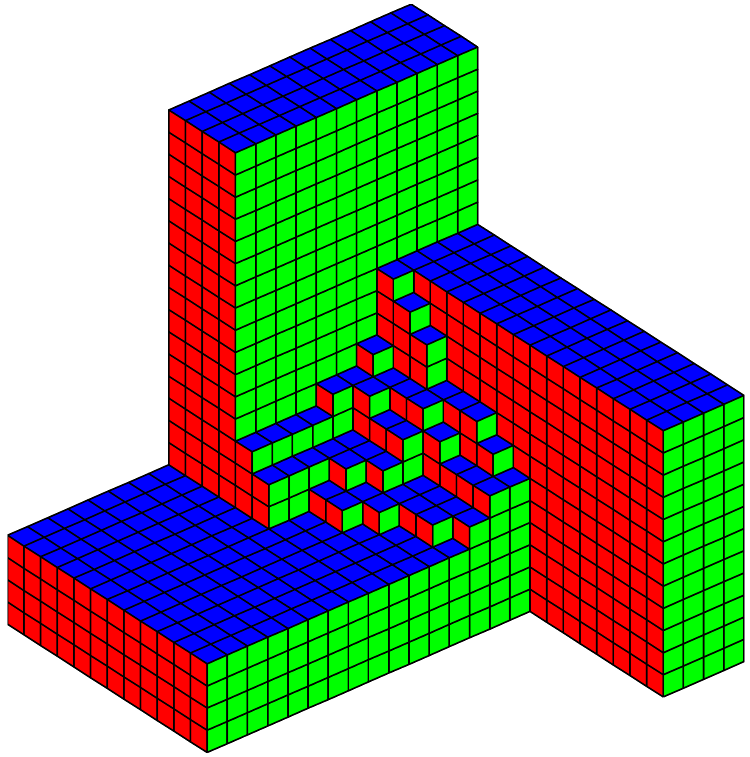
Slow Relaxation of Blinkers

Slow Relaxation of Blinkers

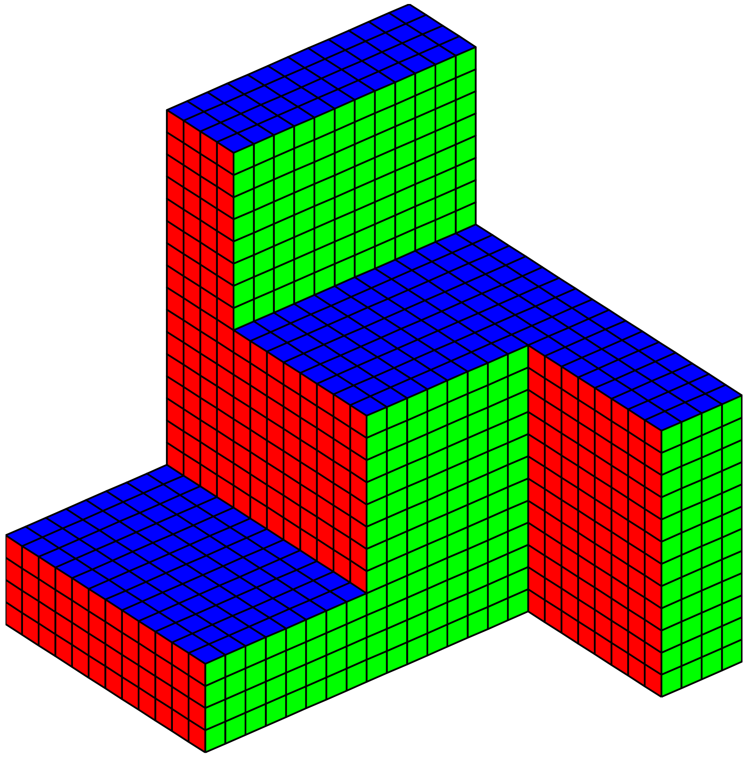
synthetic blinker configuration



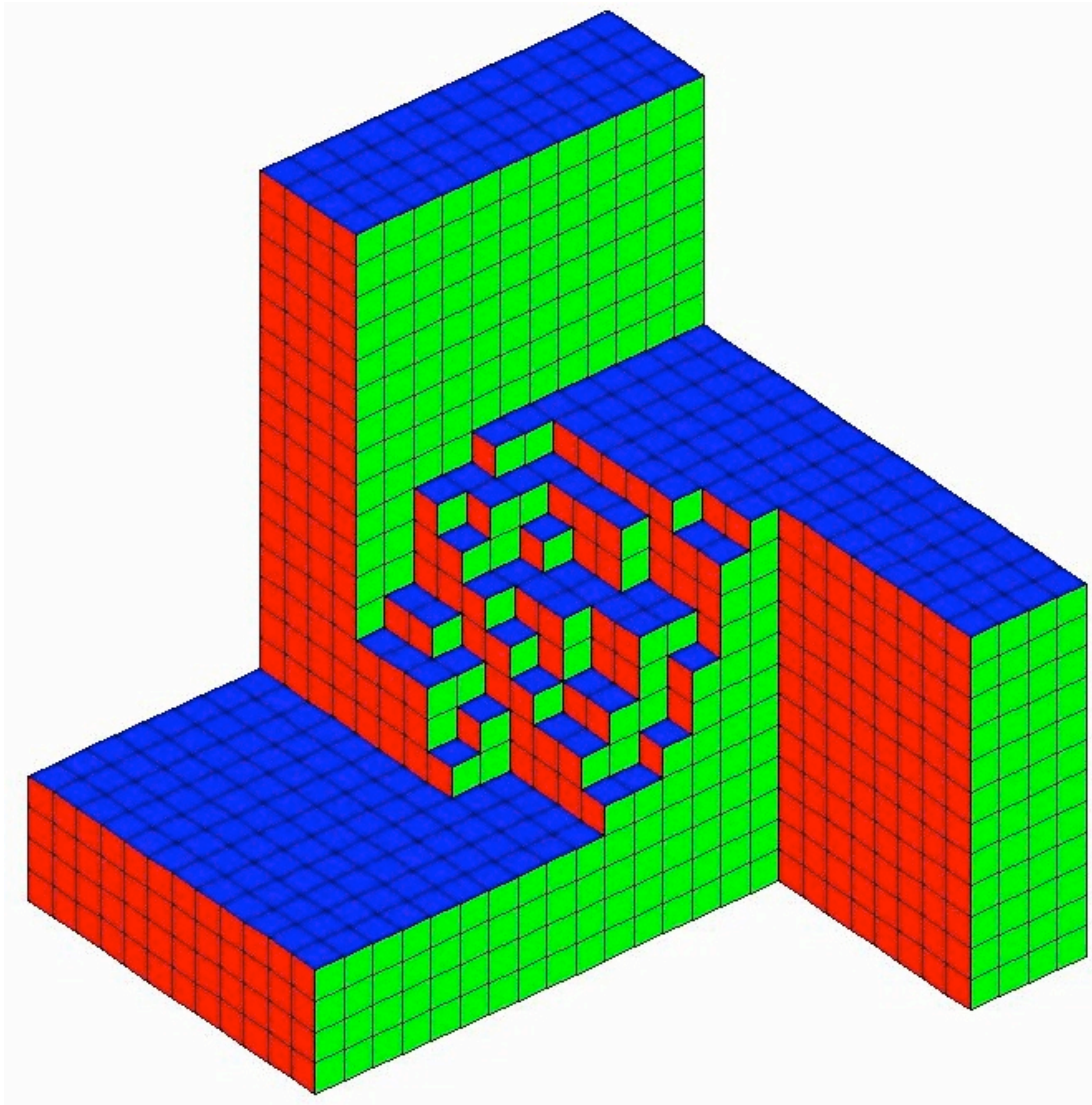
deflated

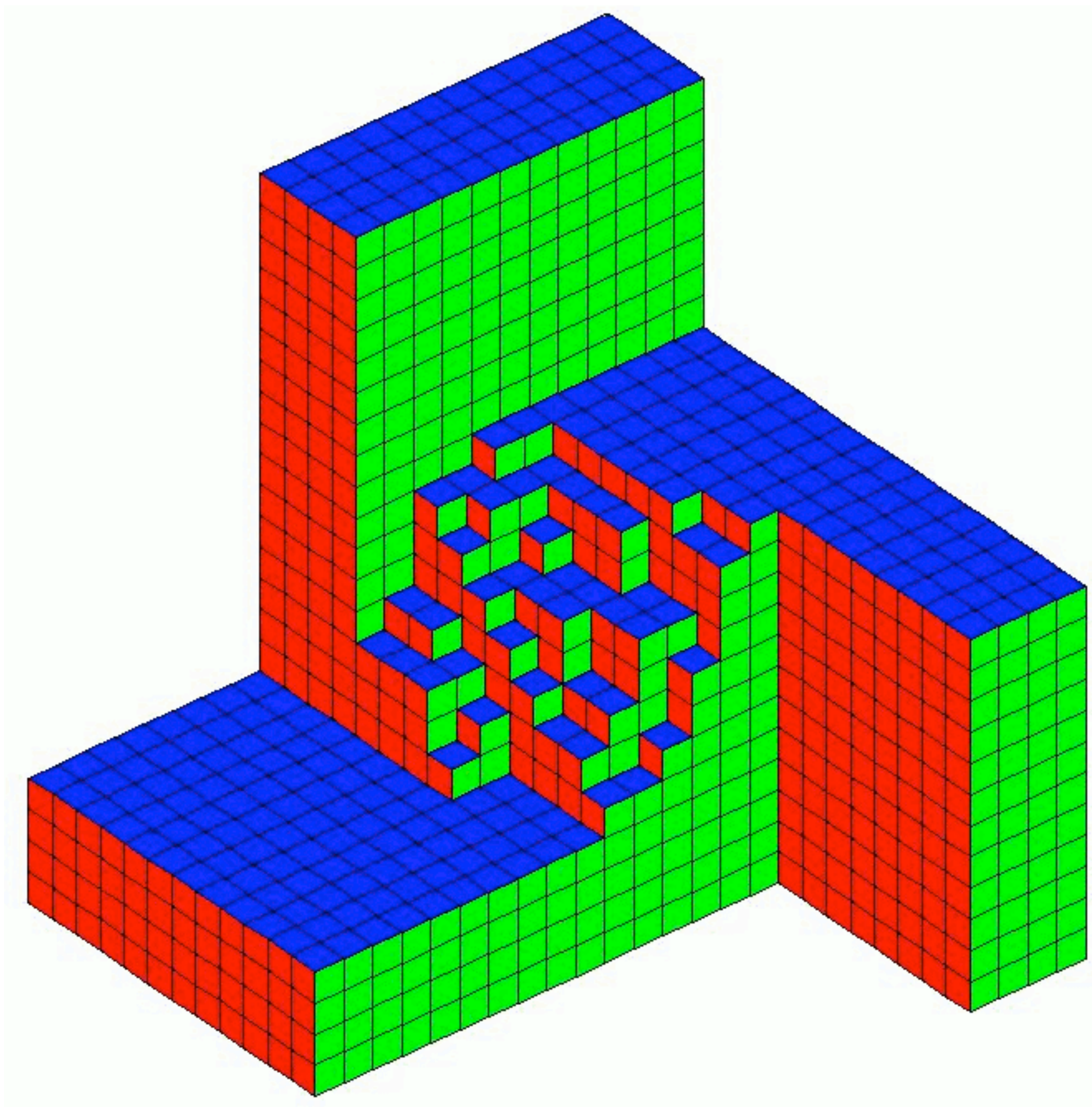


intermediate

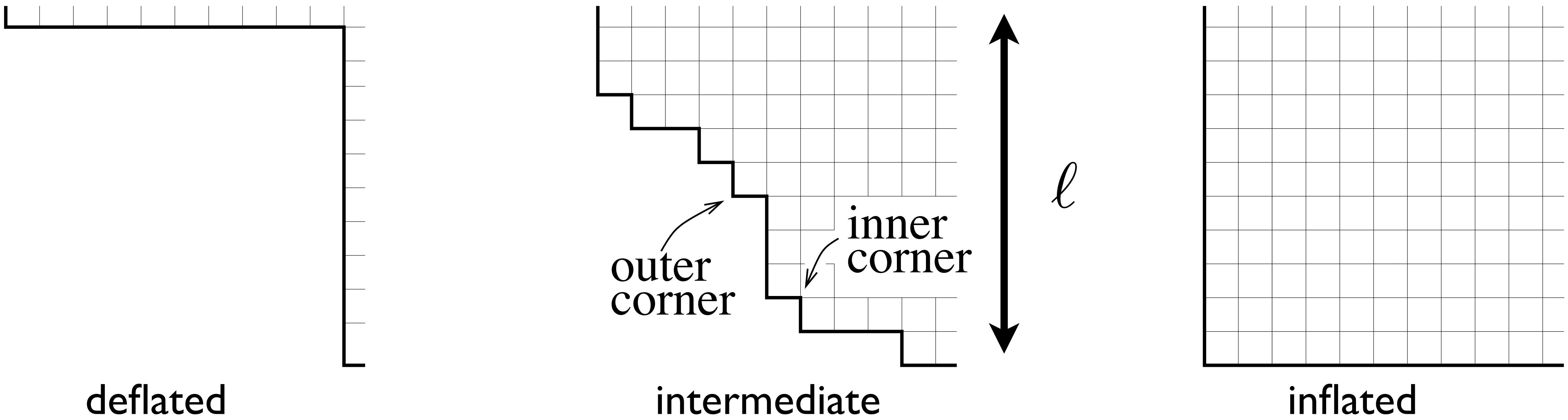


inflated





2d analog:

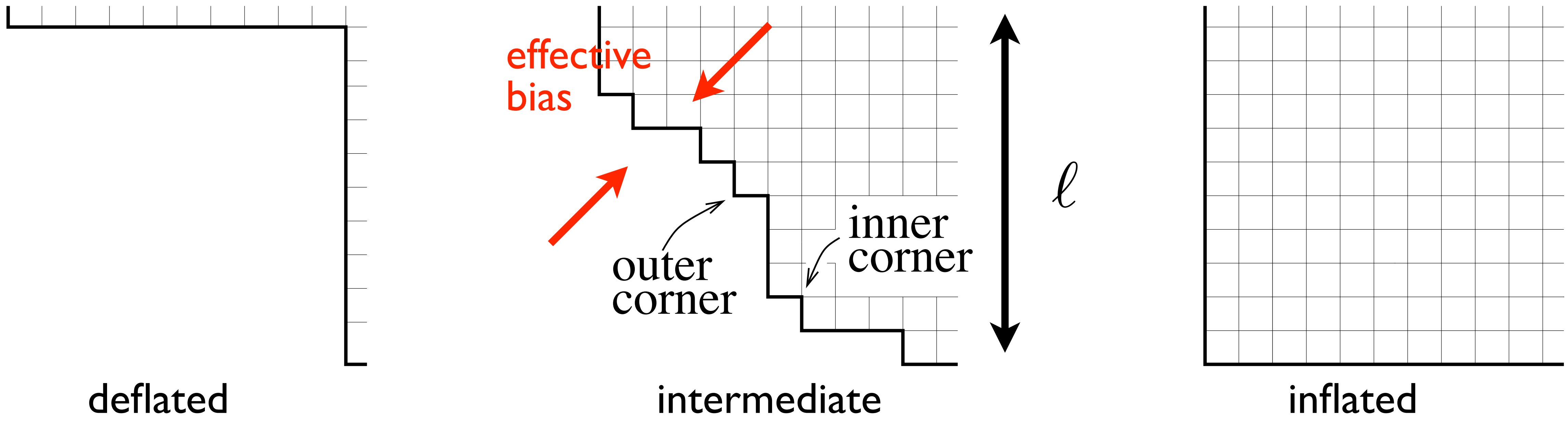


N_+ outer corners

N_- inner corners

$N_+ - N_- = 1$

2d analog:



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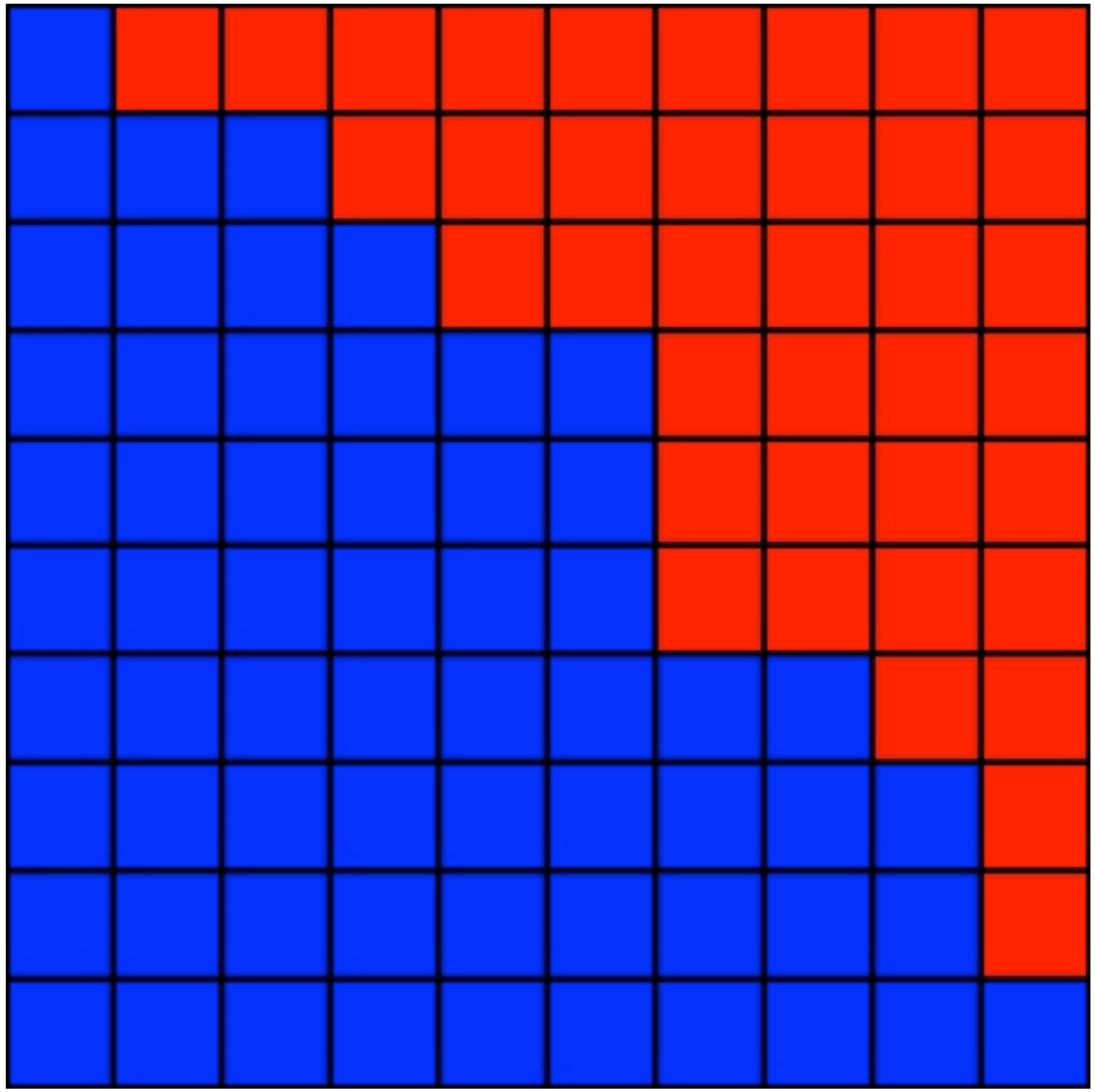

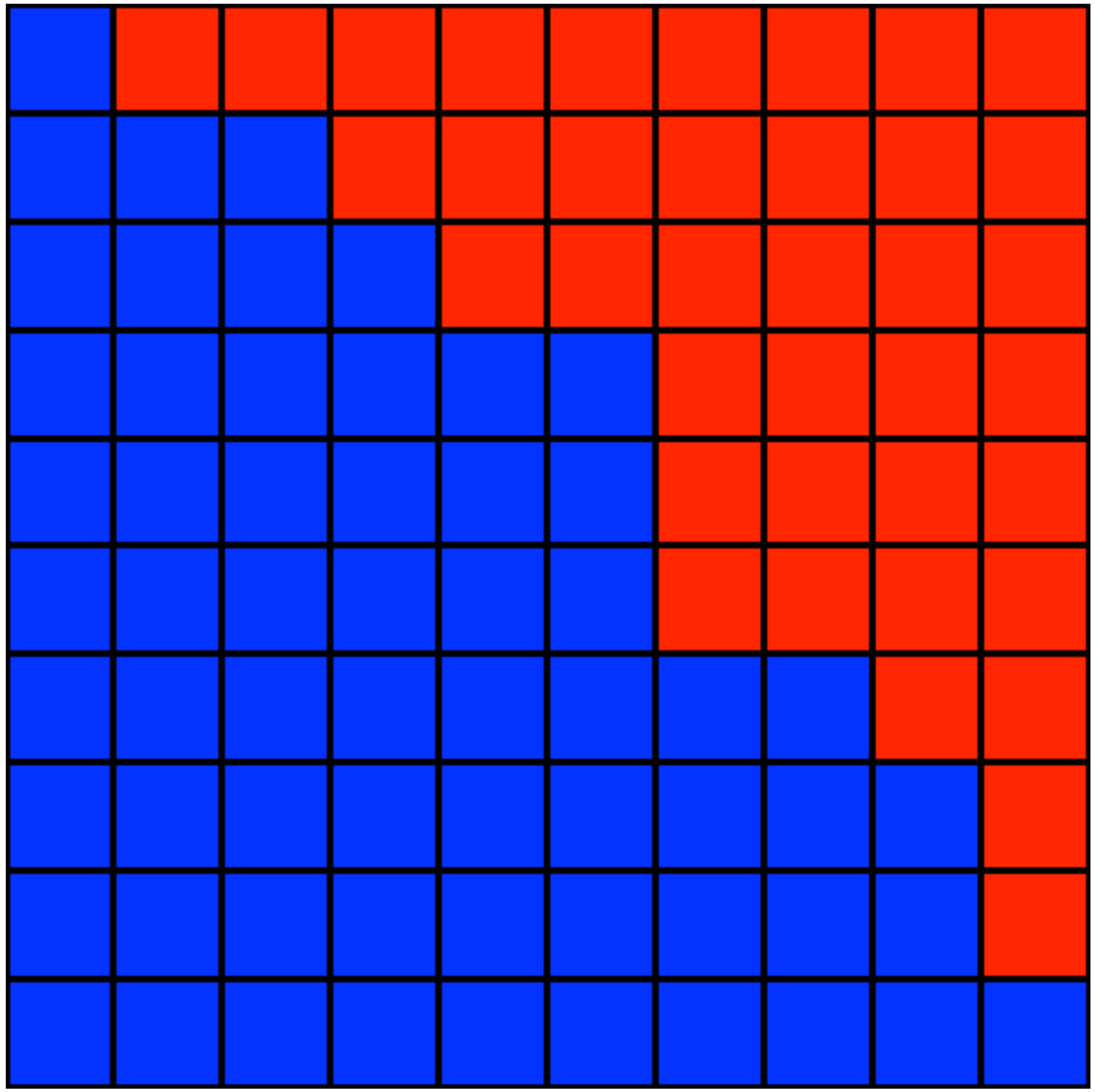

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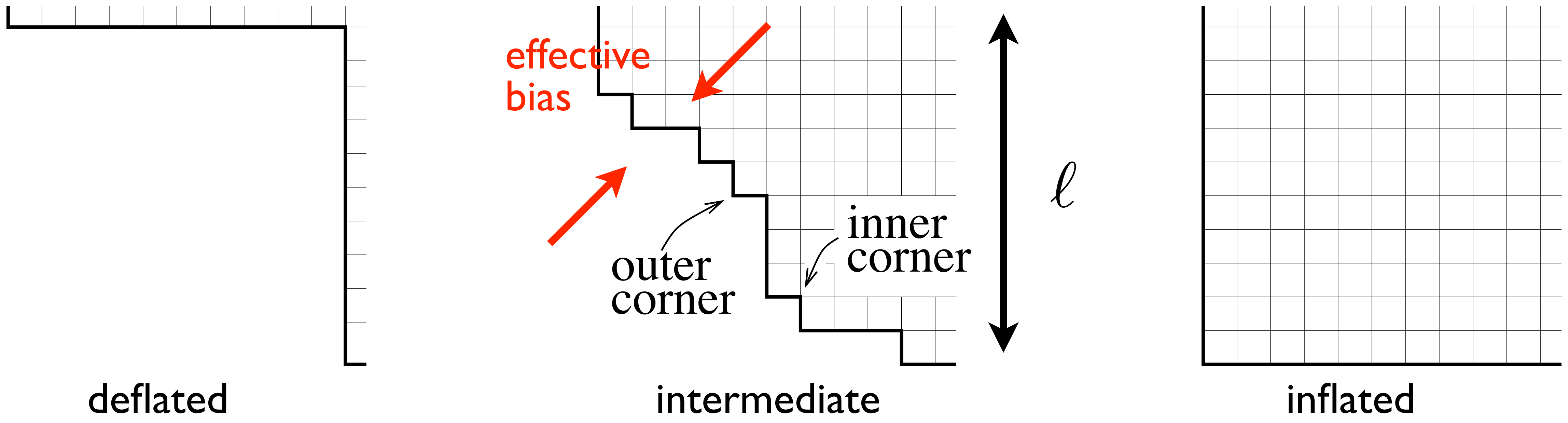

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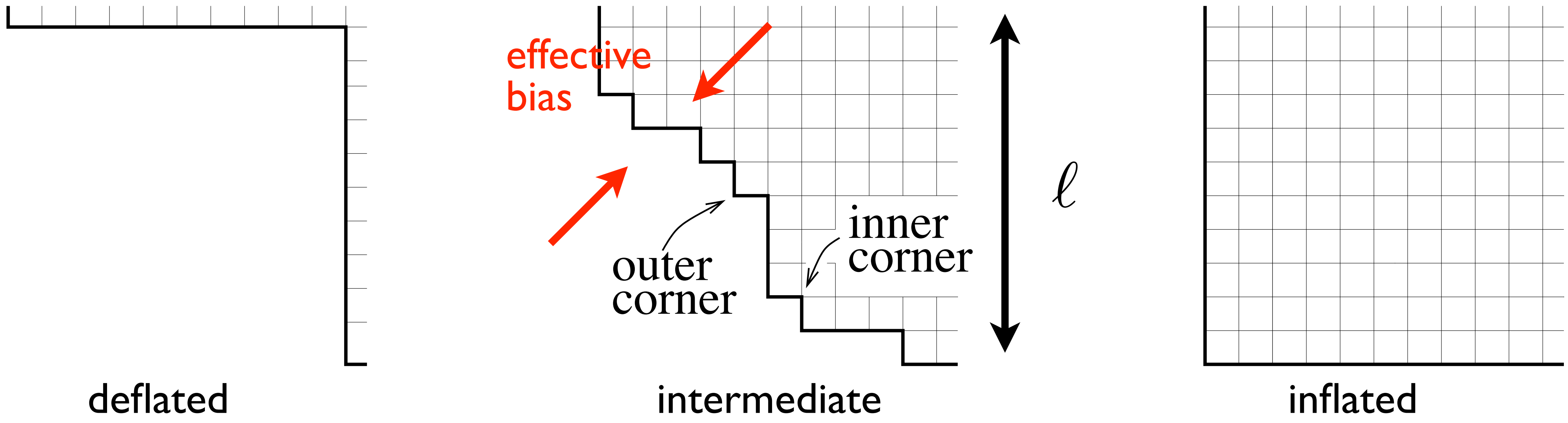


N_+ outer corners

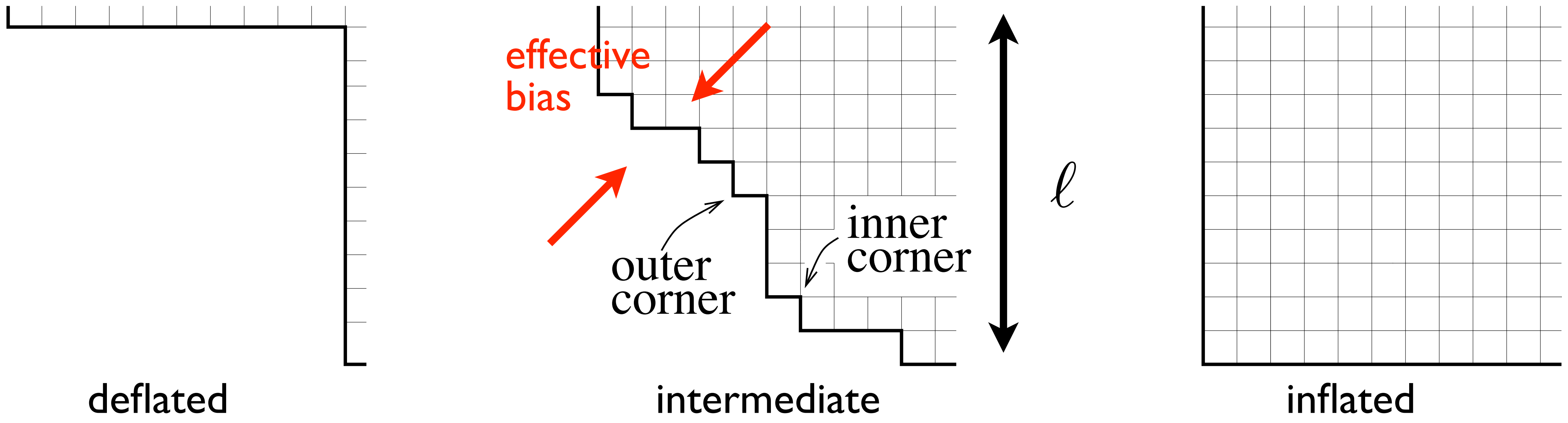
N_- inner corners

$N_+ - N_- = 1$

$$u = \frac{\Delta A}{\Delta t} = -1$$

$$D = \frac{(\Delta A)^2}{\Delta t} \sim N_\pm \sim \ell$$

2d analog:



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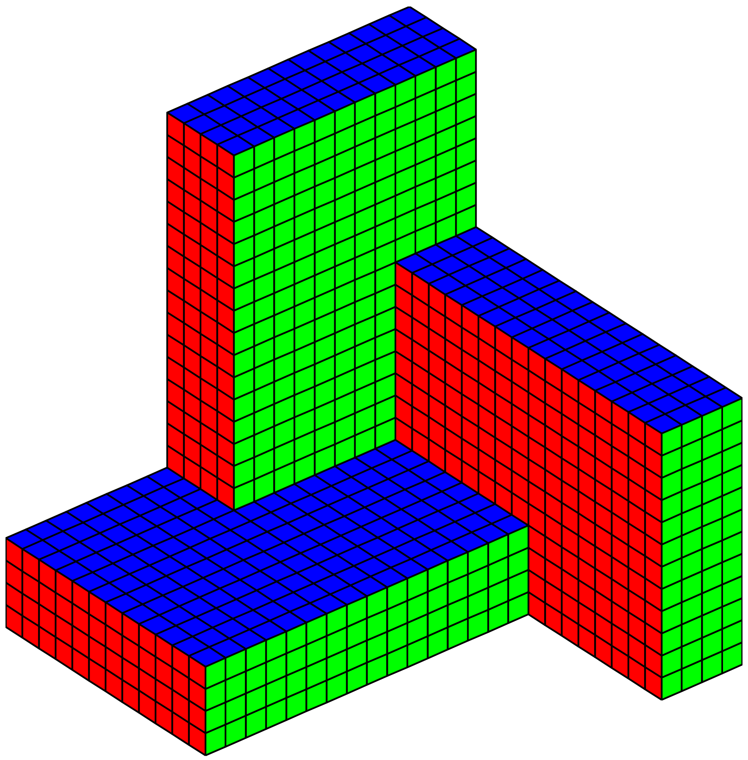
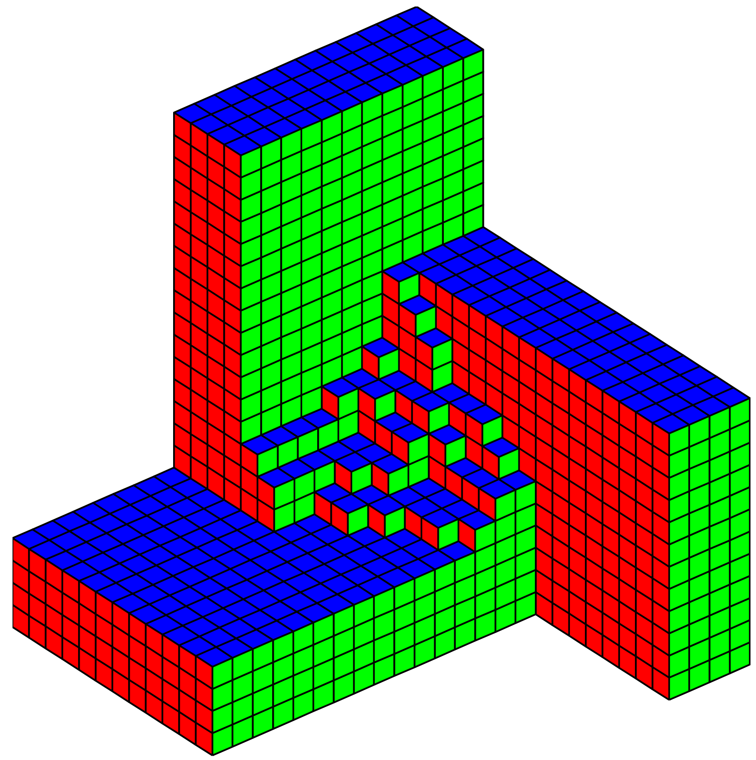
$$\tau \sim e^{|u|\ell^2/D} \sim e^{\ell}$$

$$D = \frac{(\Delta A)^2}{\Delta t} \sim N_\pm \sim \ell$$

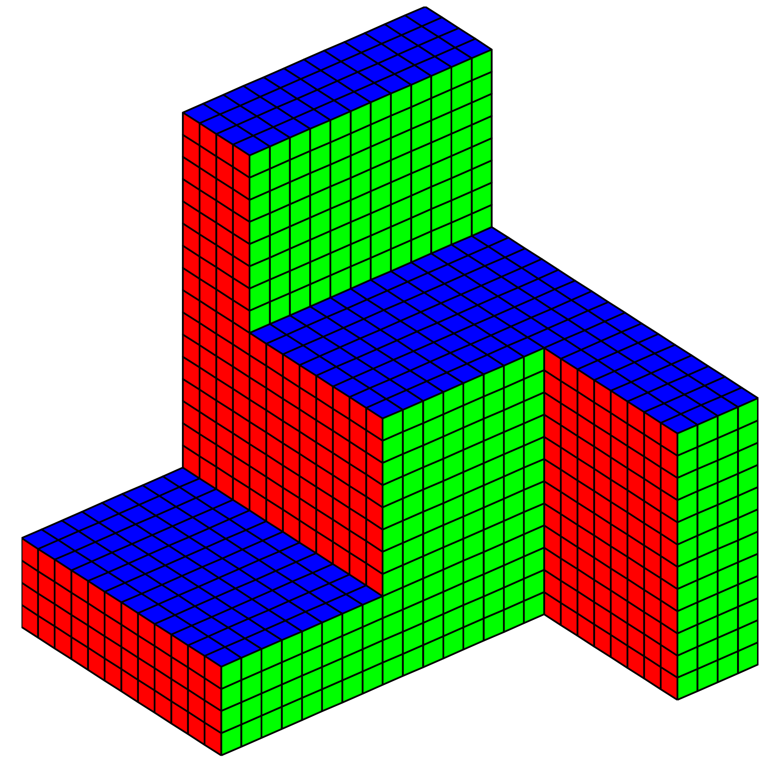
Slow Blinker Relaxation in 3d



deflated



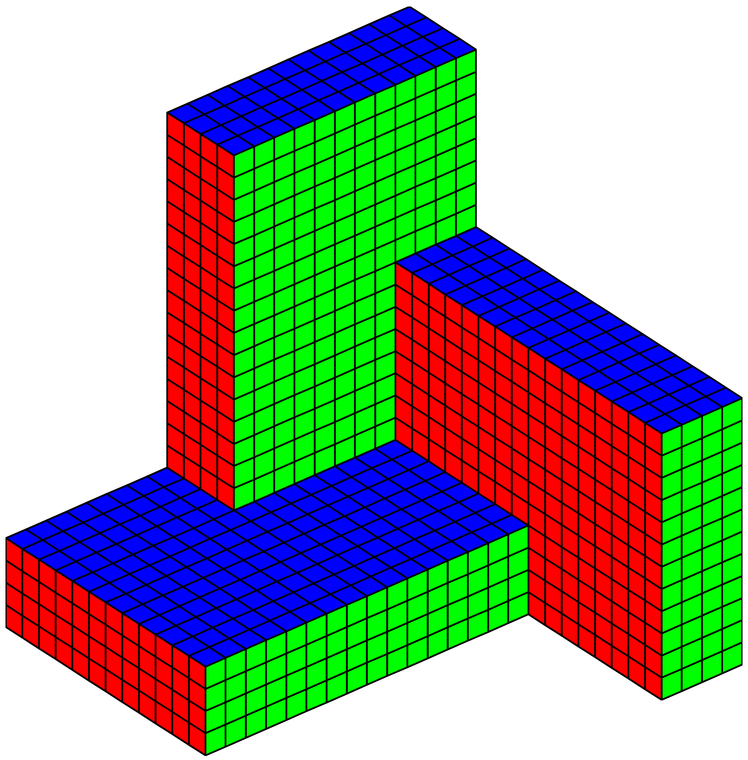
intermediate



inflated

$$N_{\pm} \sim \ell^2$$
$$N_+ - N_- \sim \ell$$

Slow Blinker Relaxation in 3d



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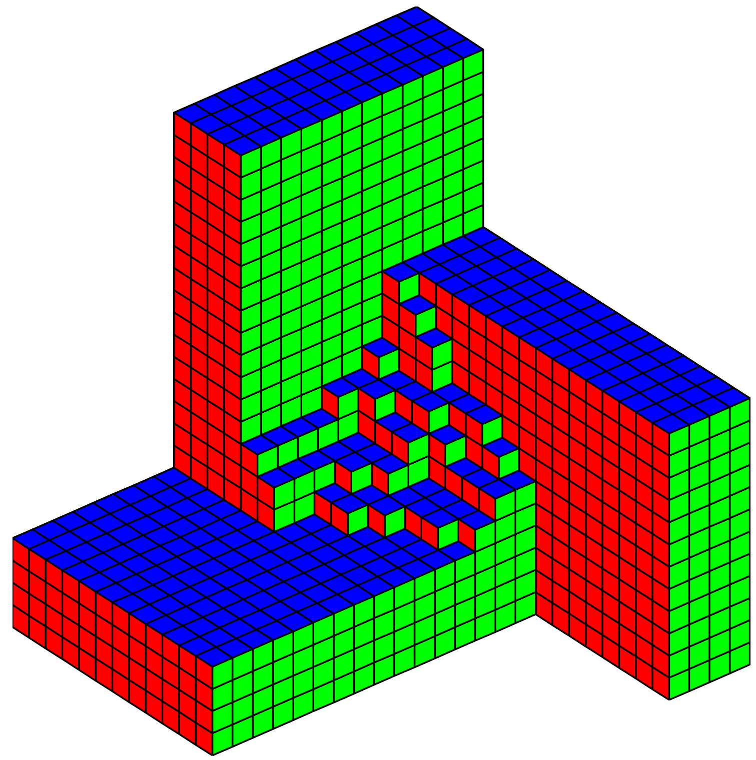

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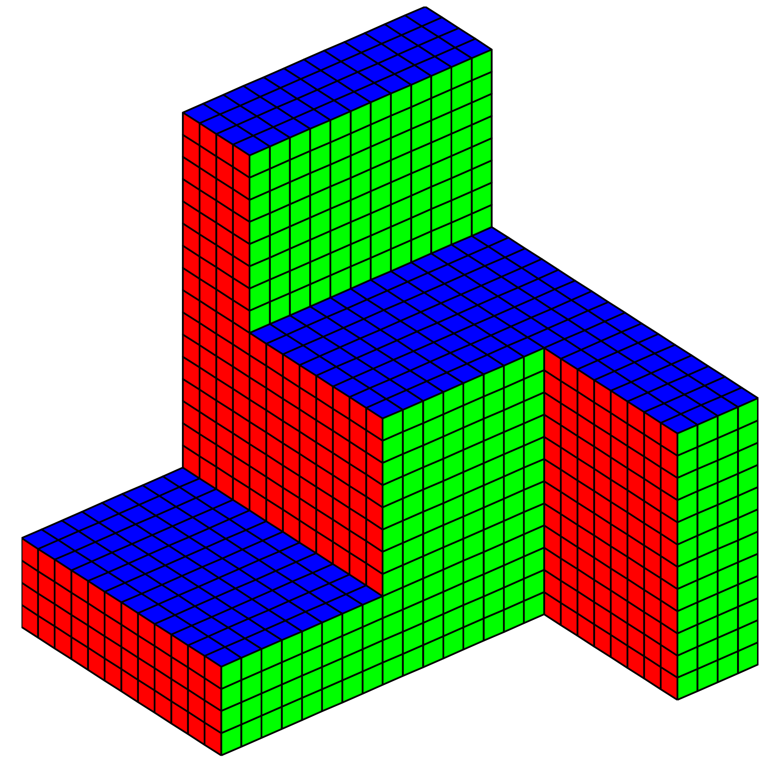

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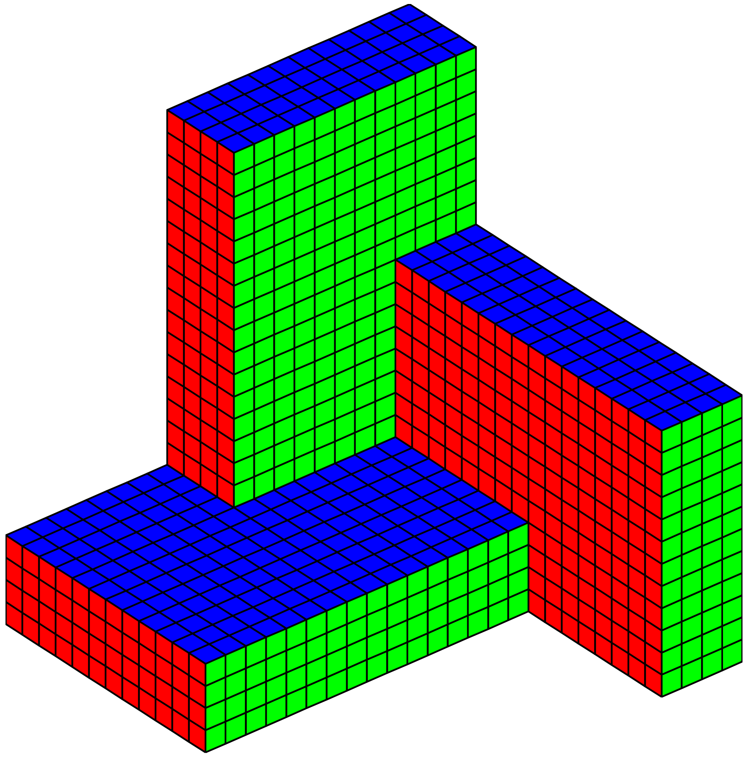

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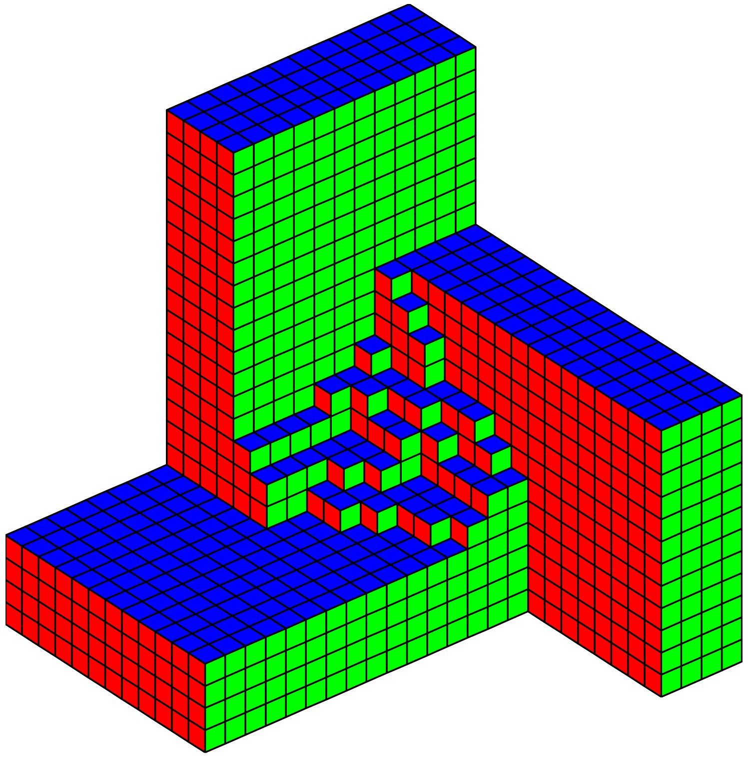

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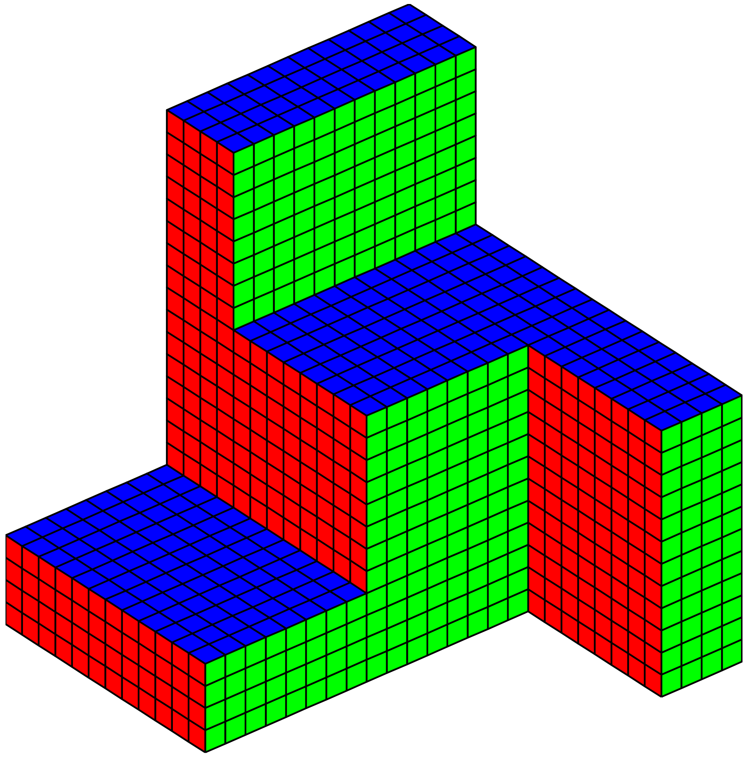

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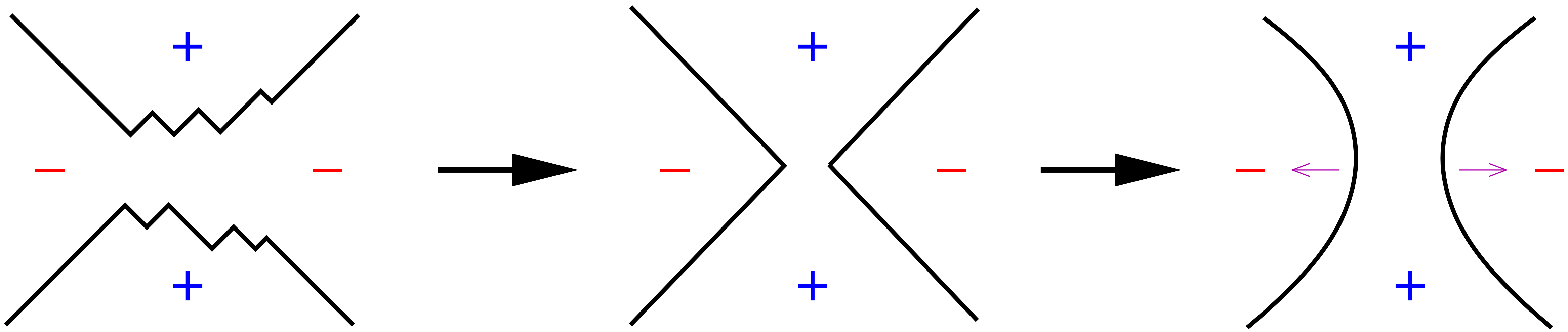
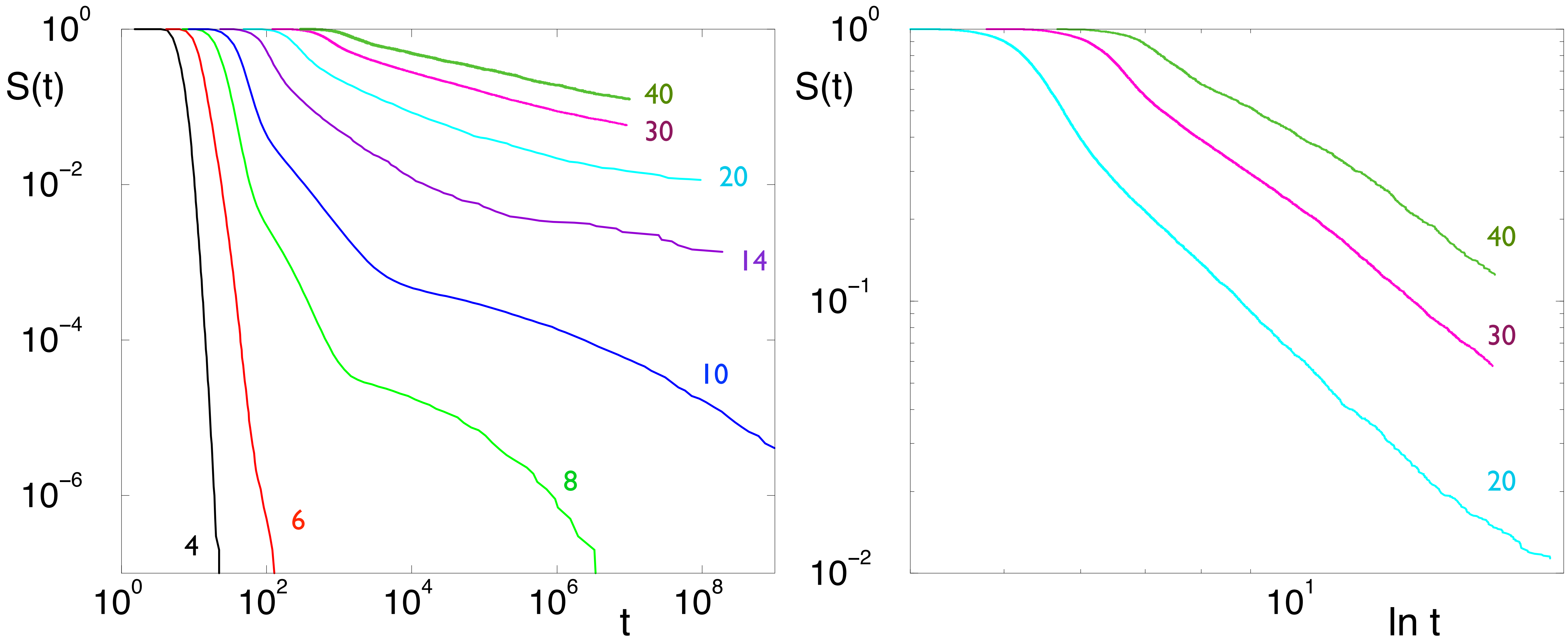
Merging of 2 Inflated Blinkers

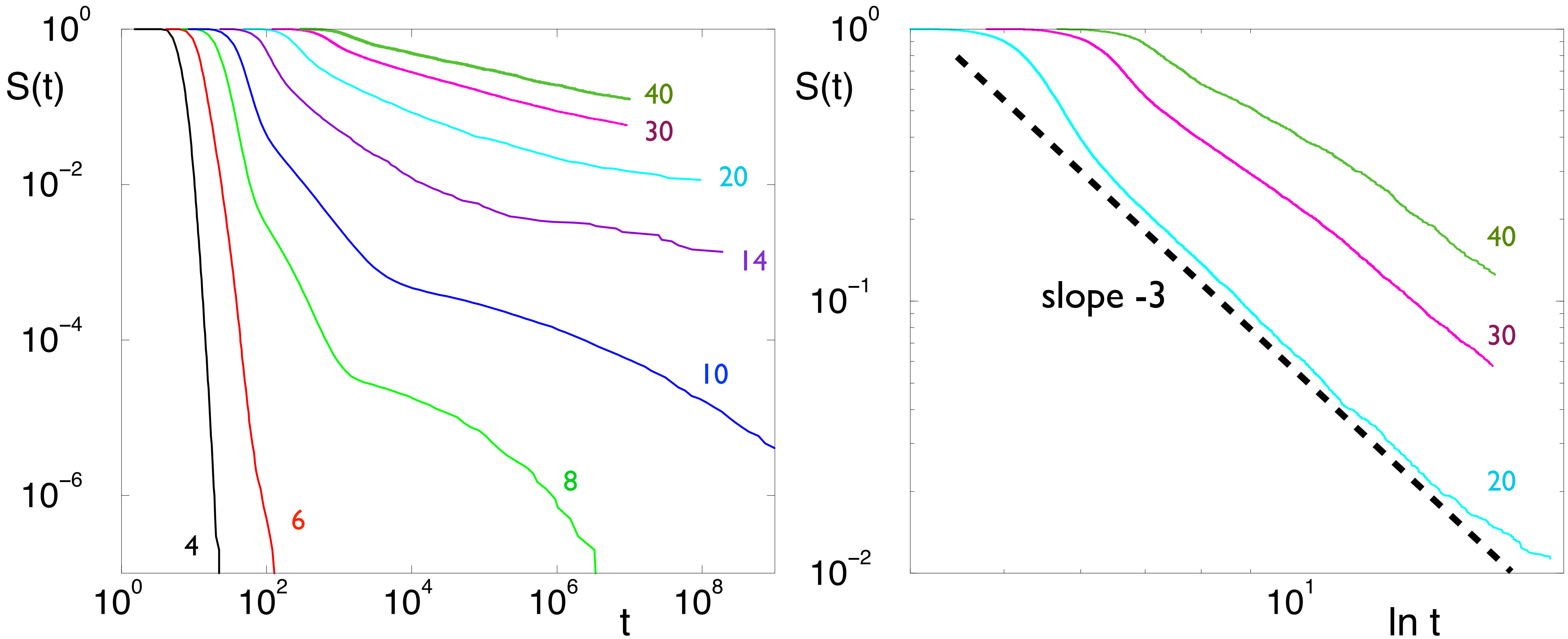
controls the long-time relaxation



Slow Relaxation in 3d



Slow Relaxation in 3d



$$S(t) \sim (\ln t)^{-3} \ ?$$

Summary & Open Problems

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finite temperature

corner geometry

Finite Temperature

lifetime of stripe state:

Finite Temperature

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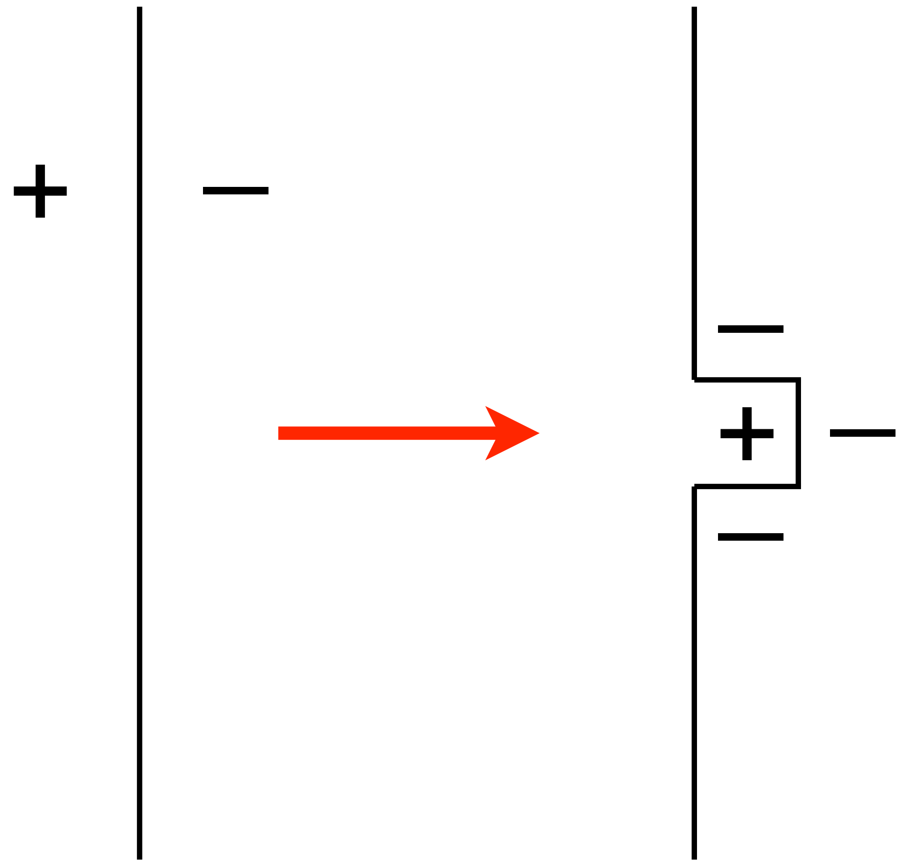

$e^{4J/T}$

Finite Temperature

lifetime of stripe state:

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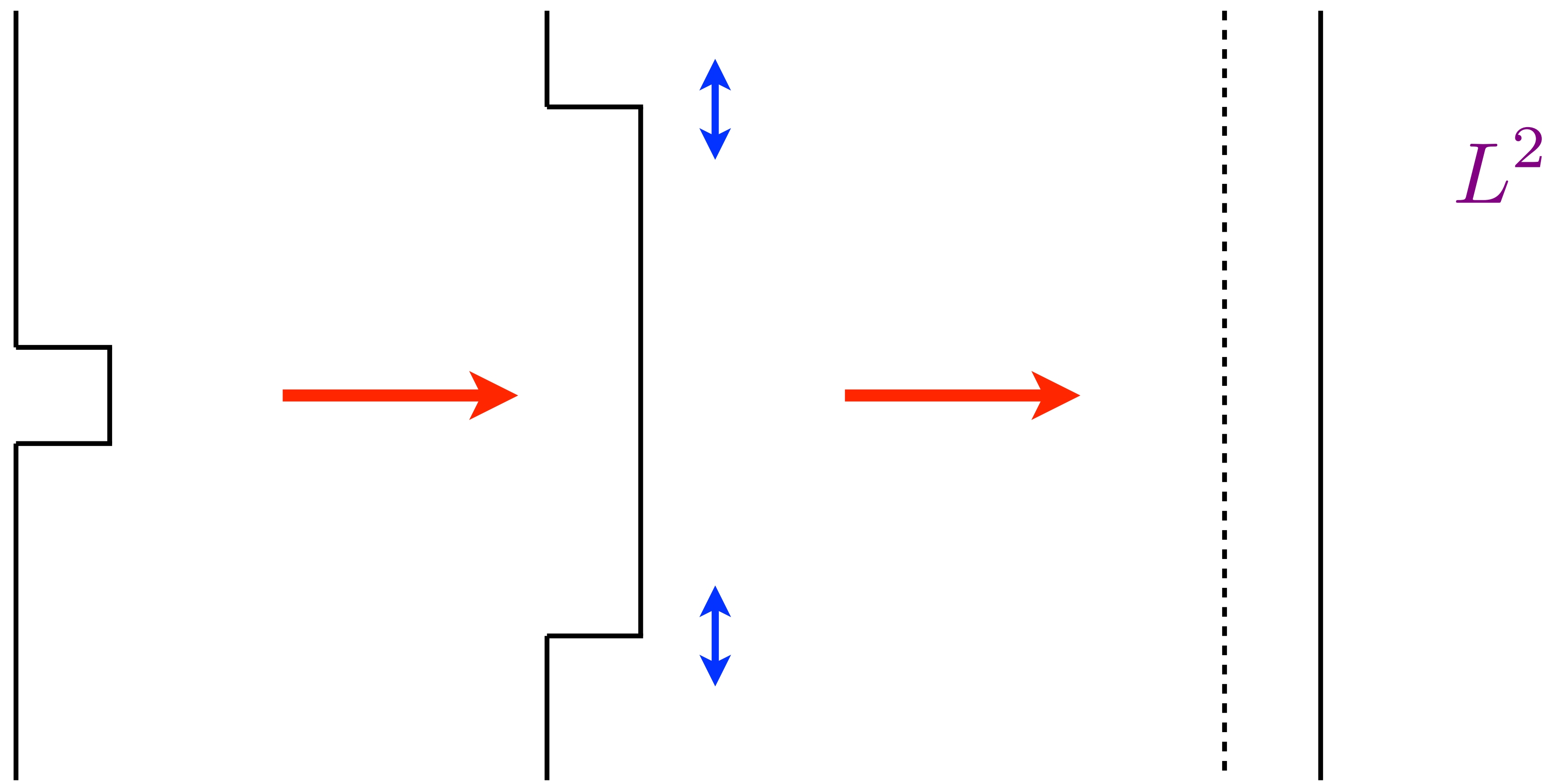
2. defect propagation, stripe translates by 1



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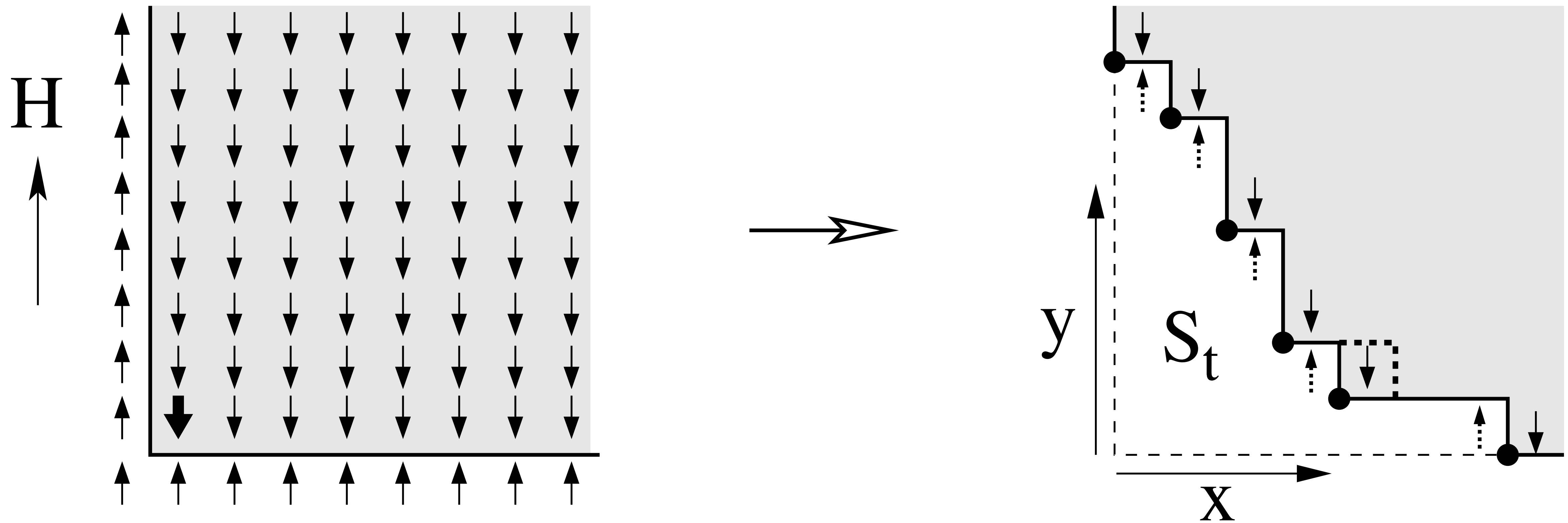
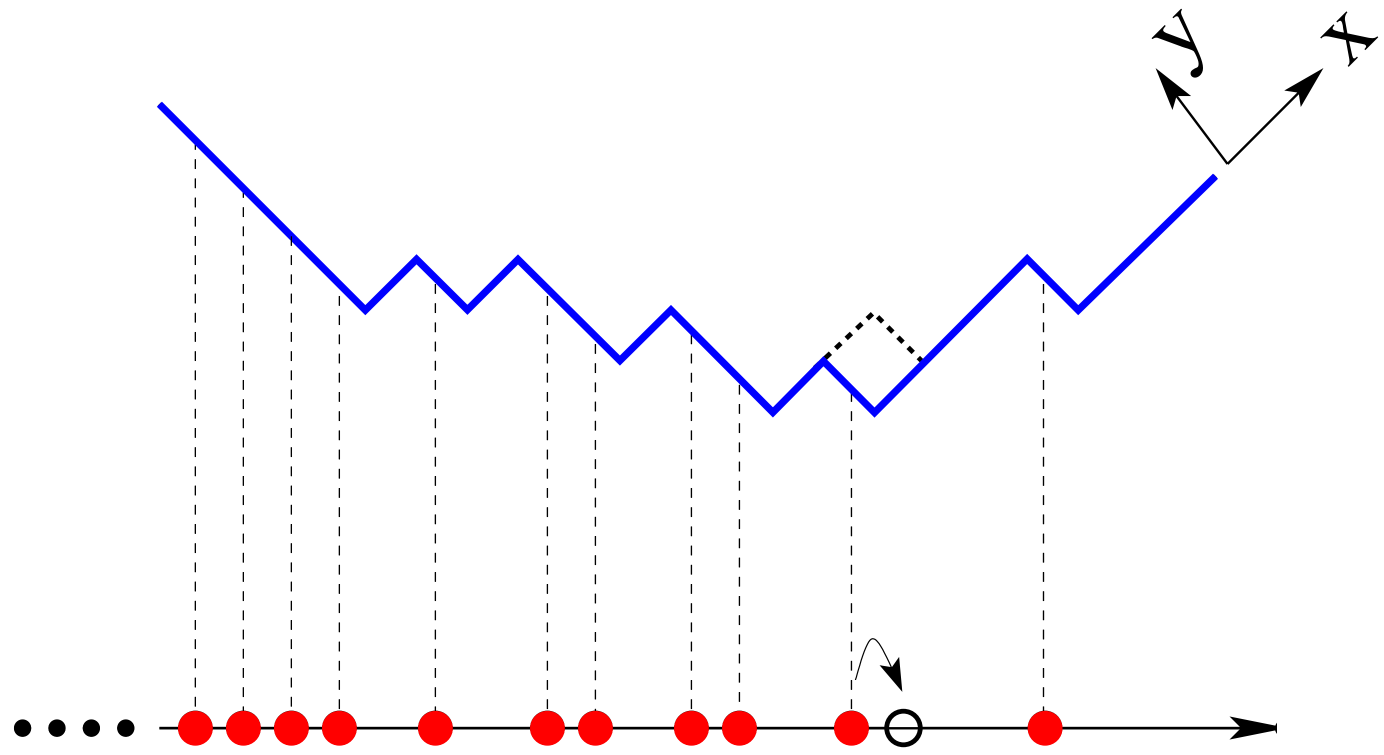
Open: what is the optimal cooling schedule
to ensure that ground state is reached?

Corner Geometry (driven interface)



(a) (b)

ASEP Correspondence



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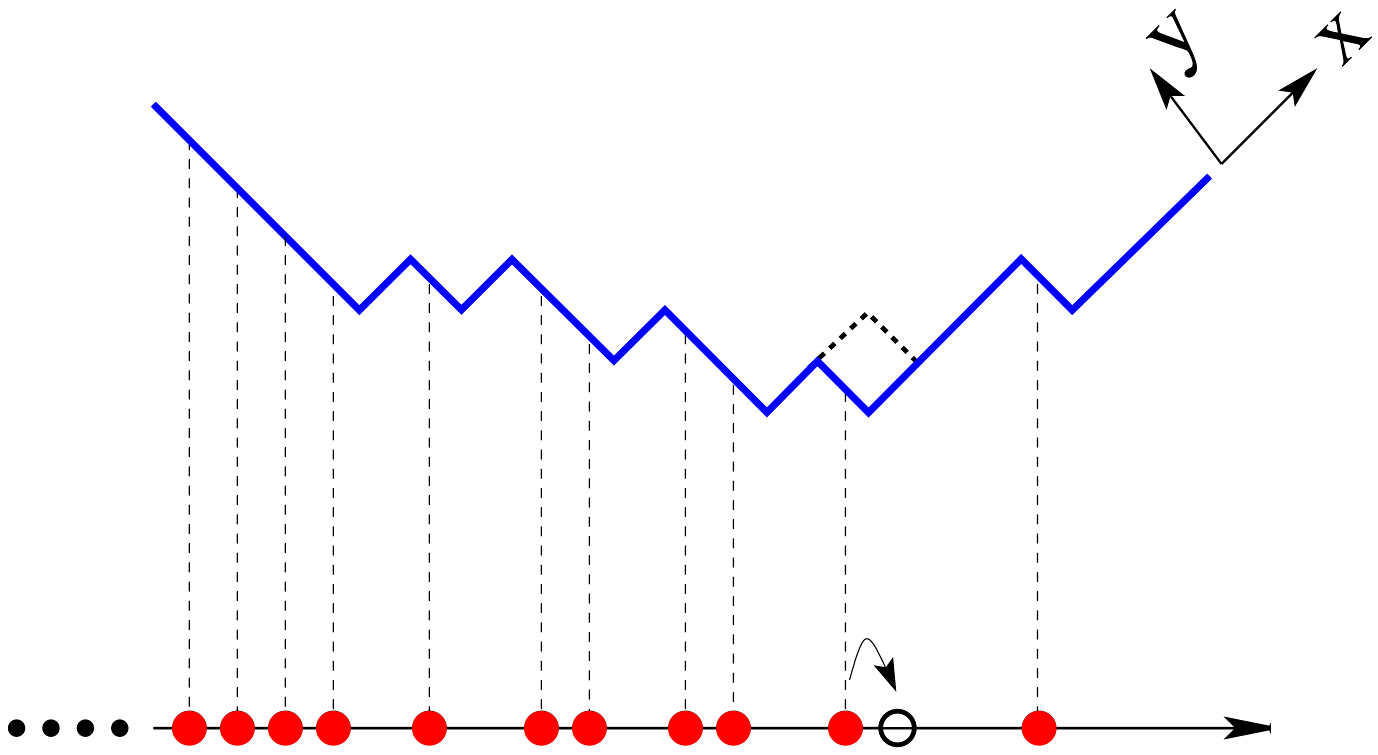

particle equation of motion:

$$\frac{\partial n}{\partial t} + \frac{\partial [n(1-n)]}{\partial z} = \frac{\partial^2 n}{\partial z^2} = 0$$

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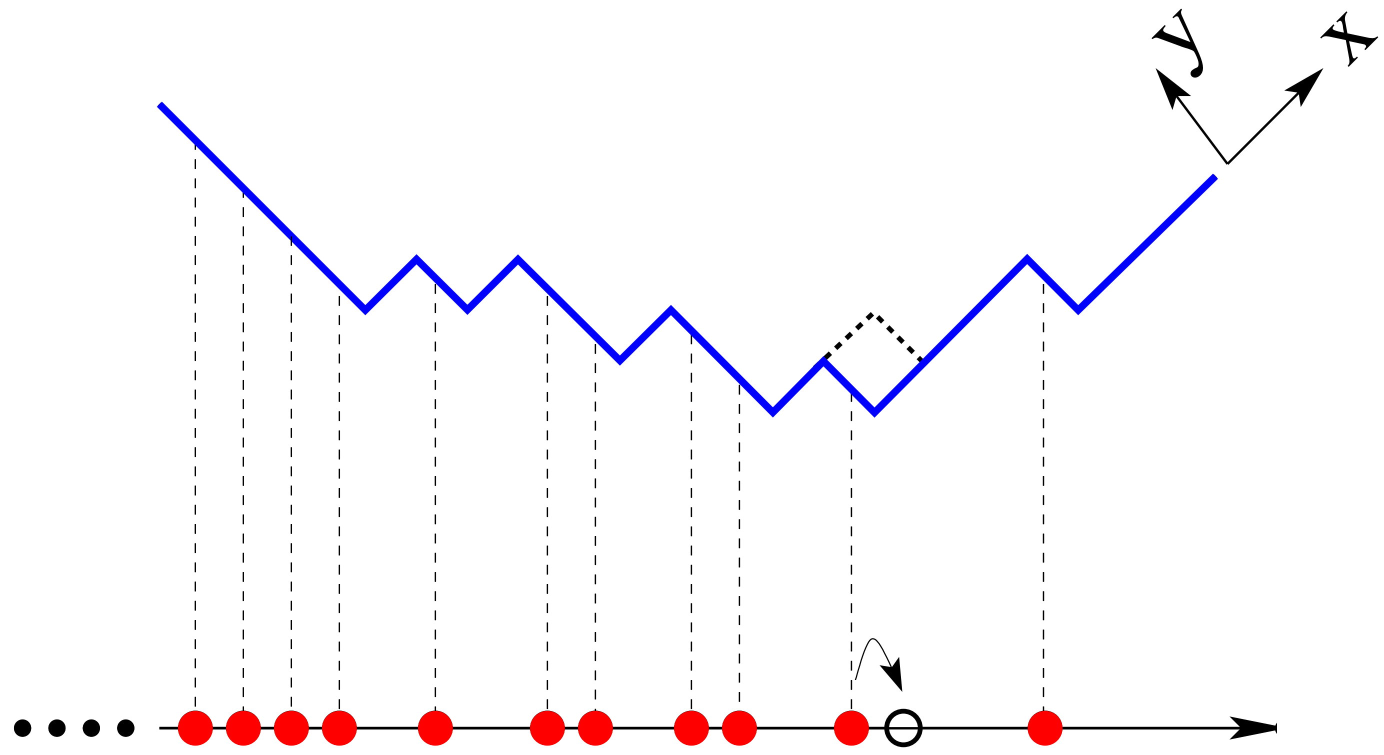

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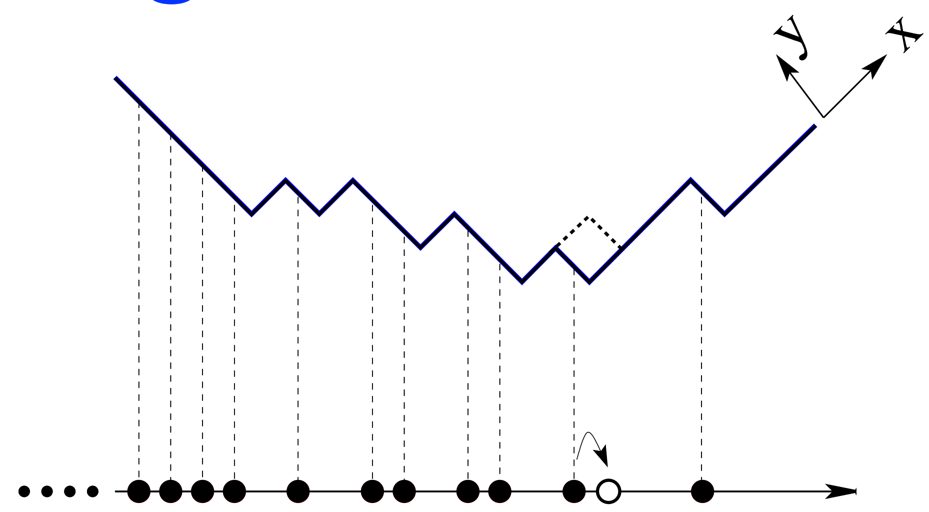
solution for step IC:

$$n(z,t) = \begin{cases} 1 & z < -t \\ \frac{1}{2}\left(1 - \frac{z}{t}\right) & |z| < t \\ 0 & z > t \end{cases}$$

Mapping to *Driven* Ising Interface

$$n(z) = -y'(x) \qquad z = x - y$$



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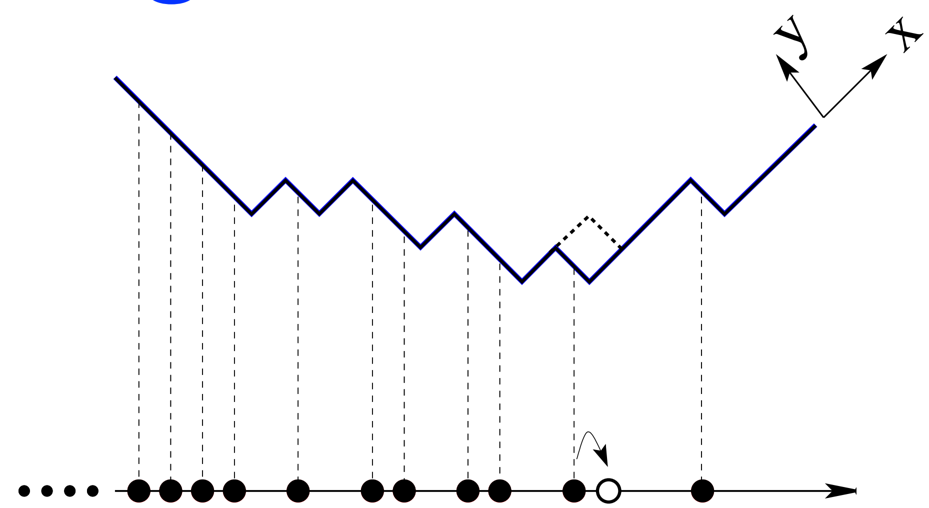

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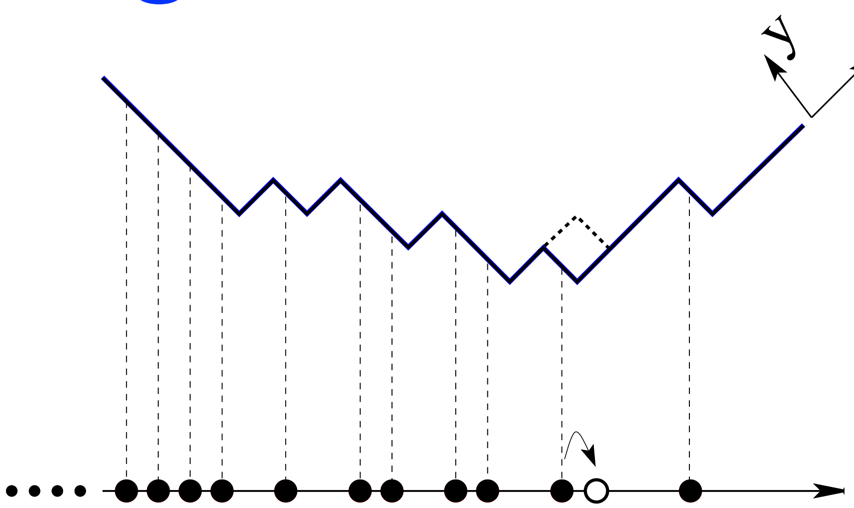

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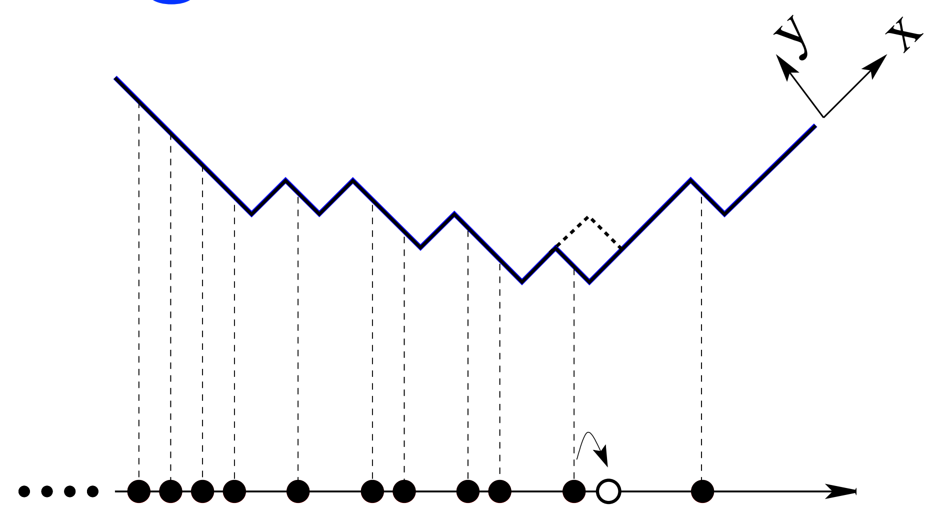

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$$\sqrt{x} + \sqrt{y} = \sqrt{t} \qquad\qquad 0 < x, y < t$$

(Rost, 1981)

Undriven Ising Interface

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Note: *equilibrium* Ising interface related to partitions of the integers

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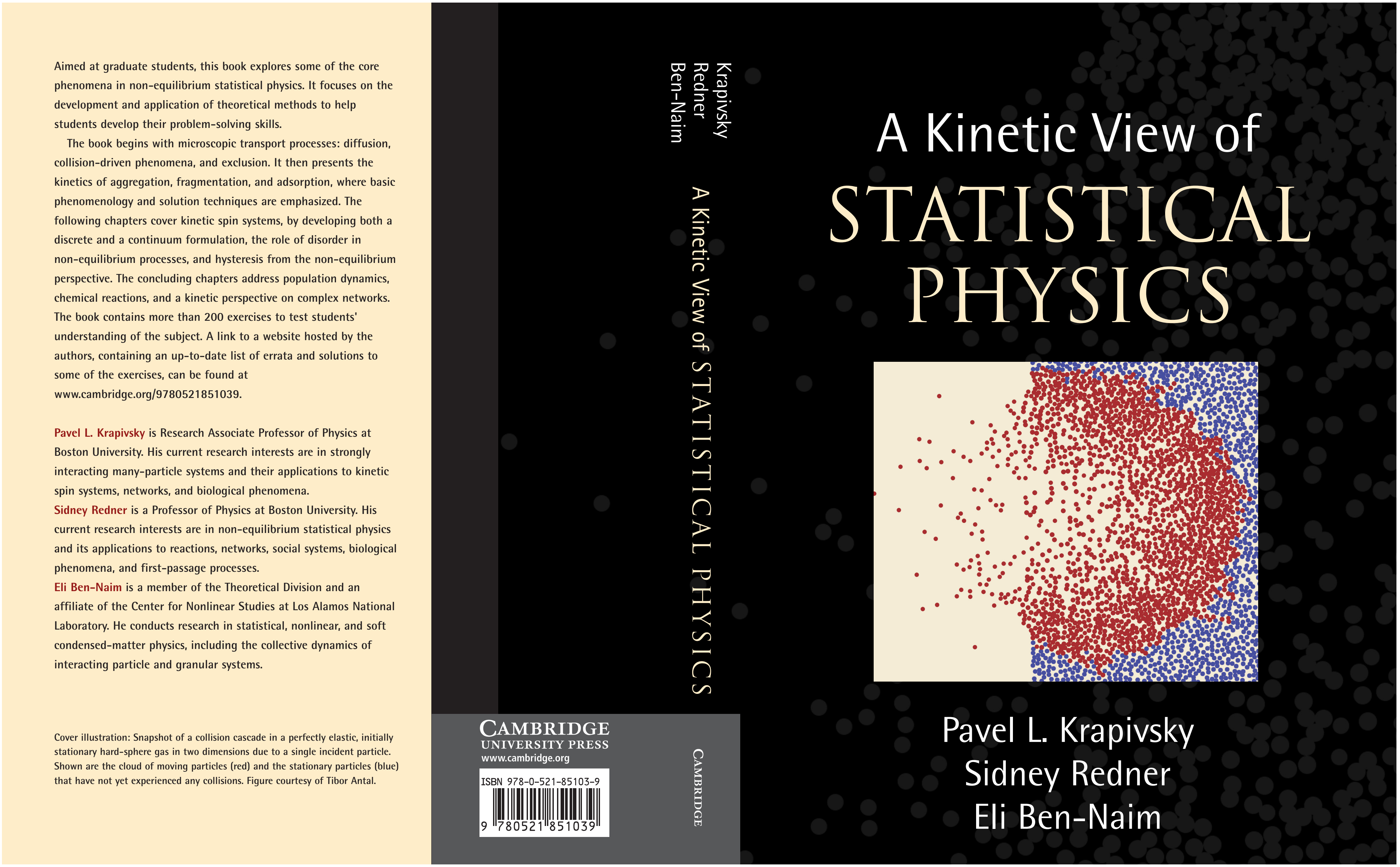
d≥3: rich state space structure

topologically complex final state
topological connection between energy & genus
perpetually blinking spins
ultra-slow relaxation *whose functional form is unknown*
finite temperature
corner geometry

Crass Commercialism



published Dec. 2010

1. Aperitifs	6. Fragmentation	11. Hysteresis
2. Diffusion	7. Adsorption	12. Population Dynamics
3. Collisions	8. Spin Dynamics	13. Diffusion Reactions
4. Exclusion	9. Coarsening	14. Complex Networks
5. Aggregation	10. Disorder	> 200 problems