

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner

collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner

collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Basic question: What is the final state of the Ising-Glauber model @ $T=0$ with symmetric initial conditions?

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Basic question: What is the final state of the Ising-Glauber model @ $T=0$ with symmetric initial conditions?

Expectation:

- Ground state is approached as $t \rightarrow \infty$
- Power-law coarsening

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Basic question: What is the final state of the Ising-Glauber model @ $T=0$ with symmetric initial conditions?

Expectation:

- Ground state is approached as $t \rightarrow \infty$
- Power-law coarsening

The result:

dimension	expectation
1	absolutely correct

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Basic question: What is the final state of the Ising-Glauber model @ $T=0$ with symmetric initial conditions?

Expectation:

- Ground state is approached as $t \rightarrow \infty$
- Power-law coarsening

The result:

dimension	expectation
1	absolutely correct
2	“sort of” correct

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Basic question: What is the final state of the Ising-Glauber model @ $T=0$ with symmetric initial conditions?

Expectation:

- Ground state is approached as $t \rightarrow \infty$
- Power-law coarsening

The result:

dimension	expectation
1	absolutely correct
2	“sort of” correct
≥ 3	wrong

Central Dogma of Spin Dynamics

Start at $T \gg T_c$ and suddenly quench to T_f .

1. **Supercritical dynamics** $T_f > T_c$
2. **Critical** $T_f = T_c$
3. **Subcritical** $T_f < T_c$ *universal*, same as $T_f = 0$.

Central Dogma of Spin Dynamics

Start at $T \gg T_c$ and suddenly quench to T_f .

1. **Supercritical dynamics** $T_f > T_c$
2. **Critical** $T_f = T_c$
3. **Subcritical** $T_f < T_c$ *universal*, same as $T_f = 0$.

Dynamic scaling hypothesis:

1. **Single length scale** $L(t) \rightarrow$ coarsening
2. **Algebraic scaling** $L(t) \sim t^z$
3. **Universality** z independent of most details.

The System

Ising Hamiltonian $\mathcal{H} = - \sum_{\langle i,j \rangle} \sigma_i \sigma_j \quad \sigma_i = \pm 1$

The System

Ising Hamiltonian $\mathcal{H} = - \sum_{\langle i,j \rangle} \sigma_i \sigma_j$ $\sigma_i = \pm 1$ *homogeneous*

The System

Ising Hamiltonian $\mathcal{H} = - \sum_{\langle i,j \rangle} \sigma_i \sigma_j$ $\sigma_i = \pm 1$ *homogeneous*

Lattice:

- *even co-ordination number*
- *periodic boundaries*

The System

Ising Hamiltonian $\mathcal{H} = - \sum_{\langle i,j \rangle} \sigma_i \sigma_j$ $\sigma_i = \pm 1$ *homogeneous*

Lattice:

- *even co-ordination number*
- periodic boundaries

Initial state:

- antiferromagnetic
- \uparrow with prob. $1/2$, \downarrow with prob. $1/2$
- $1/2$ spins \uparrow , $1/2$ spins \downarrow

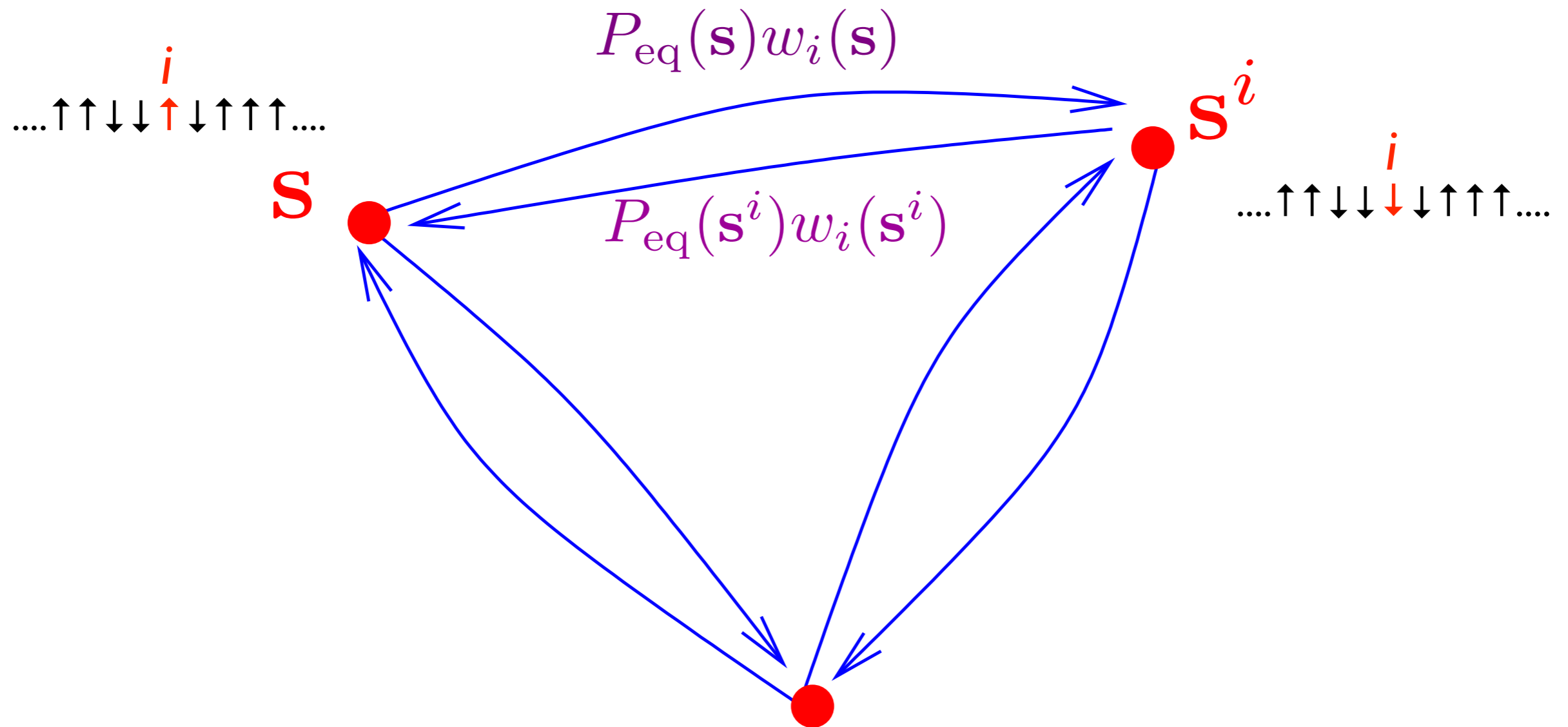
Endowing Spins with a Dynamics

Endowing Spins with a Dynamics

detailed balance condition:

$$\frac{w_i(\mathbf{s})}{w_i(\mathbf{s}^i)} = \frac{P_{\text{eq}}(\mathbf{s}^i)}{P_{\text{eq}}(\mathbf{s})} = \frac{e^{-\beta s_i \sum J_{ij} s_j}}{e^{\beta s_i \sum J_{ij} s_j}} = \frac{1 - s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}{1 + s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}$$

detailed balance condition in state space:



Endowing Spins with a Dynamics

detailed balance condition:

$$\frac{w_i(\mathbf{s})}{w_i(\mathbf{s}^i)} = \frac{P_{\text{eq}}(\mathbf{s}^i)}{P_{\text{eq}}(\mathbf{s})} = \frac{e^{-\beta s_i \sum J_{ij} s_j}}{e^{\beta s_i \sum J_{ij} s_j}} = \frac{1 - s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}{1 + s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}$$

Endowing Spins with a Dynamics

detailed balance condition:

$$\frac{w_i(\mathbf{s})}{w_i(\mathbf{s}^i)} = \frac{P_{\text{eq}}(\mathbf{s}^i)}{P_{\text{eq}}(\mathbf{s})} = \frac{e^{-\beta s_i \sum J_{ij} s_j}}{e^{\beta s_i \sum J_{ij} s_j}} = \frac{1 - s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}{1 + s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}$$

flip rate in one dimension:

$$w_i = \frac{1}{2} \left\{ 1 - s_i \tanh[\beta J (s_{i-1} + s_{i+1})] \right\}$$
$$\rightarrow \frac{1}{2} \left[1 - s_i \frac{s_{i-1} + s_{i+1}}{2} \right] \quad T \rightarrow 0$$

Endowing Spins with a Dynamics

detailed balance condition:

$$\frac{w_i(\mathbf{s})}{w_i(\mathbf{s}^i)} = \frac{P_{\text{eq}}(\mathbf{s}^i)}{P_{\text{eq}}(\mathbf{s})} = \frac{e^{-\beta s_i \sum J_{ij} s_j}}{e^{\beta s_i \sum J_{ij} s_j}} = \frac{1 - s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}{1 + s_i \tanh(\beta \sum_{j \in \langle i \rangle} J_{ij} s_j)}$$

flip rate in one dimension:

$$w_i = \frac{1}{2} \left\{ 1 - s_i \tanh[\beta J (s_{i-1} + s_{i+1})] \right\}$$
$$\rightarrow \frac{1}{2} \left[1 - s_i \frac{s_{i-1} + s_{i+1}}{2} \right] \quad T \rightarrow 0$$

Glauber dynamics at $T=0$: Pick a random spin and consider outcome of reversing it

- if $\Delta E < 0$ **do it**
- if $\Delta E > 0$ **don't do it**
- if $\Delta E = 0$ **do it with prob. 1/2**

Equations of Motion

Correlation functions: $S_i = \langle s_i \rangle$, $S_{i,j} = \langle s_i s_j \rangle$

Equations of Motion

Correlation functions: $S_i = \langle s_i \rangle$, $S_{i,j} = \langle s_i s_j \rangle$

$$\frac{dS_j}{dt} = -2\langle s_j w_j \rangle$$

$$\frac{dS_{i,j}}{dt} = -2\langle s_i s_j [w_i + w_j] \rangle$$

Equations of Motion

Correlation functions: $S_i = \langle s_i \rangle$, $S_{i,j} = \langle s_i s_j \rangle$

$$\frac{dS_j}{dt} = -2 \langle s_j w_j \rangle$$

$$\frac{dS_{i,j}}{dt} = -2 \langle s_i s_j [w_i + w_j] \rangle$$

mean spin:

$$\begin{aligned} \frac{dS_j}{dt} &= -2 \left\langle s_j \left\{ \frac{1}{2} \left[1 - s_j \frac{s_{j-1} + s_{j+1}}{2} \right] \right\} \right\rangle \\ &= -S_j + \frac{1}{2} (S_{j-1} + S_{j+1}) \end{aligned}$$

mean spin:

$$\frac{dS_j}{dt} = -S_j + \frac{1}{2} (S_{j-1} + S_{j+1})$$

mean spin:

$$\frac{dS_j}{dt} = -S_j + \frac{1}{2} (S_{j-1} + S_{j+1})$$

$$S_j(t) = I_j(t) e^{-t}$$

mean spin:

$$\frac{dS_j}{dt} = -S_j + \frac{1}{2} (S_{j-1} + S_{j+1})$$

$$S_j(t) = I_j(t) e^{-t}$$

kth-neighbor correlation function: $G_k \equiv S_{i,i+k}$

$$\frac{dG_k}{dt} = -2G_k + (G_{k-1} + G_{k+1}) \quad G_0(t) = \langle s_i^2 \rangle = 1$$

mean spin:

$$\frac{dS_j}{dt} = -S_j + \frac{1}{2} (S_{j-1} + S_{j+1})$$

$$S_j(t) = I_j(t) e^{-t}$$

kth-neighbor correlation function: $G_k \equiv S_{i,i+k}$

$$\frac{dG_k}{dt} = -2G_k + (G_{k-1} + G_{k+1}) \quad G_0(t) = \langle s_i^2 \rangle = 1$$

$$G_k(t) = 1 - e^{-2t} \left[I_0(2t) + I_k(2t) + 2 \sum_{j=1}^{k-1} I_j(2t) \right]$$

mean spin:

$$\frac{dS_j}{dt} = -S_j + \frac{1}{2} (S_{j-1} + S_{j+1})$$

$$S_j(t) = I_j(t) e^{-t}$$

kth-neighbor correlation function: $G_k \equiv S_{i,i+k}$

$$\frac{dG_k}{dt} = -2G_k + (G_{k-1} + G_{k+1}) \quad G_0(t) = \langle s_i^2 \rangle = 1$$

$$G_k(t) = 1 - e^{-2t} \left[I_0(2t) + I_k(2t) + 2 \sum_{j=1}^{k-1} I_j(2t) \right]$$

$$\begin{aligned} \rho(t) &= \frac{1}{2} (1 - G_1) = \frac{1}{2} e^{-2t} [I_0(2t) + I_1(2t)] \\ &\simeq (4\pi t)^{-1/2} \end{aligned}$$

$$\begin{aligned} G_1 &= \langle s_i s_{i+1} \rangle \\ &= \text{Prob}(\text{aligned}) - \text{Prob}(\text{anti-aligned}) \\ &= 1 - 2 \text{Prob}(\text{anti-aligned}) \\ &= 1 - 2 \text{Prob}(\text{domain wall particle exists}) \\ &= 1 - 2\rho \end{aligned}$$

$$\begin{aligned} G_1 &= \langle s_i s_{i+1} \rangle \\ &= \text{Prob}(\text{aligned}) - \text{Prob}(\text{anti-aligned}) \\ &= 1 - 2 \text{Prob}(\text{anti-aligned}) \\ &= 1 - 2 \text{Prob}(\text{domain wall particle exists}) \\ &= 1 - 2\rho \end{aligned}$$

ρ = density of domain-wall “particles”

mean spin:

$$\frac{dS_j}{dt} = -S_j + \frac{1}{2} (S_{j-1} + S_{j+1})$$

$$S_j(t) = I_j(t) e^{-t}$$

kth-neighbor correlation function: $G_k \equiv S_{i,i+k}$

$$\frac{dG_k}{dt} = -2G_k + (G_{k-1} + G_{k+1}) \quad G_0(t) = \langle s_i^2 \rangle = 1$$

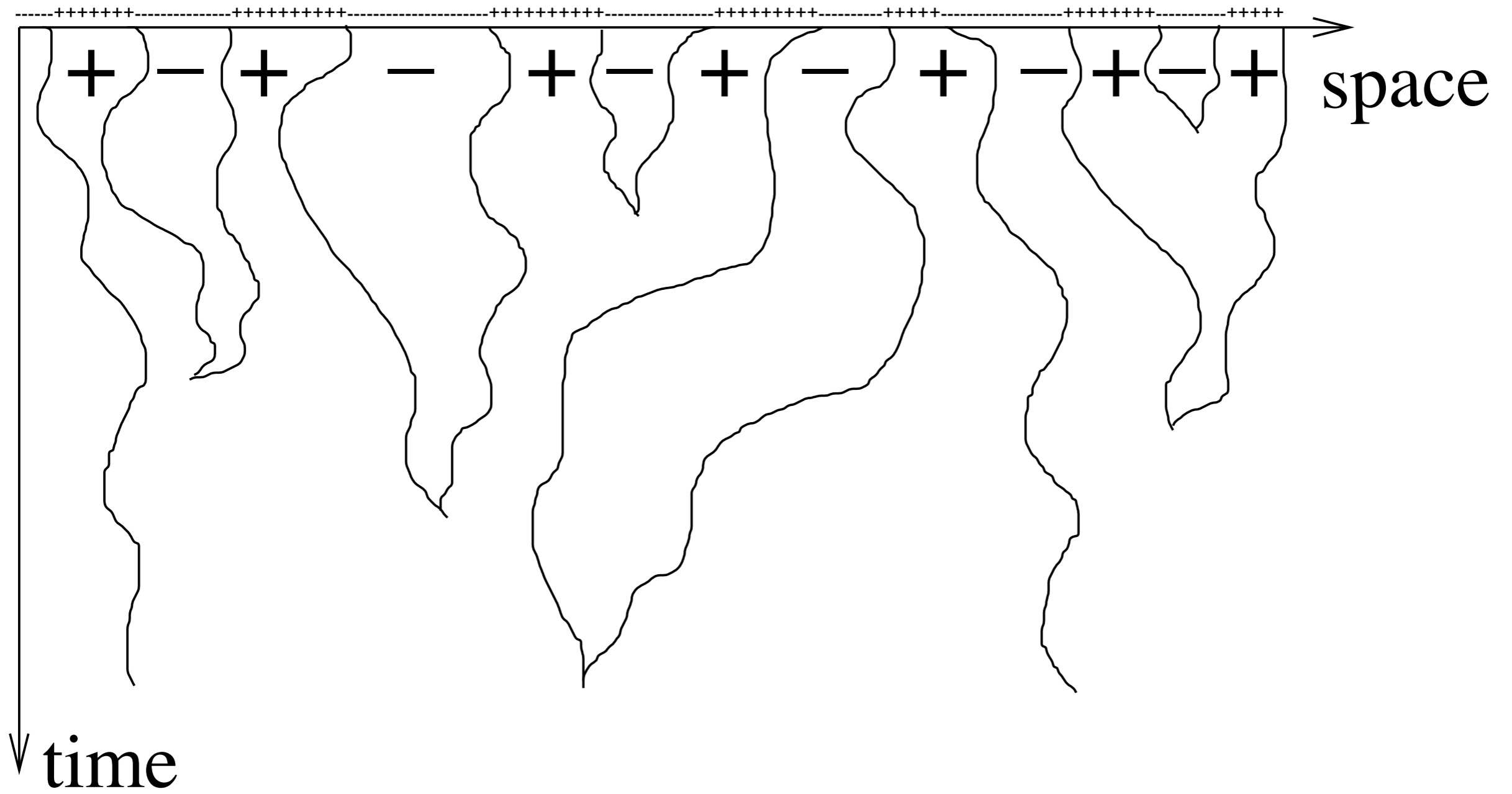
$$G_k(t) = 1 - e^{-2t} \left[I_0(2t) + I_k(2t) + 2 \sum_{j=1}^{k-1} I_j(2t) \right]$$

$$\begin{aligned} \rho(t) &= \frac{1}{2} (1 - G_1) = \frac{1}{2} e^{-2t} [I_0(2t) + I_1(2t)] \\ &\simeq (4\pi t)^{-1/2} \quad \rightarrow \text{domain wall picture} \end{aligned}$$

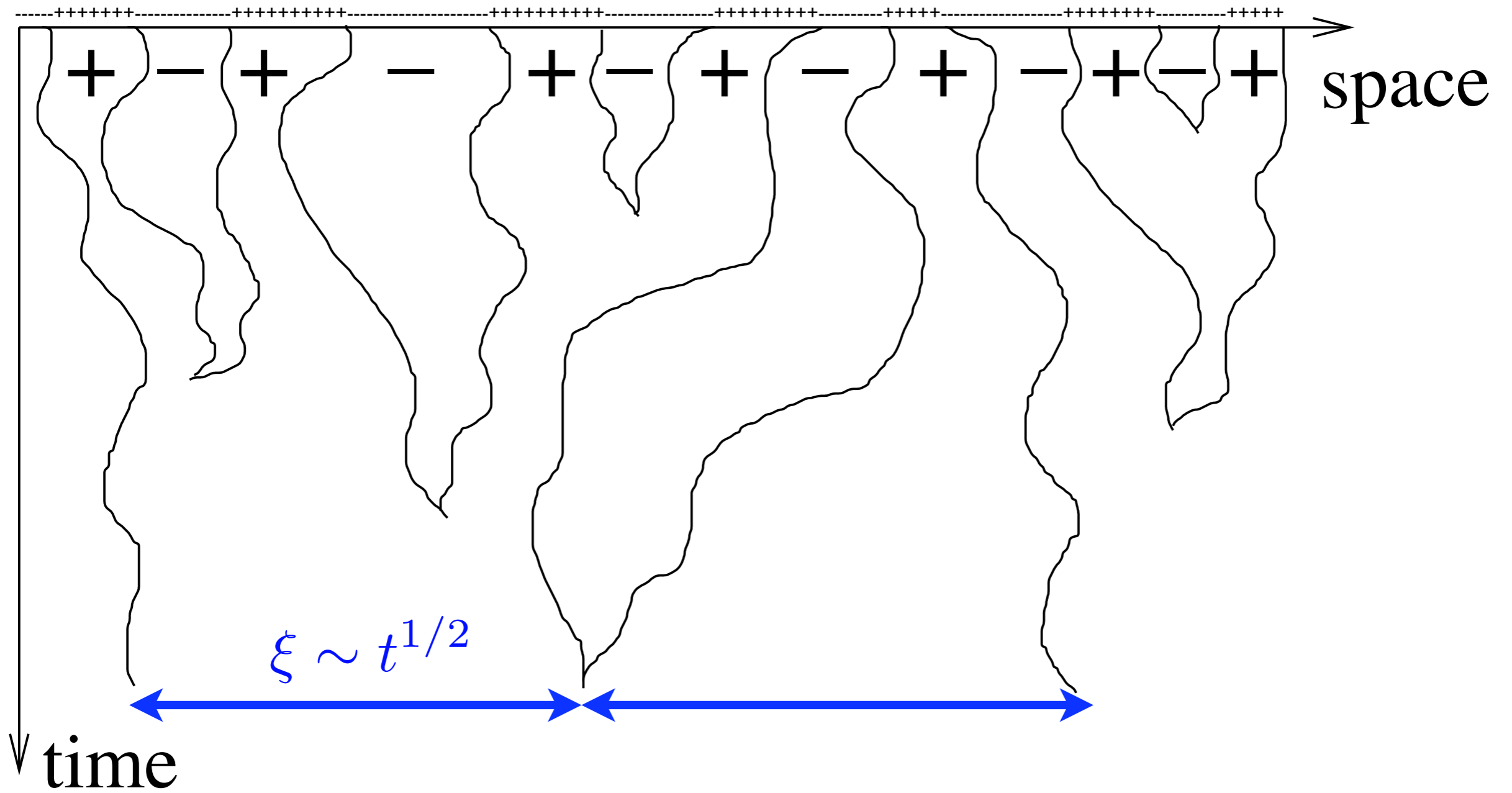
Domain Wall Picture

-----+++++-----+++++-----+++++-----+++++-----+++++-----+++++-----+++++

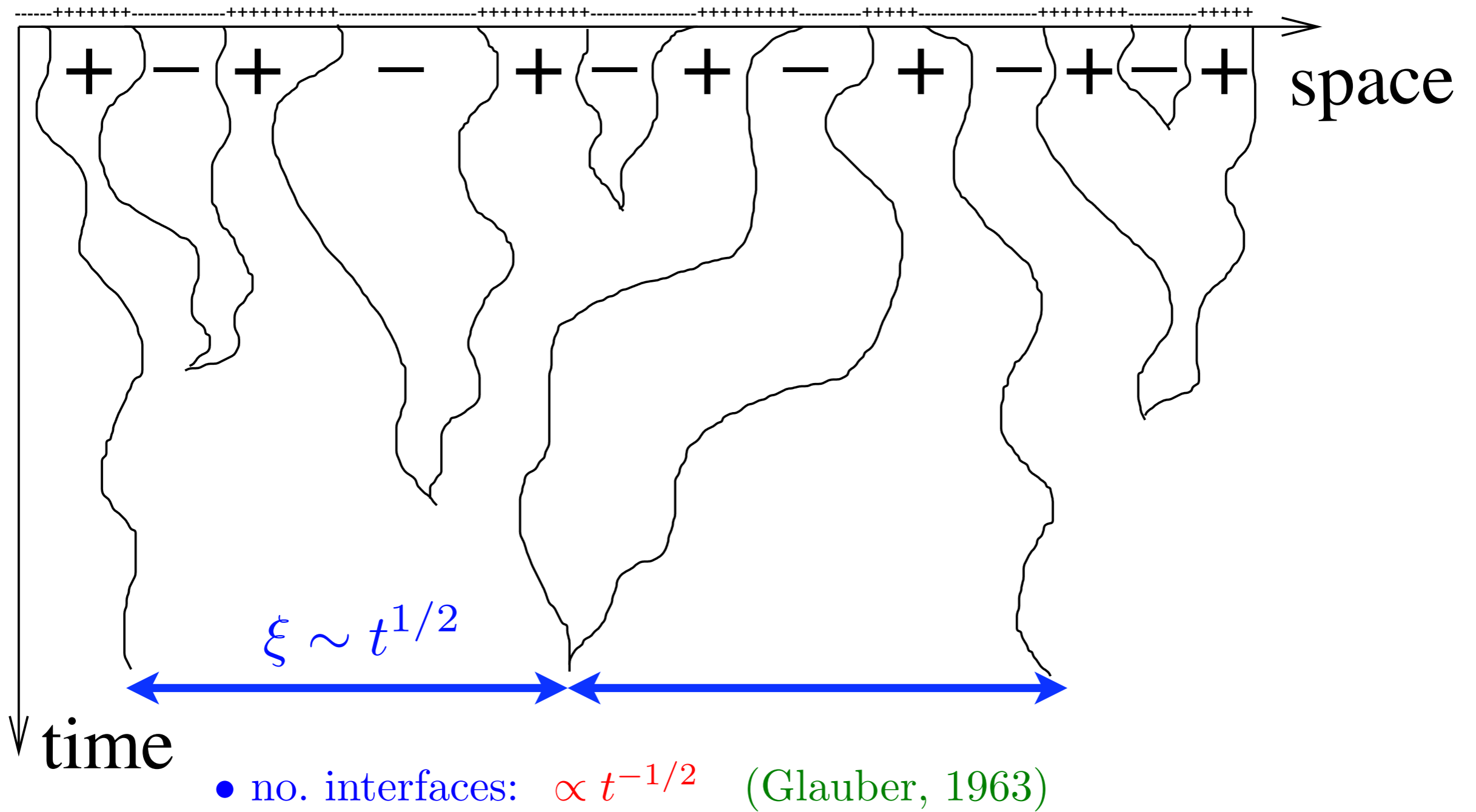
Domain Wall Picture



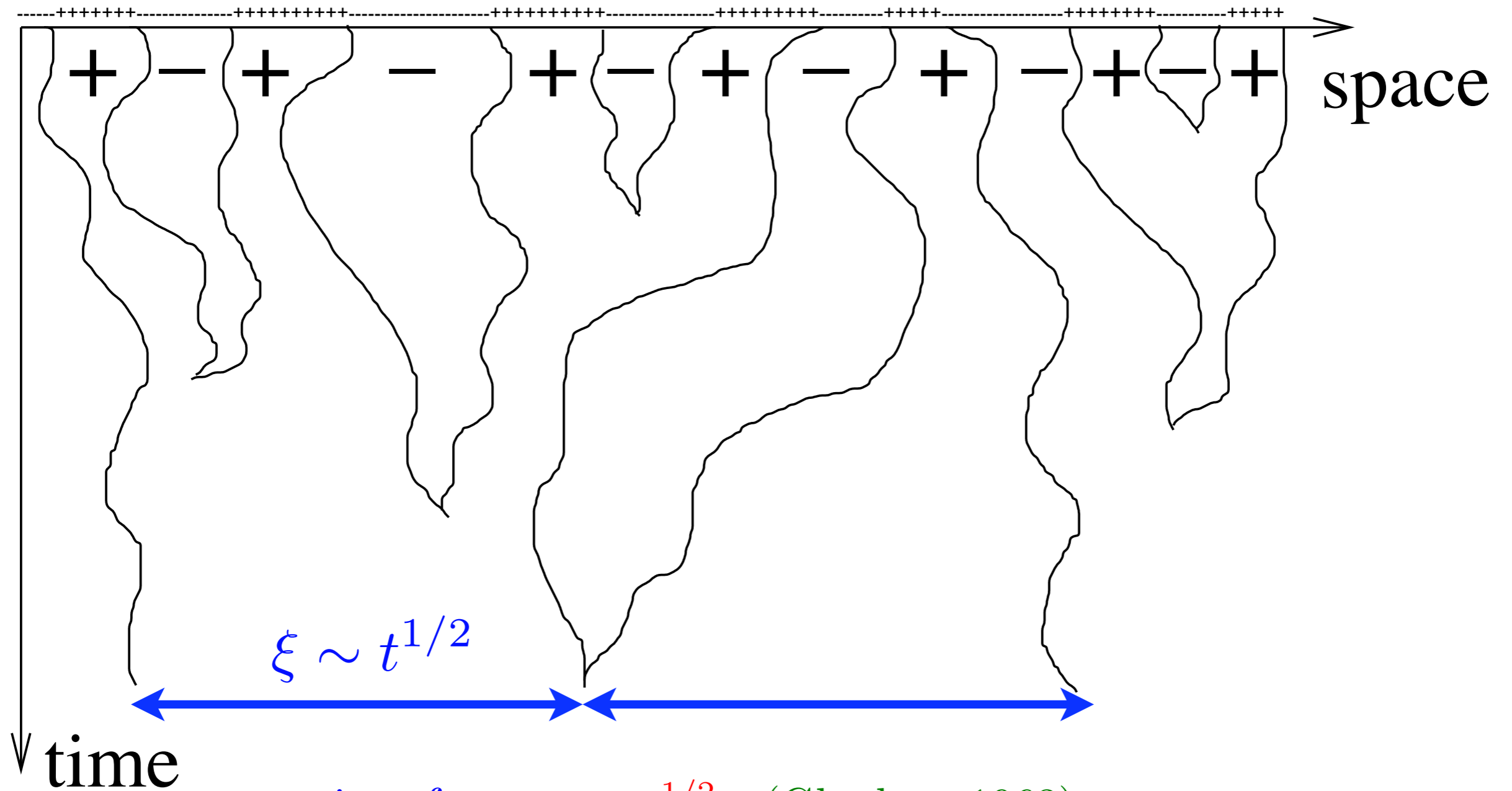
Domain Wall Picture



Domain Wall Picture

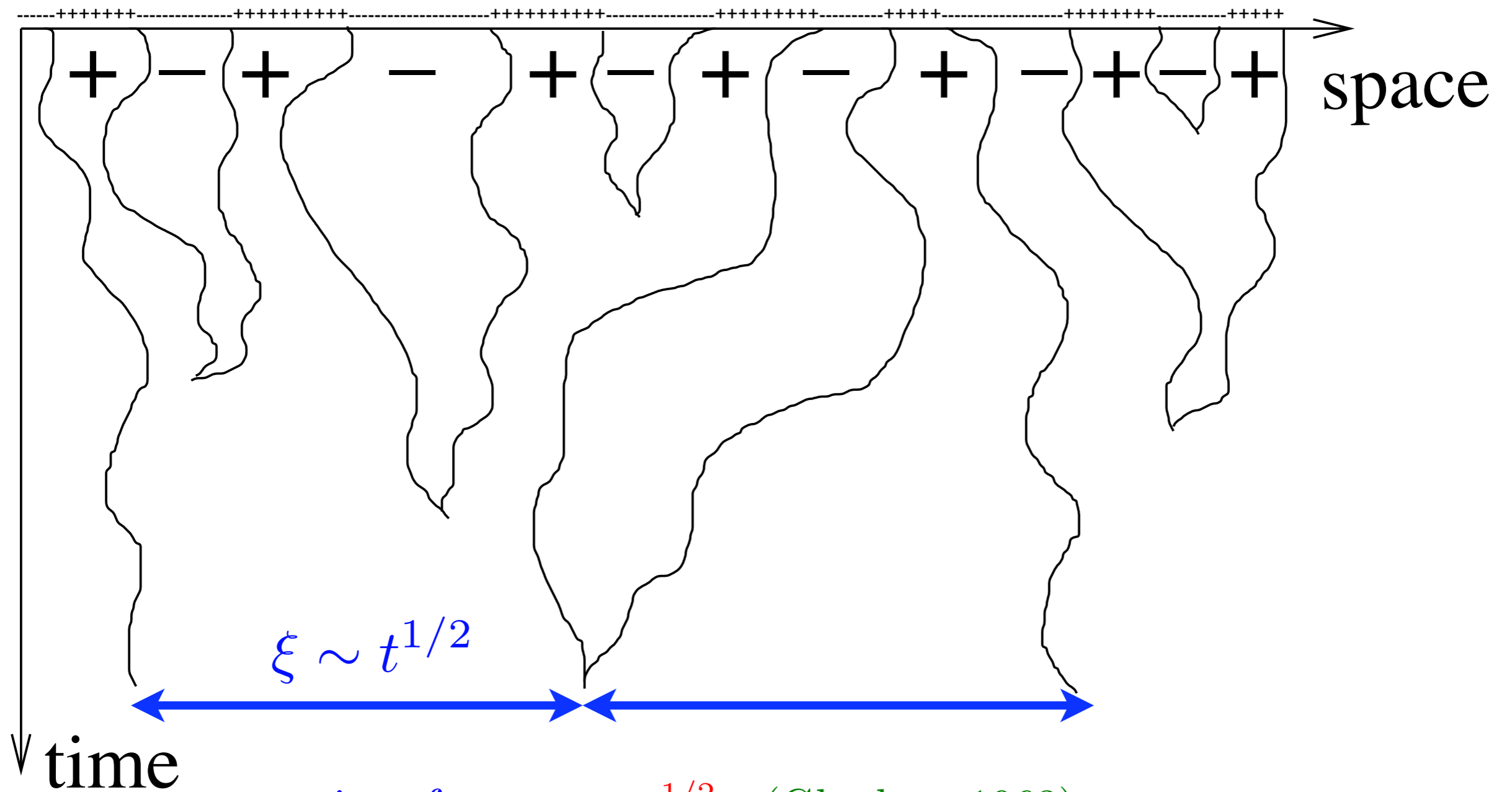


Domain Wall Picture



- no. interfaces: $\propto t^{-1/2}$ (Glauber, 1963)
- domain length dist: $\frac{x}{t^{3/2}} e^{-x^2/t}$ (Ben-Naim, Krapivsky, 1997)

Domain Wall Picture



- no. interfaces: $\propto t^{-1/2}$ (Glauber, 1963)
- domain length dist: $\frac{x}{t^{3/2}} e^{-x^2/t}$ (Ben-Naim, Krapivsky, 1997)
- time to ground state: $T \propto L^2$

Domain Length Distribution (Ben-Naim & Krapivsky 97)

Domain Length Distribution (Ben-Naim & Krapivsky 97)

scaling ansatz: $P_k(t) \simeq C t^{-1} \Phi(kt^{-1/2})$ $\int_0^\infty x \Phi(x) dx = 1$

Domain Length Distribution (Ben-Naim & Krapivsky 97)

scaling ansatz: $P_k(t) \simeq C t^{-1} \Phi(kt^{-1/2}) \quad \int_0^\infty x \Phi(x) dx = 1$

$$\rho = \sum_{k \geq 1} P_k \simeq (4\pi t)^{-1/2} \rightarrow \int_0^\infty \Phi(x) dx = (4\pi)^{-1/2} \equiv C$$

$$\frac{d\rho}{dt} = -2P_1 \rightarrow P_1 \simeq \frac{1}{4} C t^{-3/2}$$

Domain Length Distribution (Ben-Naim & Krapivsky 97)

scaling ansatz: $P_k(t) \simeq C t^{-1} \Phi(kt^{-1/2}) \quad \int_0^\infty x \Phi(x) dx = 1$

$$\rho = \sum_{k \geq 1} P_k \simeq (4\pi t)^{-1/2} \rightarrow \int_0^\infty \Phi(x) dx = (4\pi)^{-1/2} \equiv C$$

$$\frac{d\rho}{dt} = -2P_1 \rightarrow P_1 \simeq \frac{1}{4} C t^{-3/2}$$

master equation:

$$\frac{dP_k}{dt} = -2P_k + P_{k+1} + P_{k-1} \left(1 - \frac{P_1}{\rho}\right) + \frac{P_1}{\rho^2} \sum_{i+j=k-1} P_i P_j - \frac{P_1}{\rho} P_k$$

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2} \frac{d(x\Phi)}{dx} + \frac{1}{4C} \int_0^x \Phi(y) \Phi(x-y) dy = 0$$

Domain Length Distribution (Ben-Naim & Krapivsky 97)

scaling ansatz: $P_k(t) \simeq C t^{-1} \Phi(kt^{-1/2}) \quad \int_0^\infty x \Phi(x) dx = 1$

$$\rho = \sum_{k \geq 1} P_k \simeq (4\pi t)^{-1/2} \rightarrow \int_0^\infty \Phi(x) dx = (4\pi)^{-1/2} \equiv C$$

$$\frac{d\rho}{dt} = -2P_1 \rightarrow P_1 \simeq \frac{1}{4} C t^{-3/2}$$

master equation:

$$\frac{dP_k}{dt} = -2P_k + P_{k+1} + P_{k-1} \left(1 - \frac{P_1}{\rho}\right) + \frac{P_1}{\rho^2} \sum_{i+j=k-1} P_i P_j - \frac{P_1}{\rho} P_k$$

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2} \frac{d(x\Phi)}{dx} + \frac{1}{4C} \int_0^x \Phi(y) \Phi(x-y) dy = 0$$

Laplace transform: $\Phi(s) = \frac{1}{C} \int_0^\infty \Phi(x) e^{-sx} dx$

$$\frac{d\Phi}{ds} = \frac{\Phi^2}{2s} + 2s\Phi - \frac{1}{2s}$$

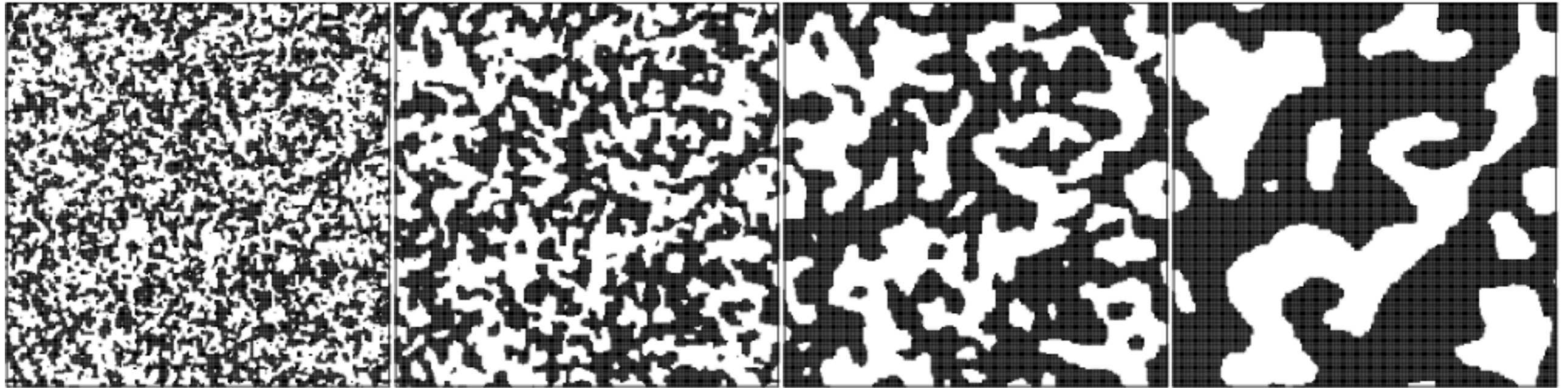
$$\Phi = 1 - 2s^2 - 2s \frac{d}{ds} \ln D_{1/2}(\sqrt{2}s) \simeq A e^{-\lambda x} \quad x \rightarrow \infty$$

Two Dimensions

Two Dimensions

coarsening of 256x256 system

(courtesy of V. Spirin)



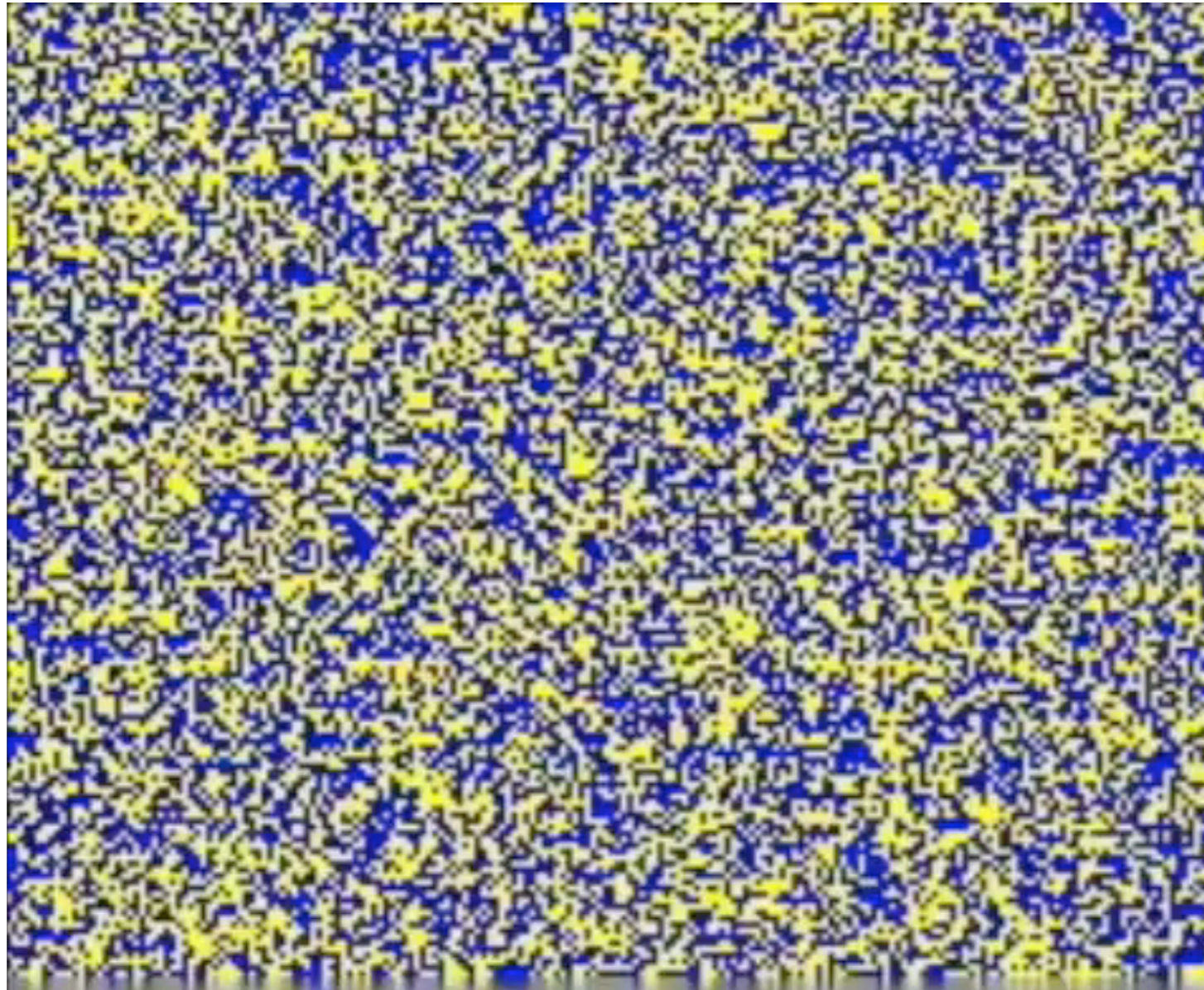
$t=4$

$t=16$

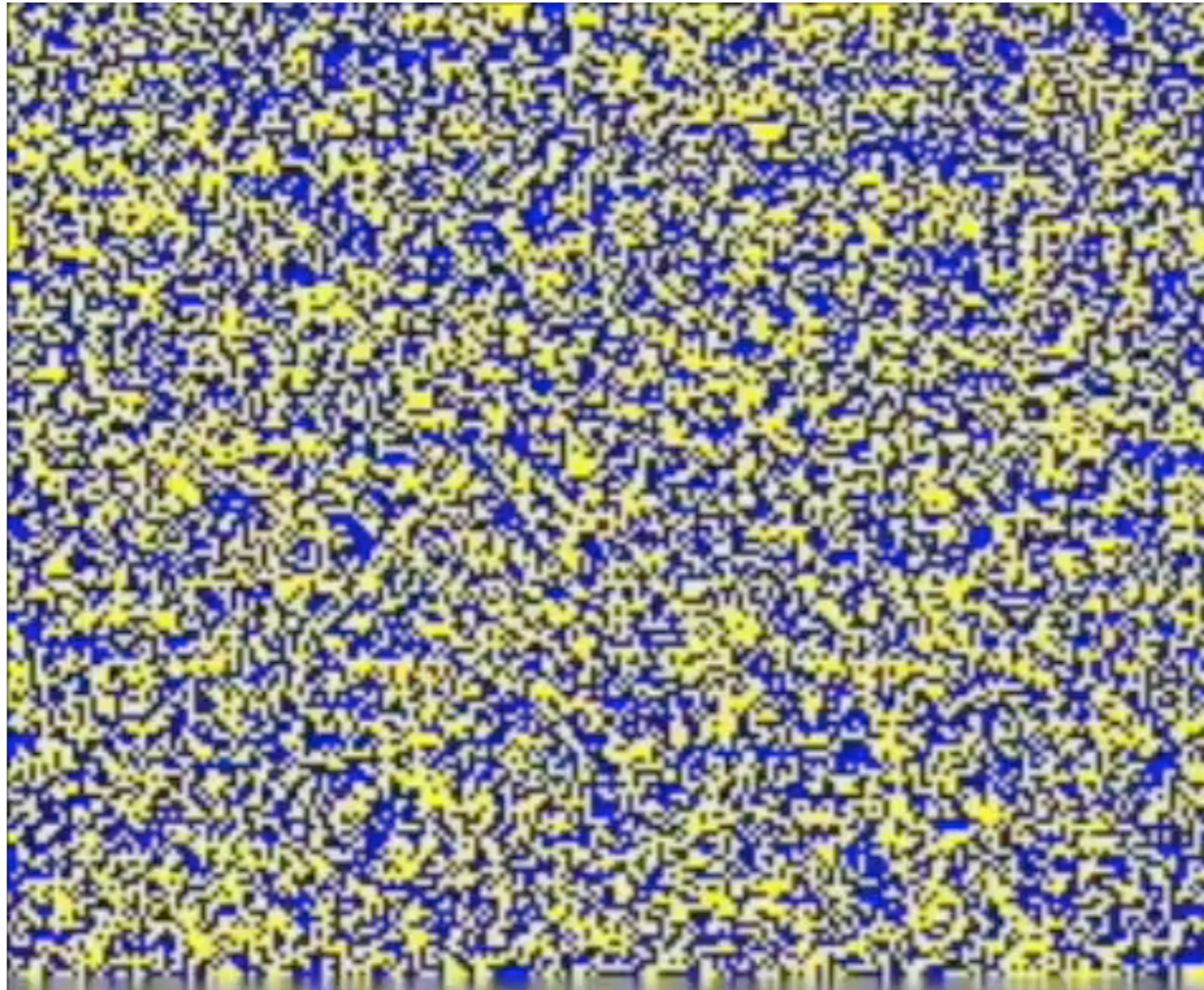
$t=64$

$t=256$

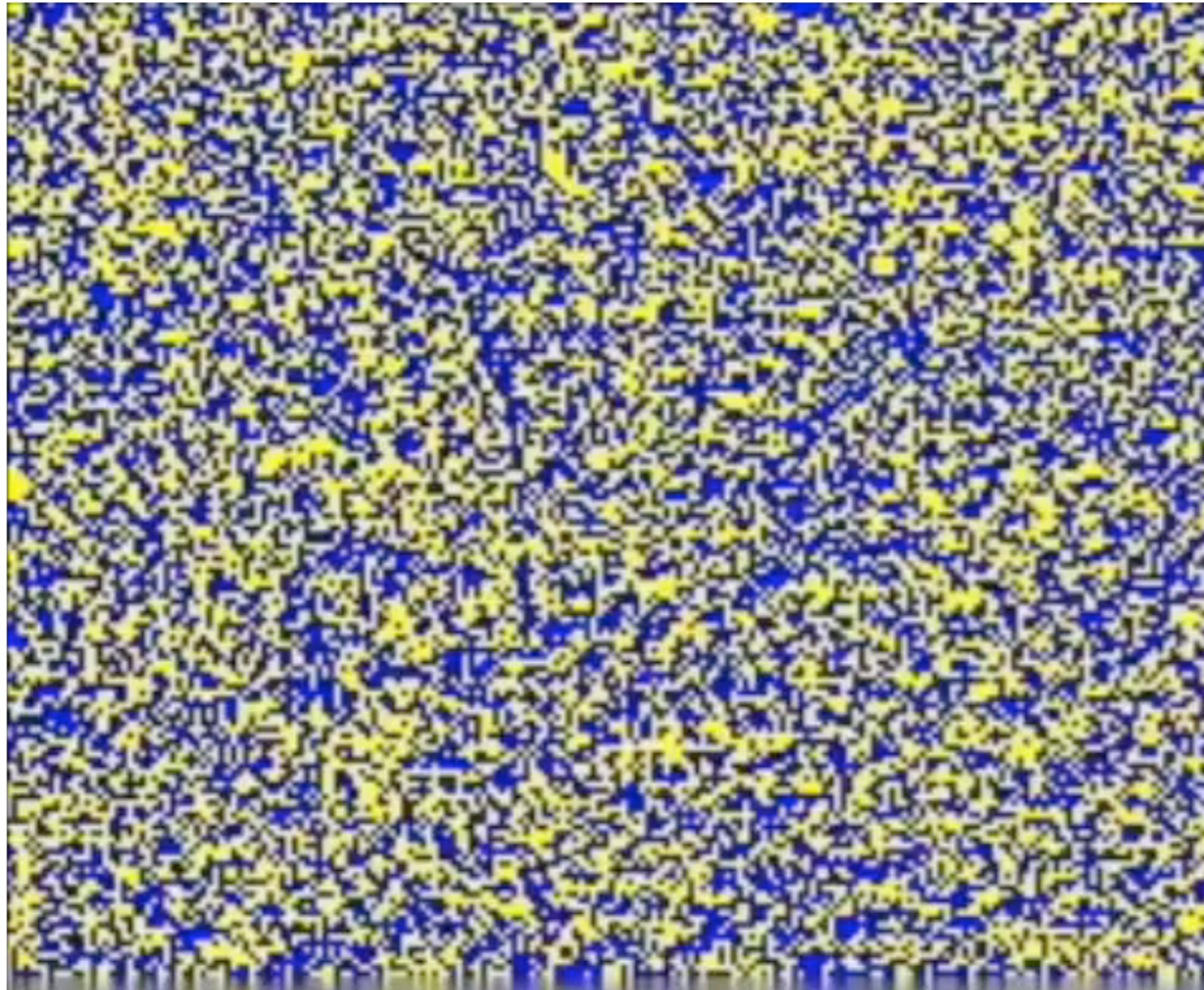
Two Dimensions: Evolution to Ground State



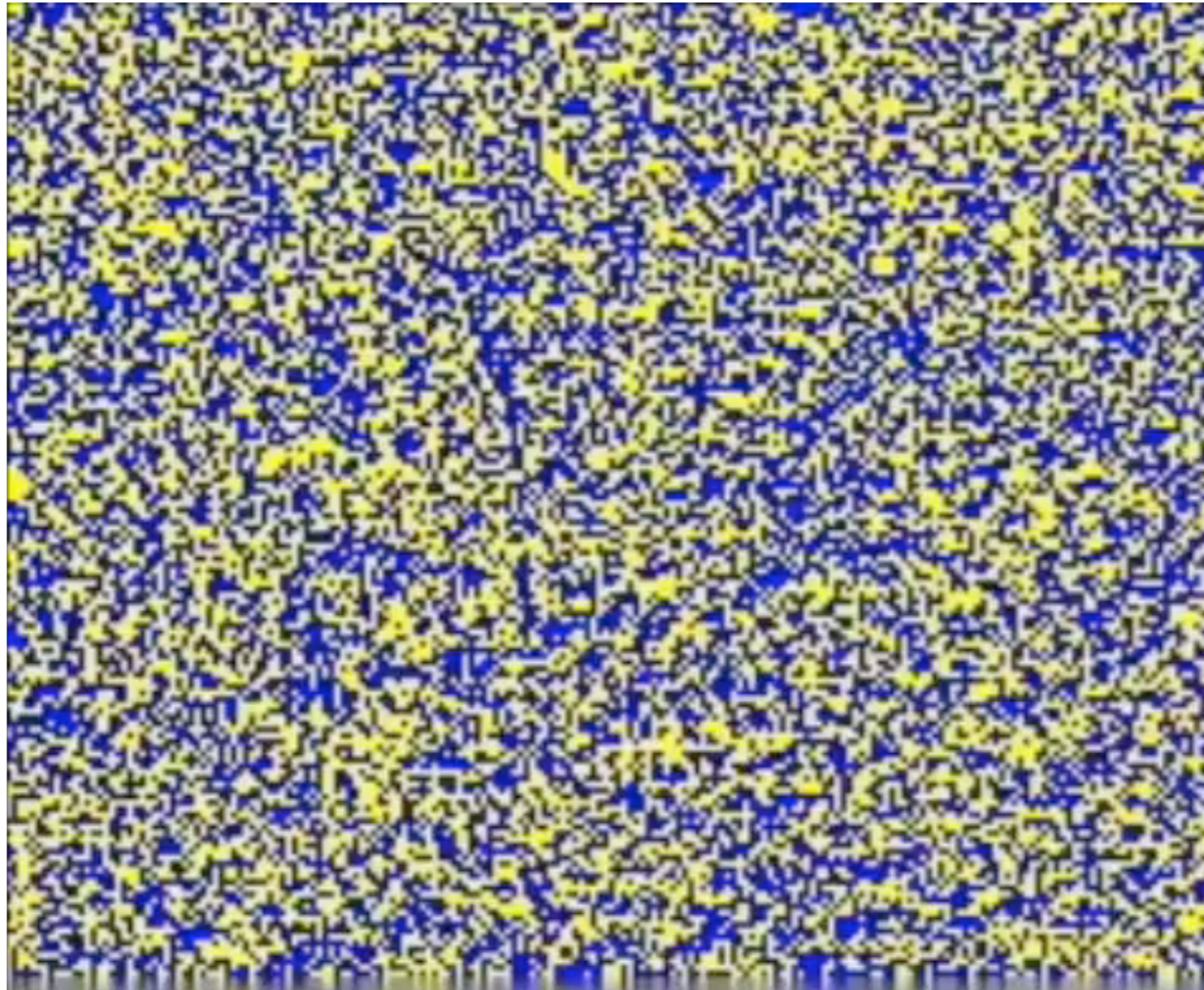
Two Dimensions: Evolution to Ground State



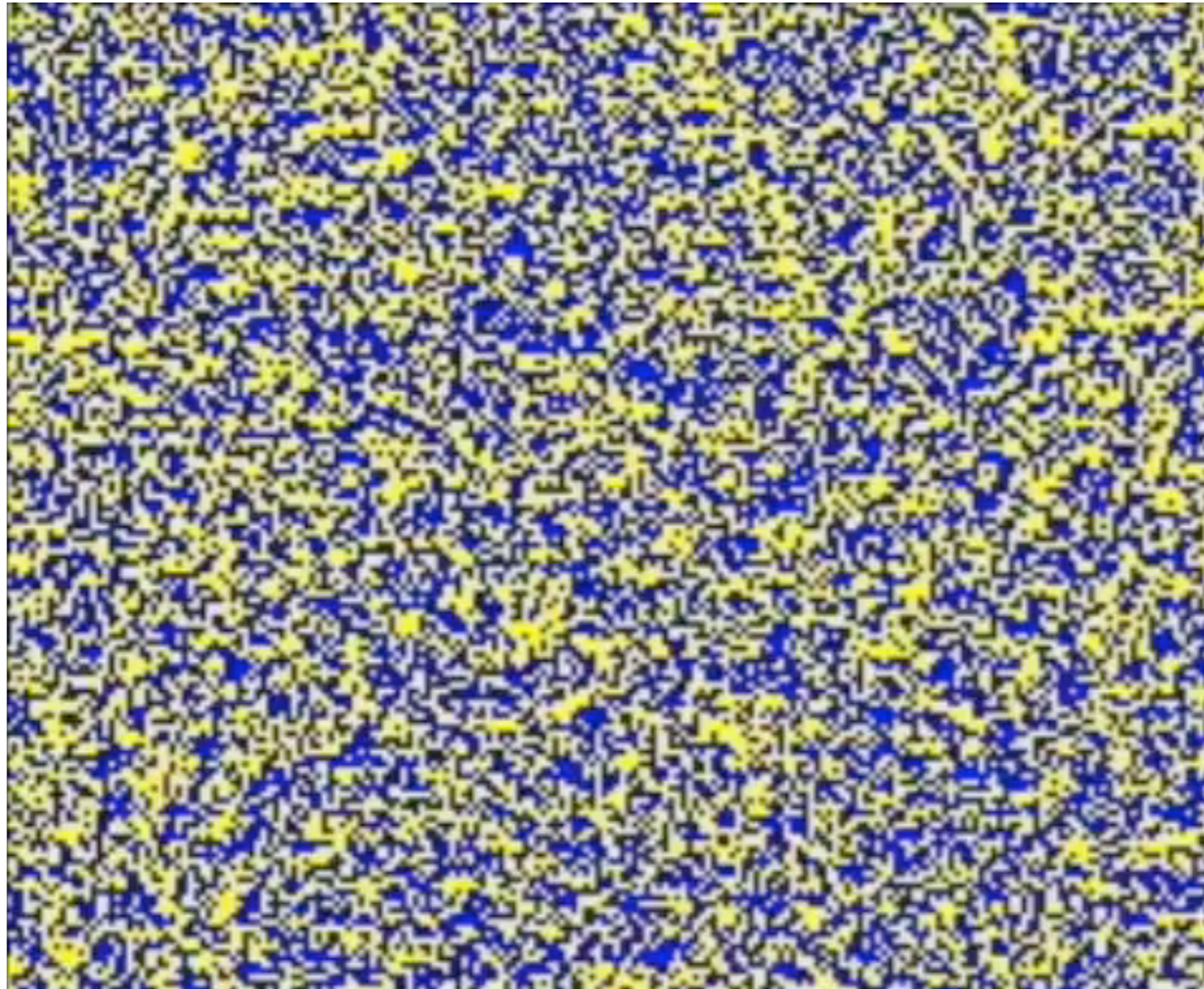
Two Dimensions: Evolution to Stripe State



Two Dimensions: Evolution to Stripe State



Two Dimensions: Evolution to Diagonal Stripe State



Two Dimensions

Question: what *is* the final state?

Two Dimensions

Question: what *is* the final state?

Answer from simulations: (Spirin, Krapivsky, & SR 01, 02)

ground state with probability $\approx 2/3$

stripe state with probability $\approx 1/3$

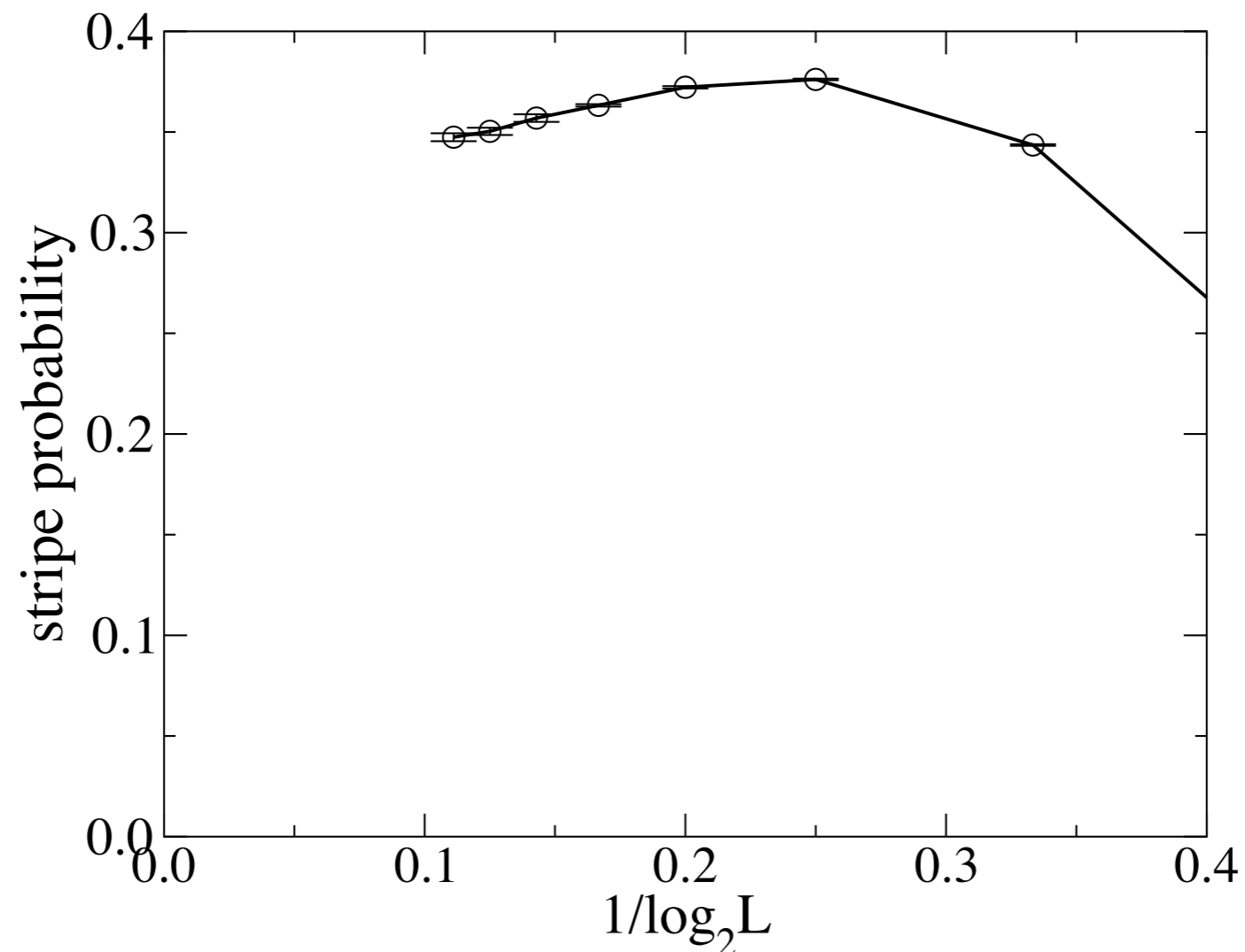
Two Dimensions

Question: what *is* the final state?

Answer from simulations: (Spirin, Krapivsky, & SR 01, 02)

ground state with probability $\approx 2/3$

stripe state with probability $\approx 1/3$



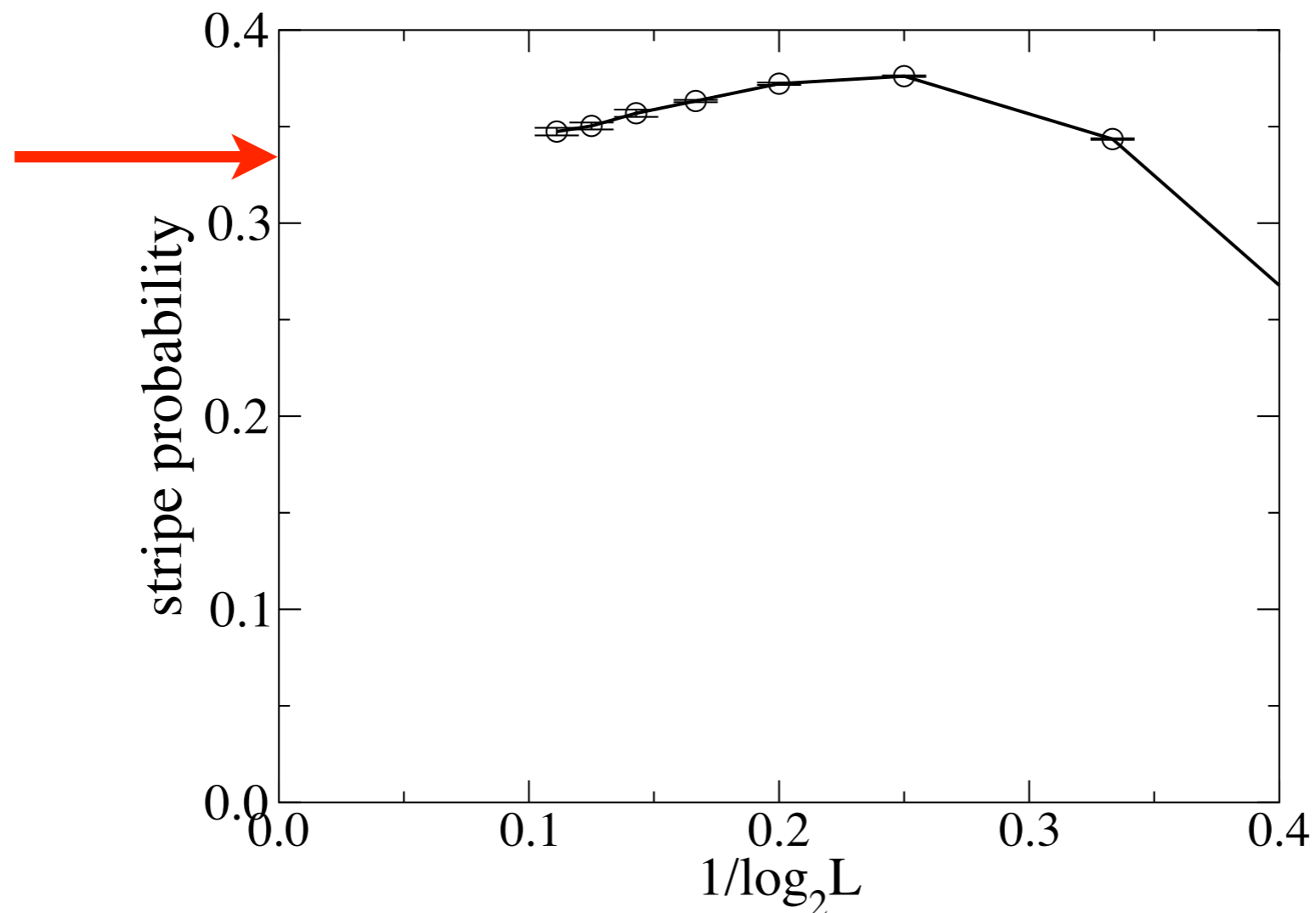
Two Dimensions

Question: what *is* the final state?

Answer from simulations: (Spirin, Krapivsky, & SR 01, 02)

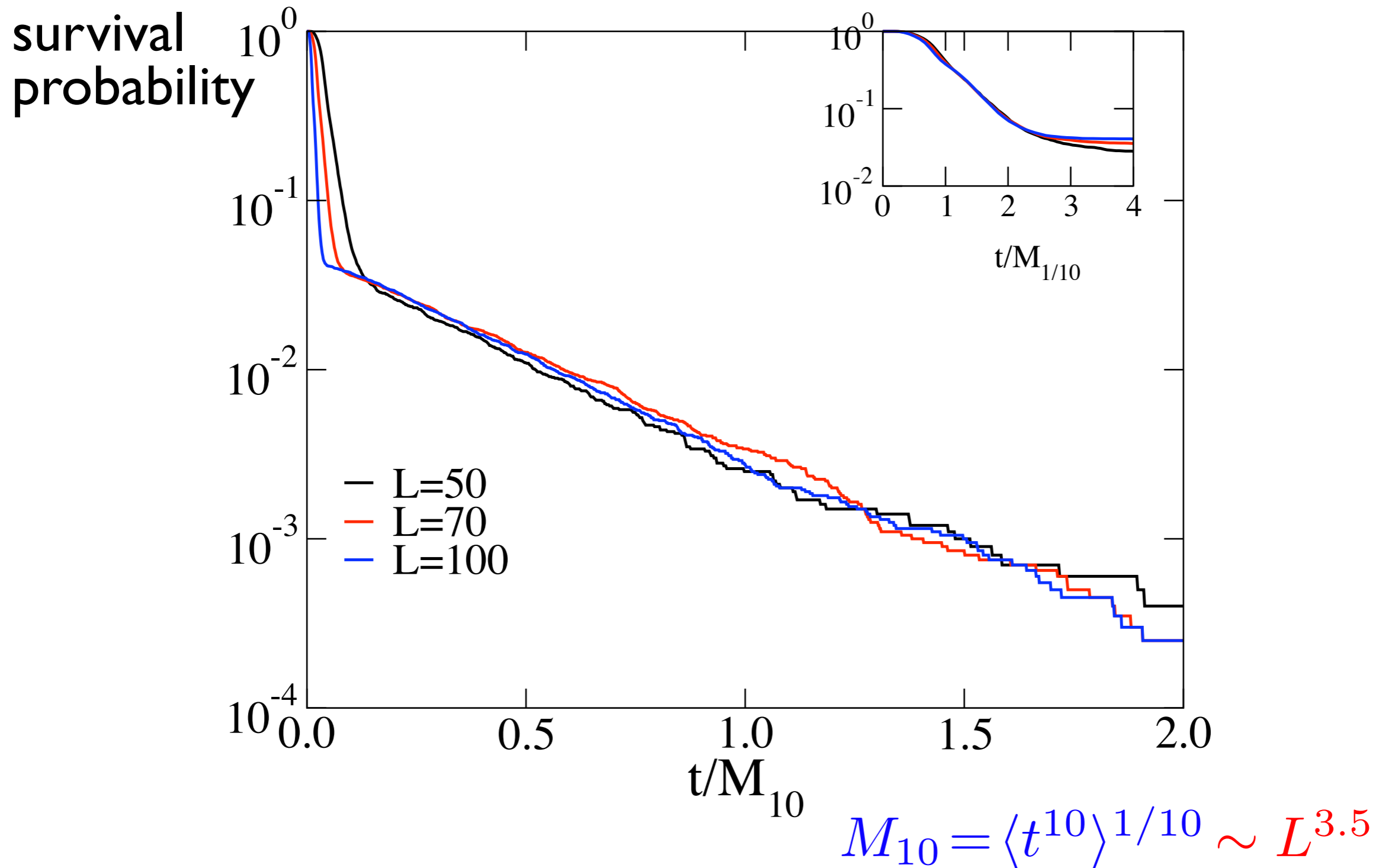
ground state with probability $\approx 2/3$

stripe state with probability $\approx 1/3$



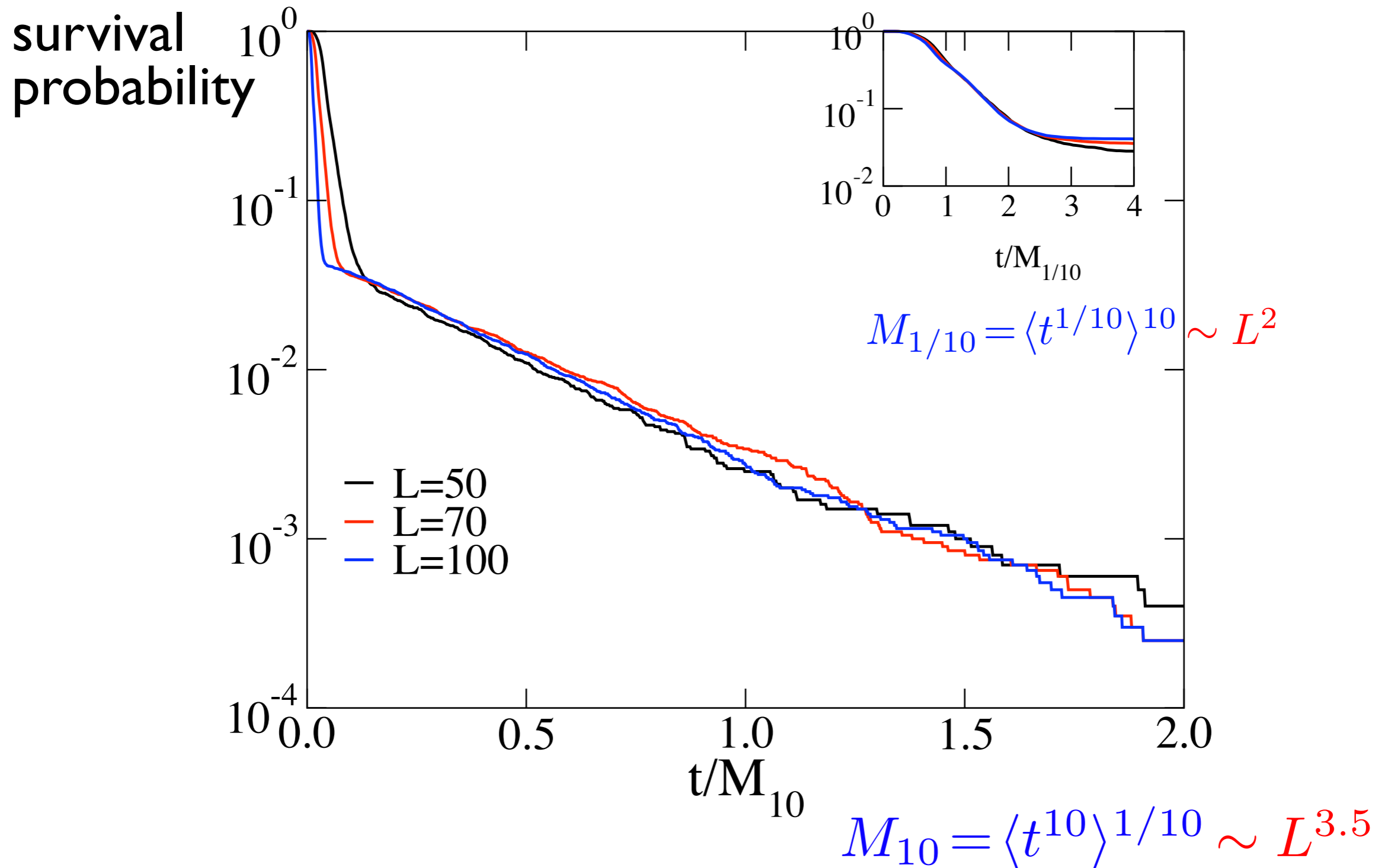
Two Dimensions

How long to reach the final state?



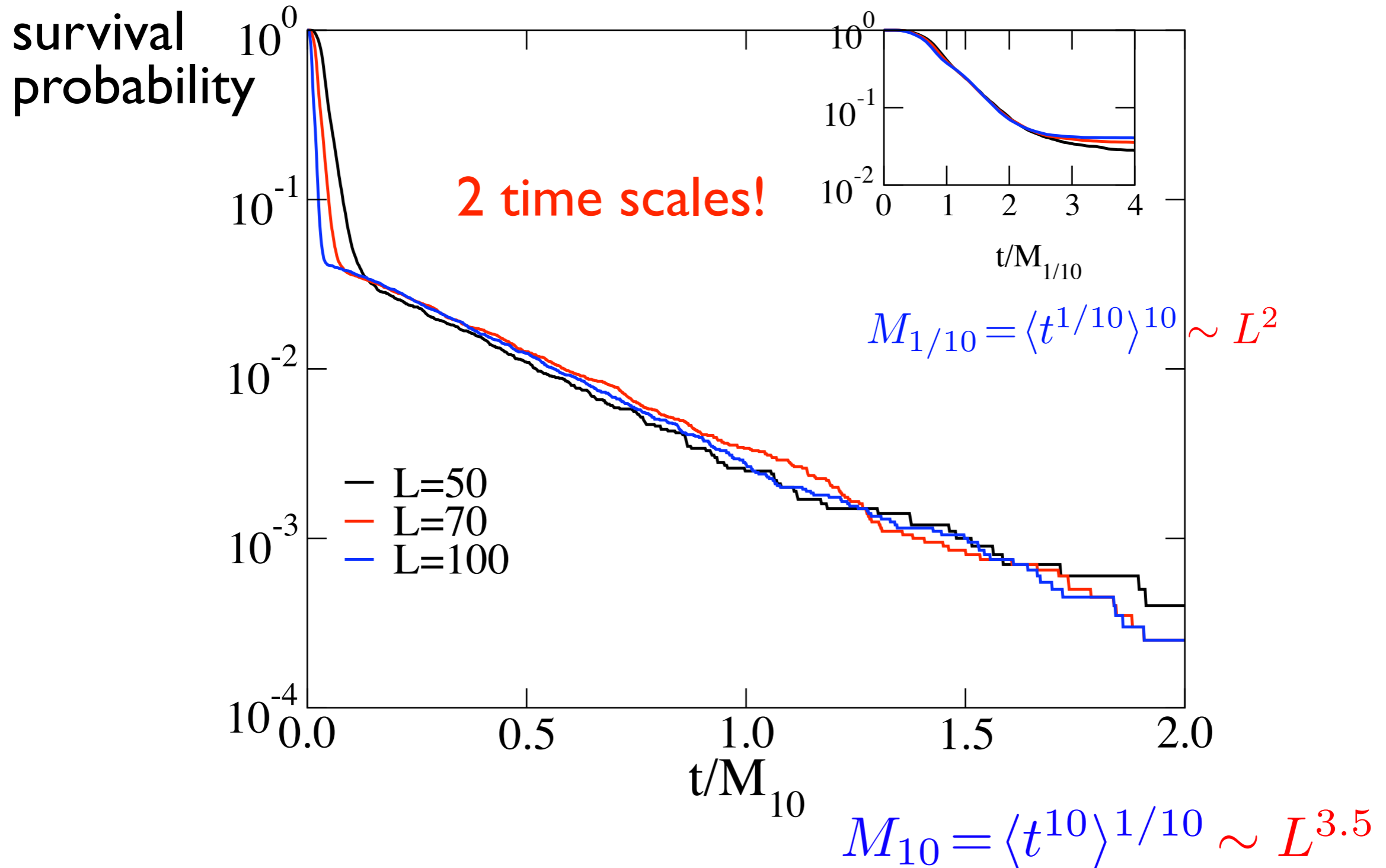
Two Dimensions

How long to reach the final state?



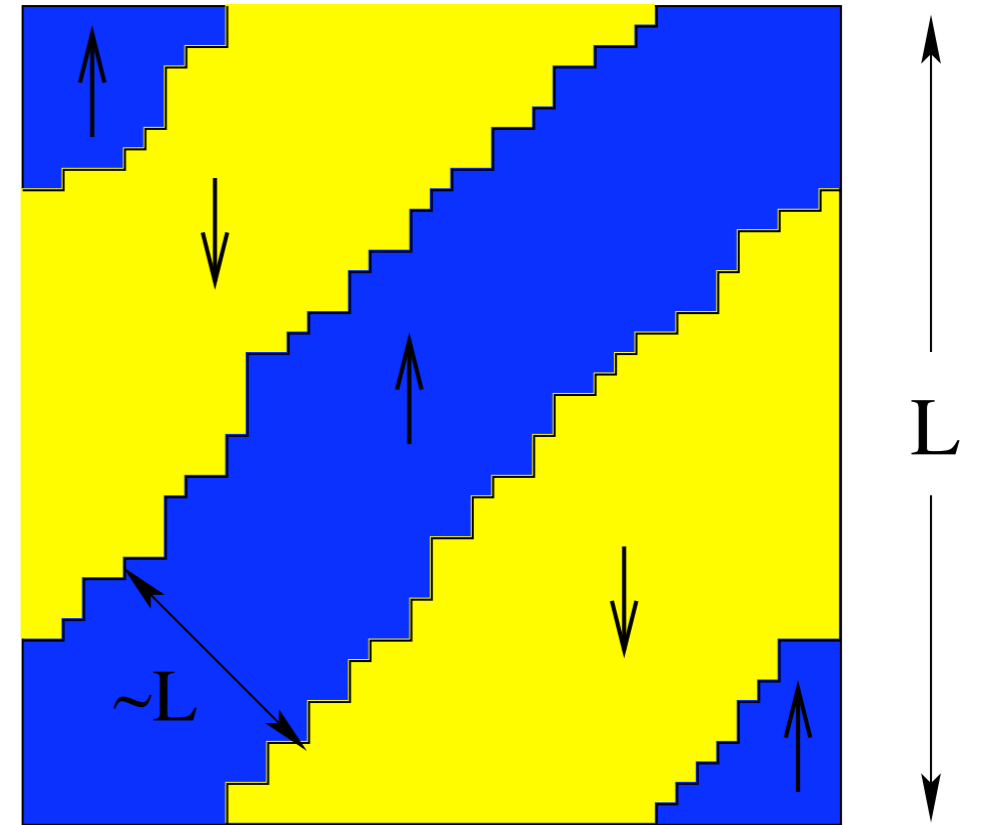
Two Dimensions

How long to reach the final state?



Two time-scale relaxation

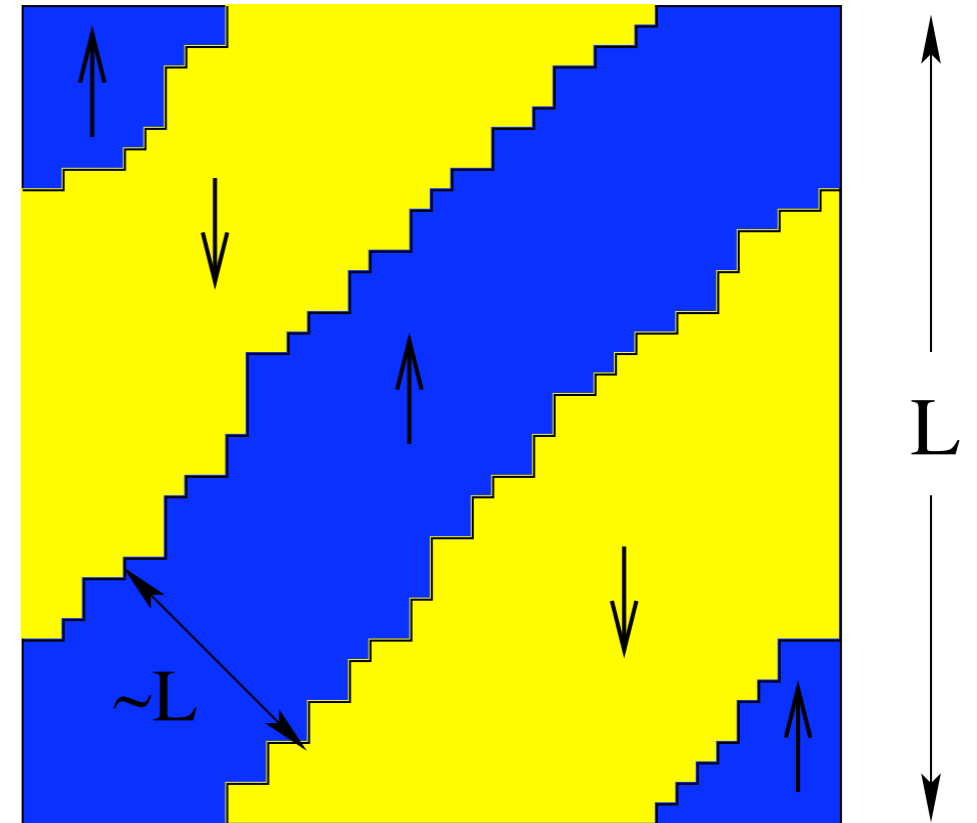
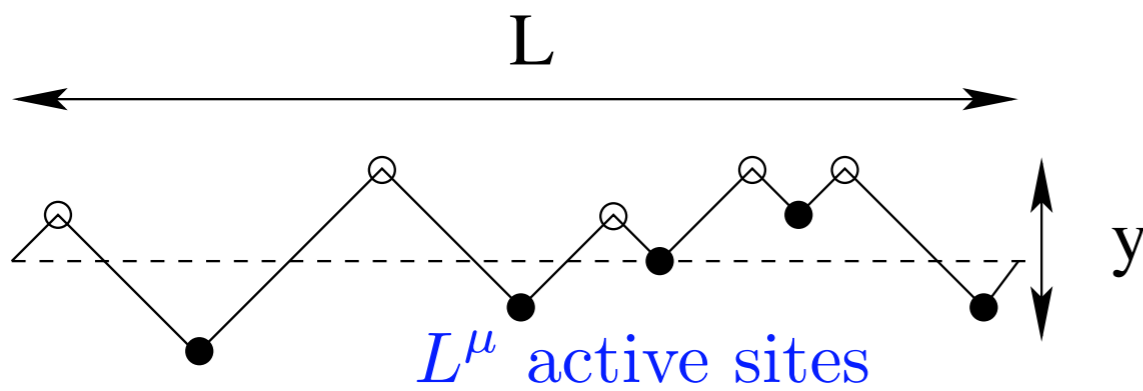
95% short-lived, but 5% long lived because of diagonal stripes!



Two time-scale relaxation

95% short-lived, but 5% long lived because of diagonal stripes!

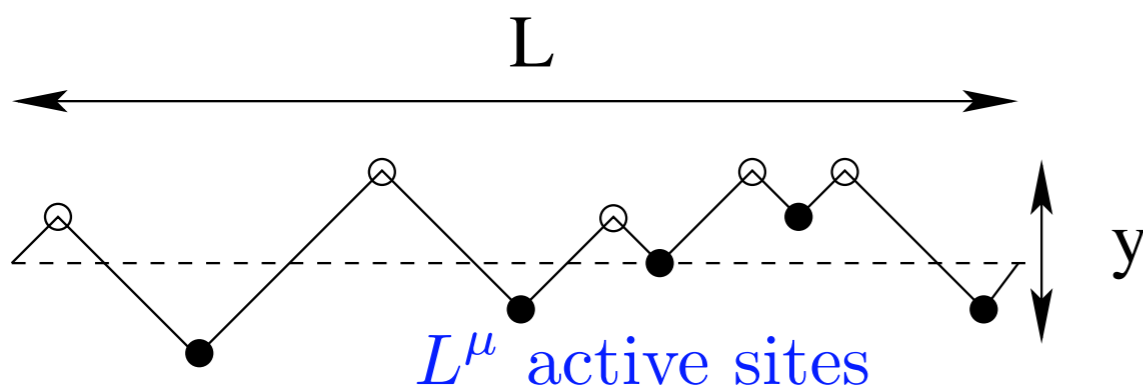
Diagonal stripe dynamics: (Plischke et al 87)



Two time-scale relaxation

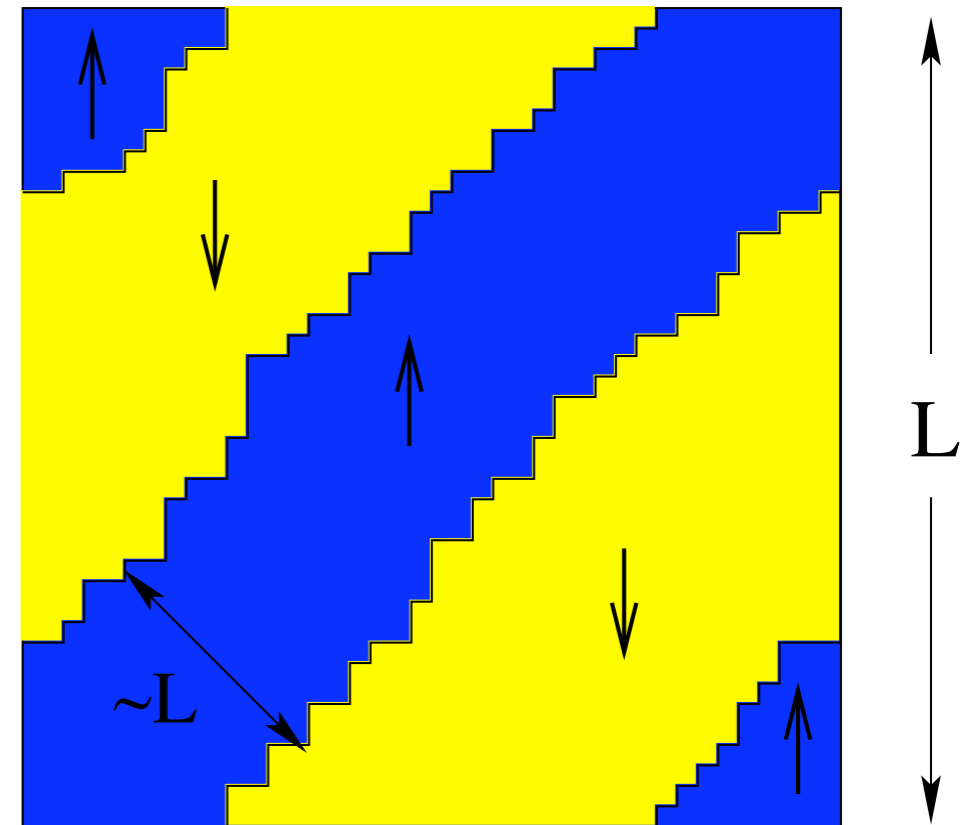
95% short-lived, but 5% long lived because of diagonal stripes!

Diagonal stripe dynamics: (Plischke et al 87)



$$\Delta t = 1, \quad L^\mu \text{ events} \quad \rightarrow \quad \Delta y_{\text{cm}} \sim L^{\mu/2} / L$$

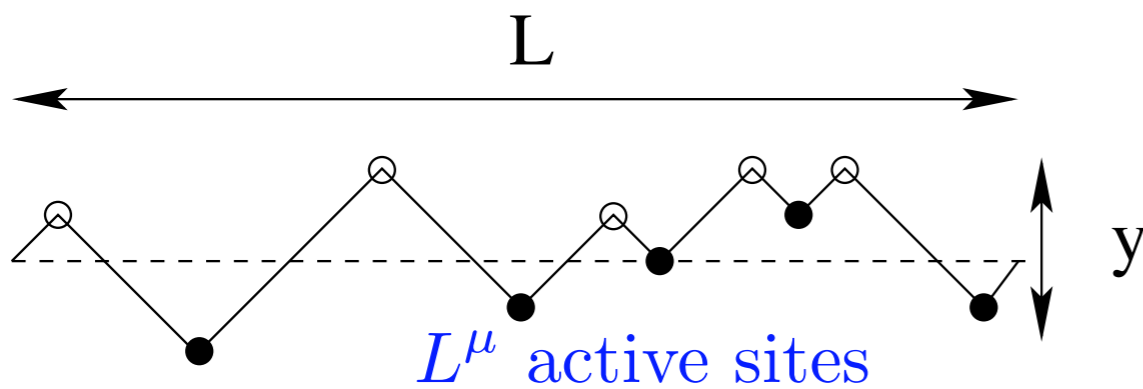
$$\rightarrow \quad D(L) \sim L^{\mu-2}$$



Two time-scale relaxation

95% short-lived, but 5% long lived because of diagonal stripes!

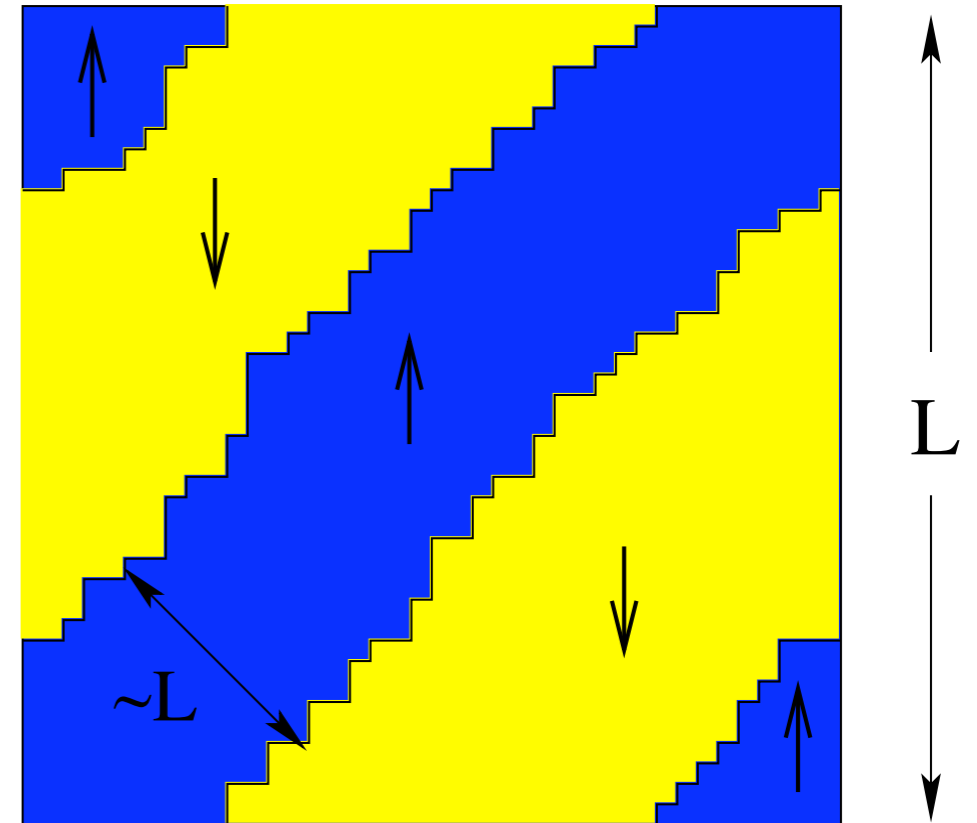
Diagonal stripe dynamics: (Plischke et al 87)



$$\Delta t = 1, \quad L^\mu \text{ events} \quad \rightarrow \quad \Delta y_{\text{cm}} \sim L^{\mu/2} / L$$

$$\rightarrow \quad D(L) \sim L^{\mu-2}$$

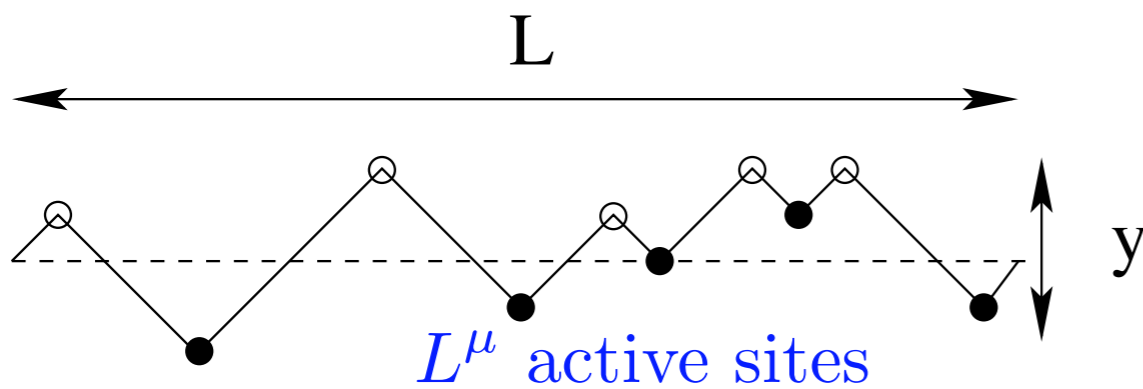
$$\text{survival time } \tau \sim L^2 / D \quad \sim \quad L^{4-\mu}$$



Two time-scale relaxation

95% short-lived, but 5% long lived because of diagonal stripes!

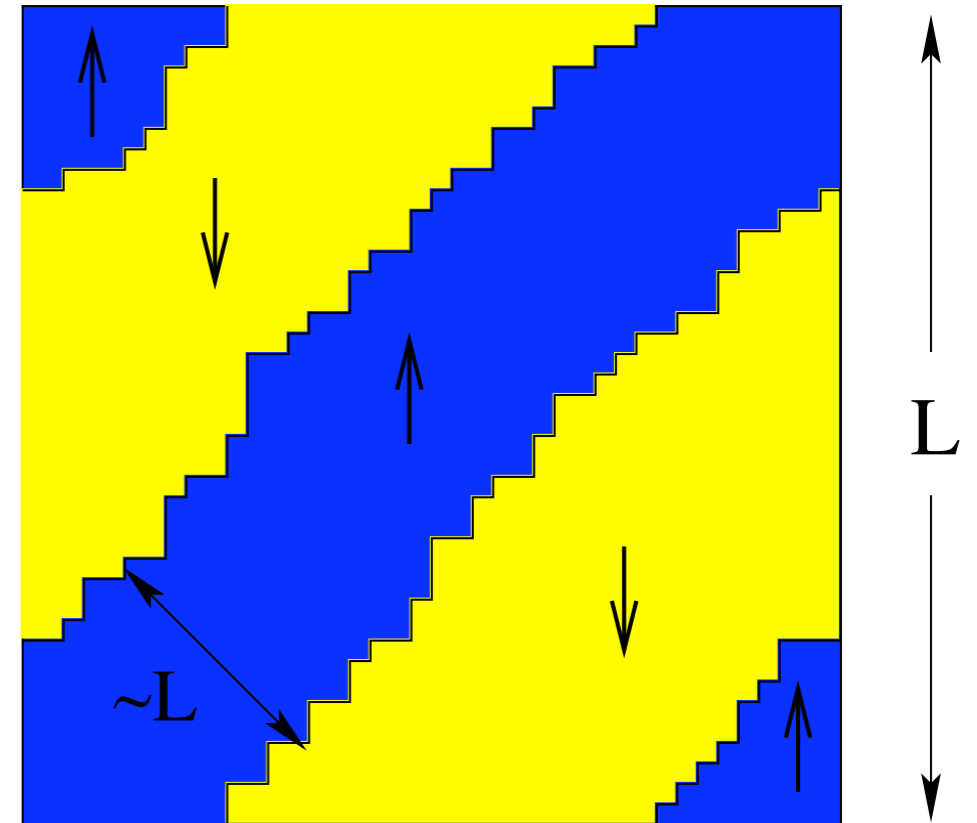
Diagonal stripe dynamics: (Plischke et al 87)



$$\Delta t = 1, \quad L^\mu \text{ events} \quad \rightarrow \quad \Delta y_{\text{cm}} \sim L^{\mu/2} / L$$

$$\rightarrow \quad D(L) \sim L^{\mu-2}$$

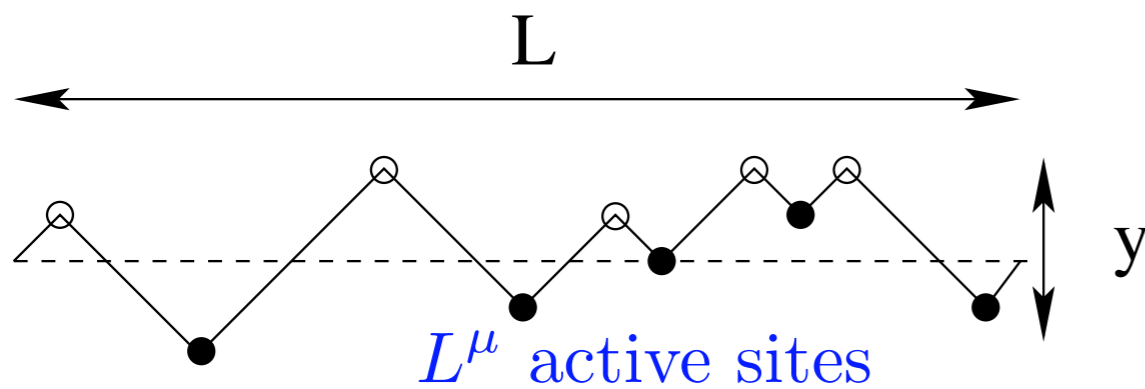
survival time $\tau \sim L^2 / D \sim L^{4-\mu}$ but $\mu = 1/2$



Two time-scale relaxation

95% short-lived, but 5% long lived because of diagonal stripes!

Diagonal stripe dynamics: (Plischke et al 87)

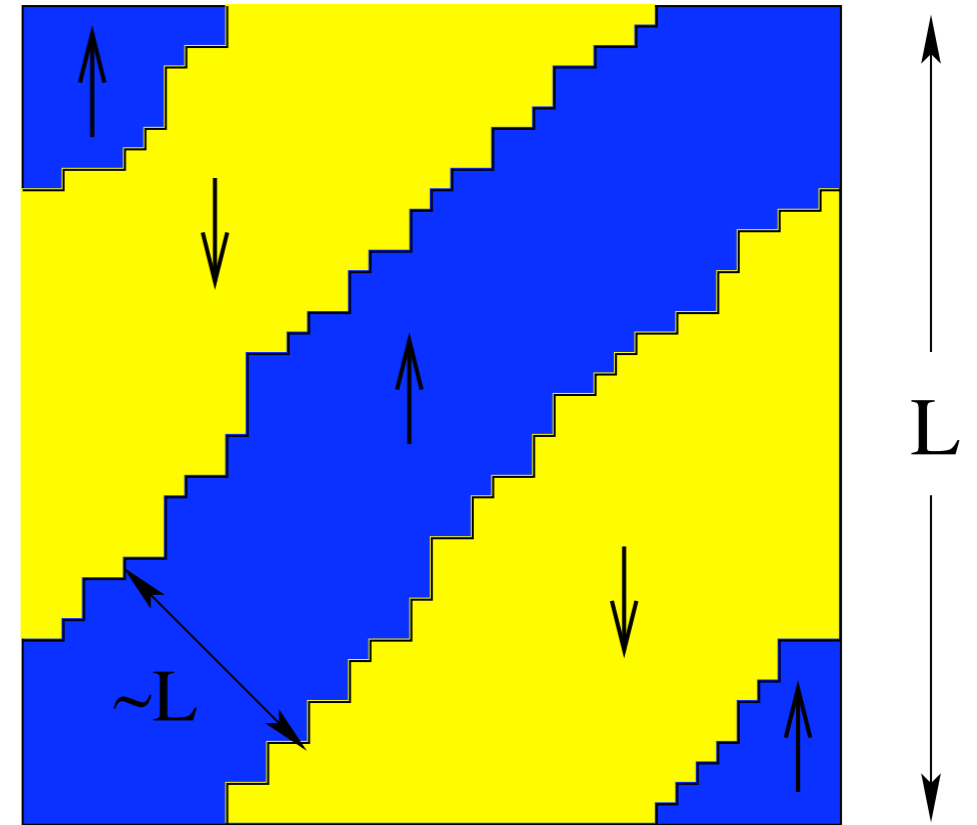


$$\Delta t = 1, \quad L^\mu \text{ events} \quad \rightarrow \quad \Delta y_{\text{cm}} \sim L^{\mu/2} / L$$

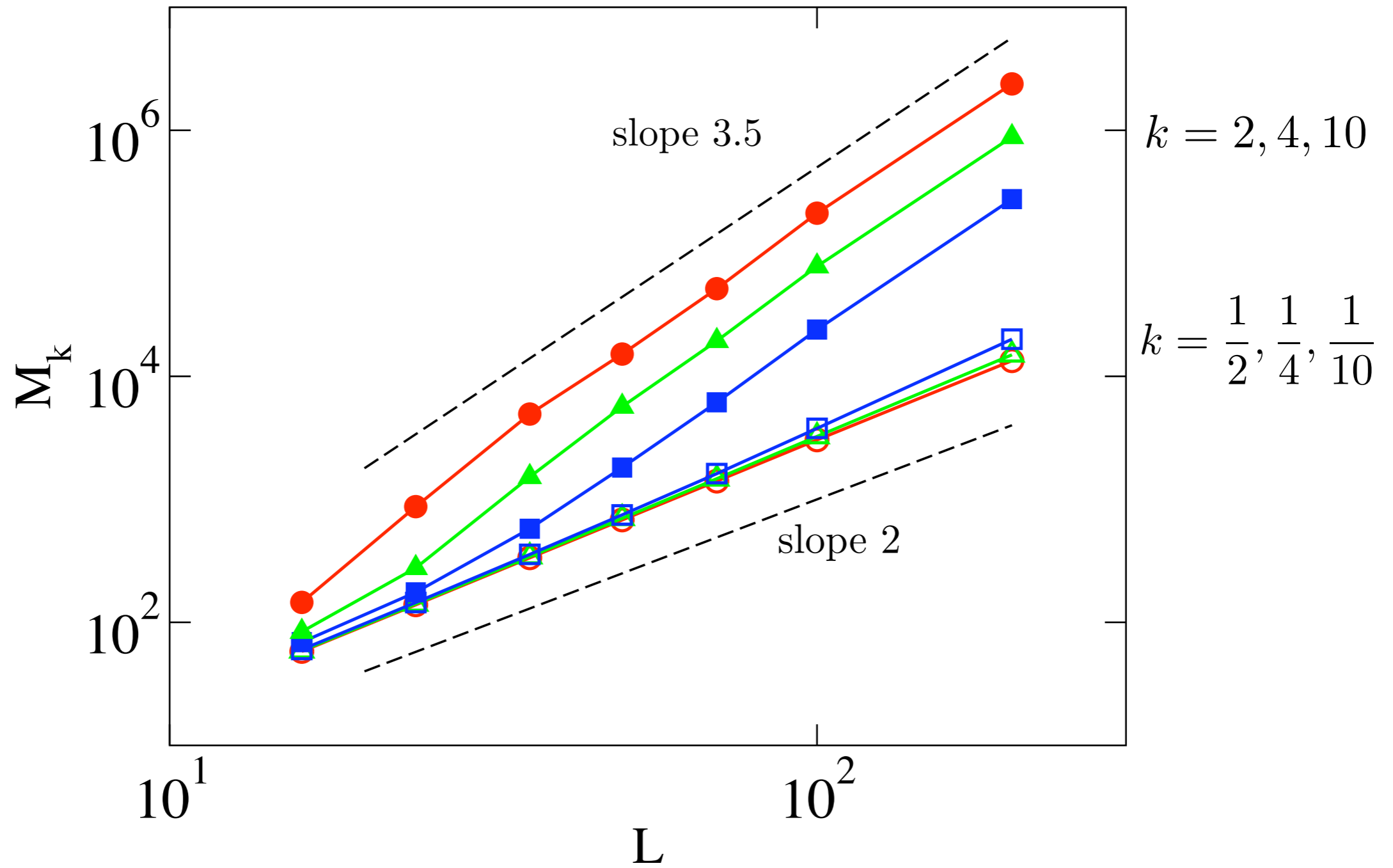
$$\rightarrow \quad D(L) \sim L^{\mu-2}$$

$$\text{survival time } \tau \sim L^2 / D \quad \sim \quad L^{4-\mu} \quad \text{but } \mu = 1/2$$

$$\sim \quad L^{3.5}$$



Multiscaling in moments of the stopping time



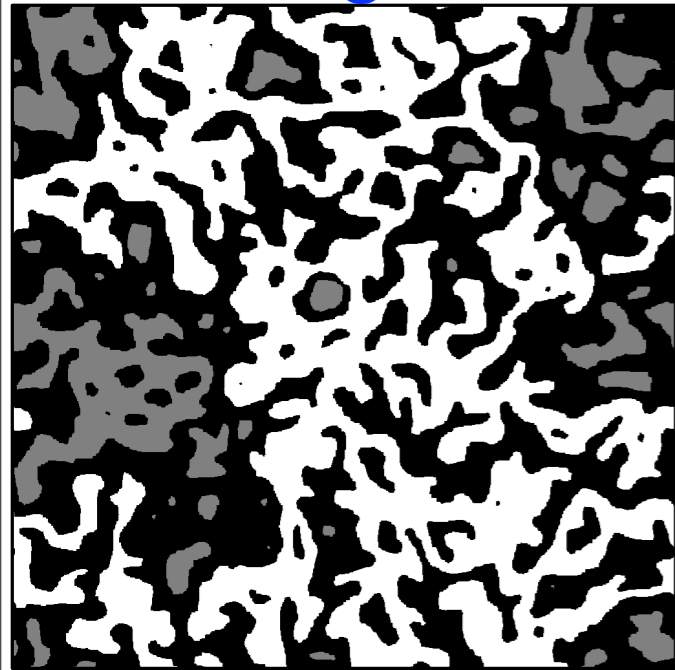
Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system

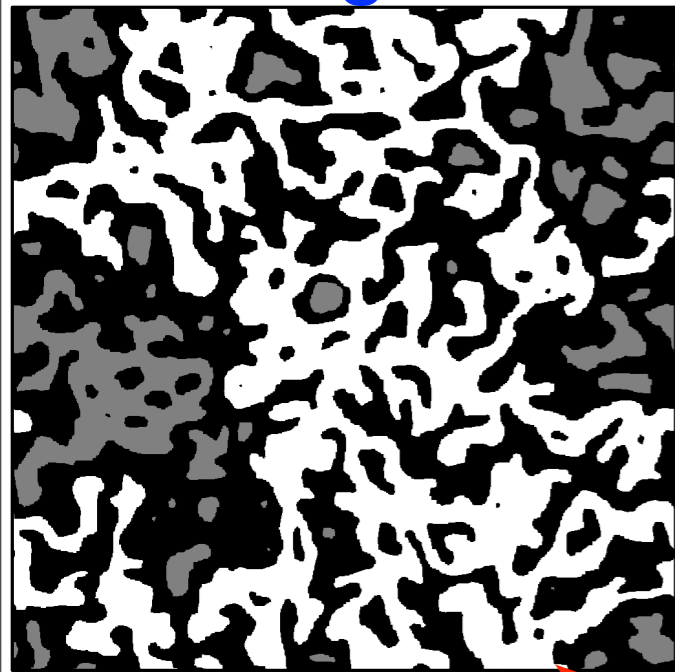


$t=200$

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system



t=200

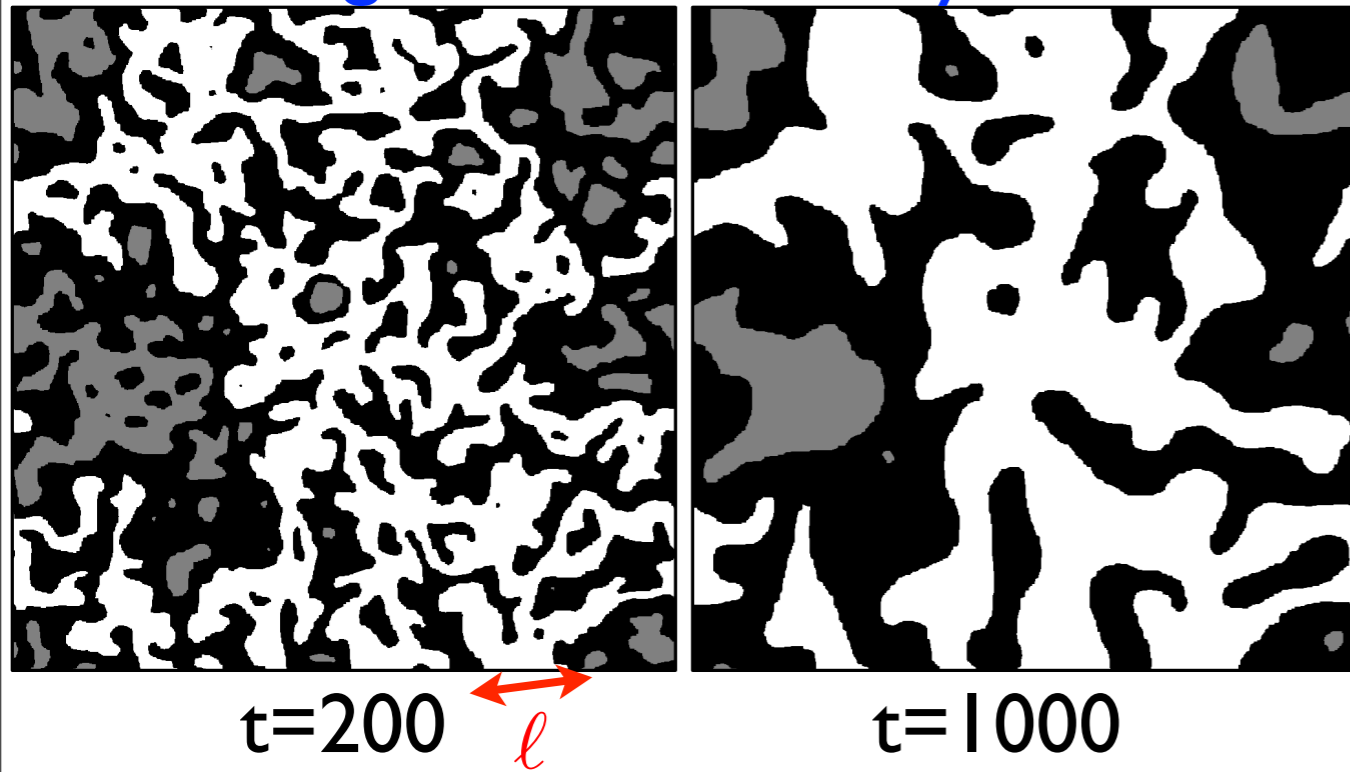


critical pt of continuum
percolation $a \ll \ell(t) \ll L$

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system

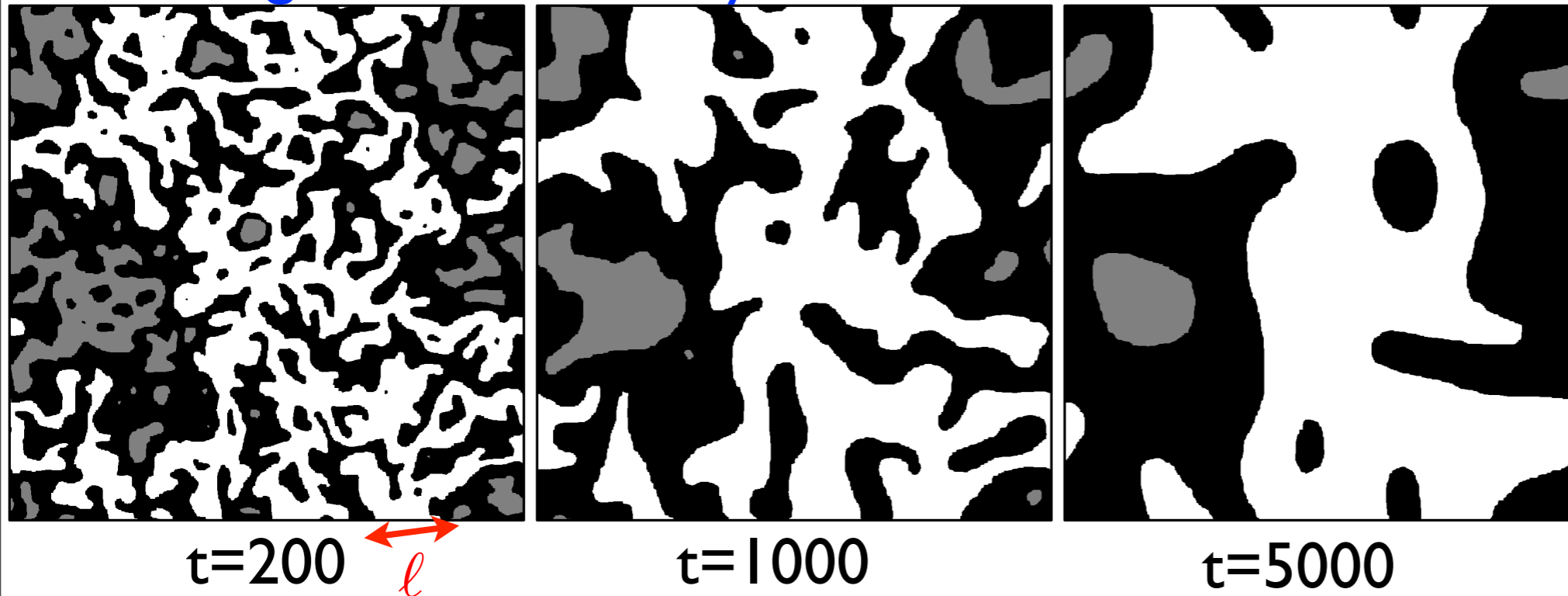


critical pt of continuum
percolation $a \ll \ell(t) \ll L$

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system

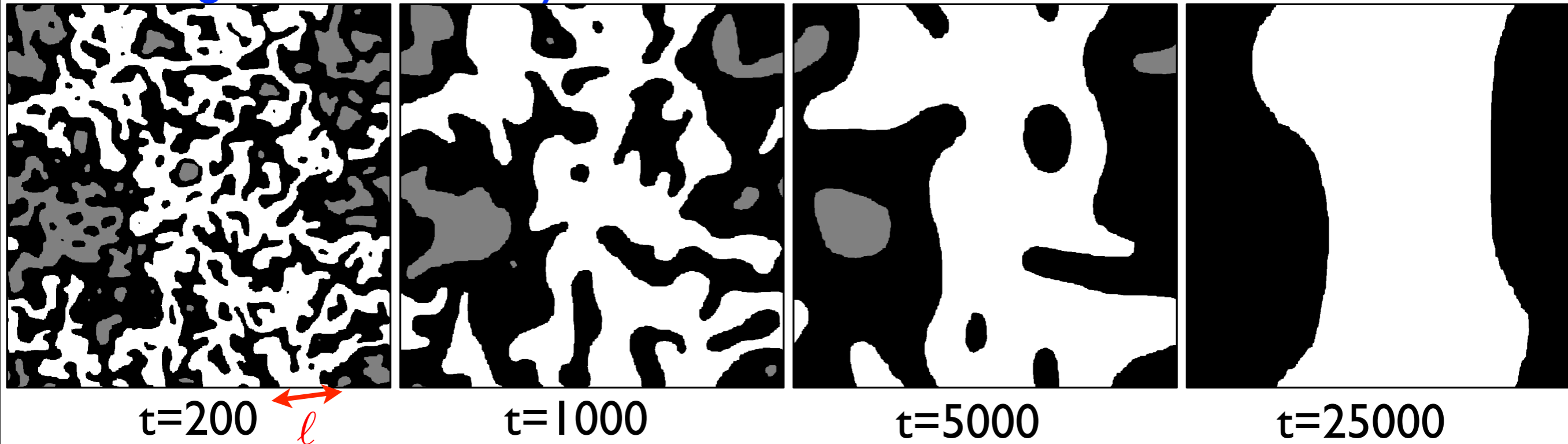


critical pt of continuum
percolation $a \ll \ell(t) \ll L$

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system

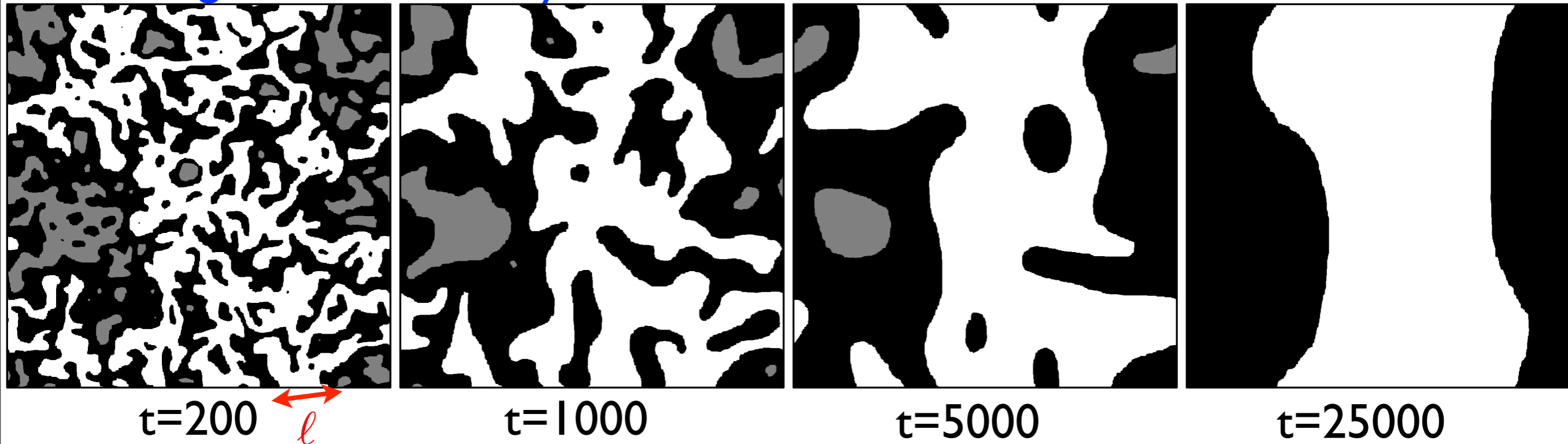


critical pt of continuum
percolation $a \ll \ell(t) \ll L$

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system



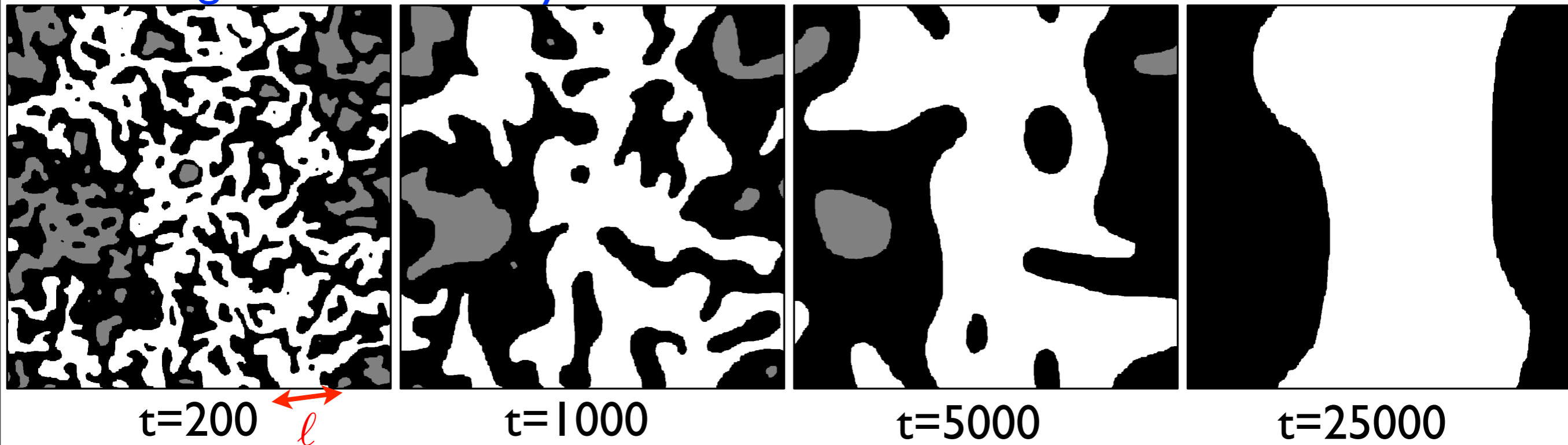
critical pt of continuum
percolation $a \ll \ell(t) \ll L$

deterministic, curvature-driven evolution for $\ell(t) \gg a$
→ Invariant topology in the coarsening regime

Two Dimensions

Answer from percolation mapping: (Barros, Krapvisky, & SR 09)

coarsening of 1024x1024 system



critical pt of continuum percolation $a \ll \ell(t) \ll L$

deterministic, curvature-driven evolution for $\ell(t) \gg a$
 \rightarrow Invariant topology in the coarsening regime

$$\text{FBC: } P_{\text{stripe}} = \frac{\sqrt{3}}{2\pi} \lambda(r) {}_3F_2 \left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda \right) \quad \lambda(r) = \left(\frac{1-k}{1+k} \right)^2$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2\pi} \ln \frac{27}{16} = 0.3558 \dots \quad r = \text{aspect ratio} = \frac{2K(k^2)}{K(1-k^2)}$$

$$\text{PBC: } P_{\text{stripe}} \approx 0.3388$$

(Cardy 1992, Watts 1996, Simmons et al. 2007)

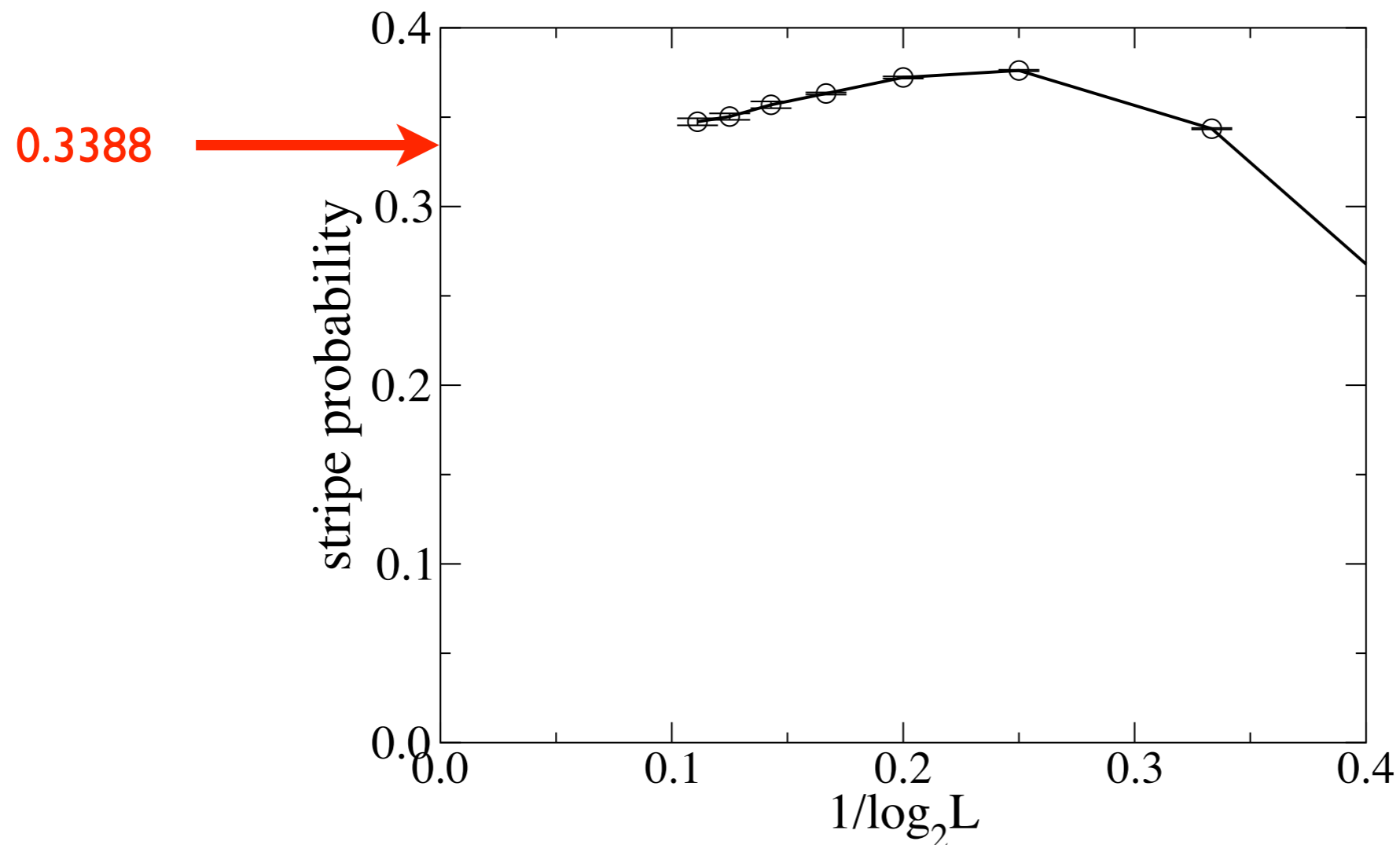
Two Dimensions

Question: what *is* the final state?

Answer from simulations: (Spirin, Krapivsky, & SR 01, 02)

ground state with probability $\approx 2/3$

stripe state with probability $\approx 1/3$



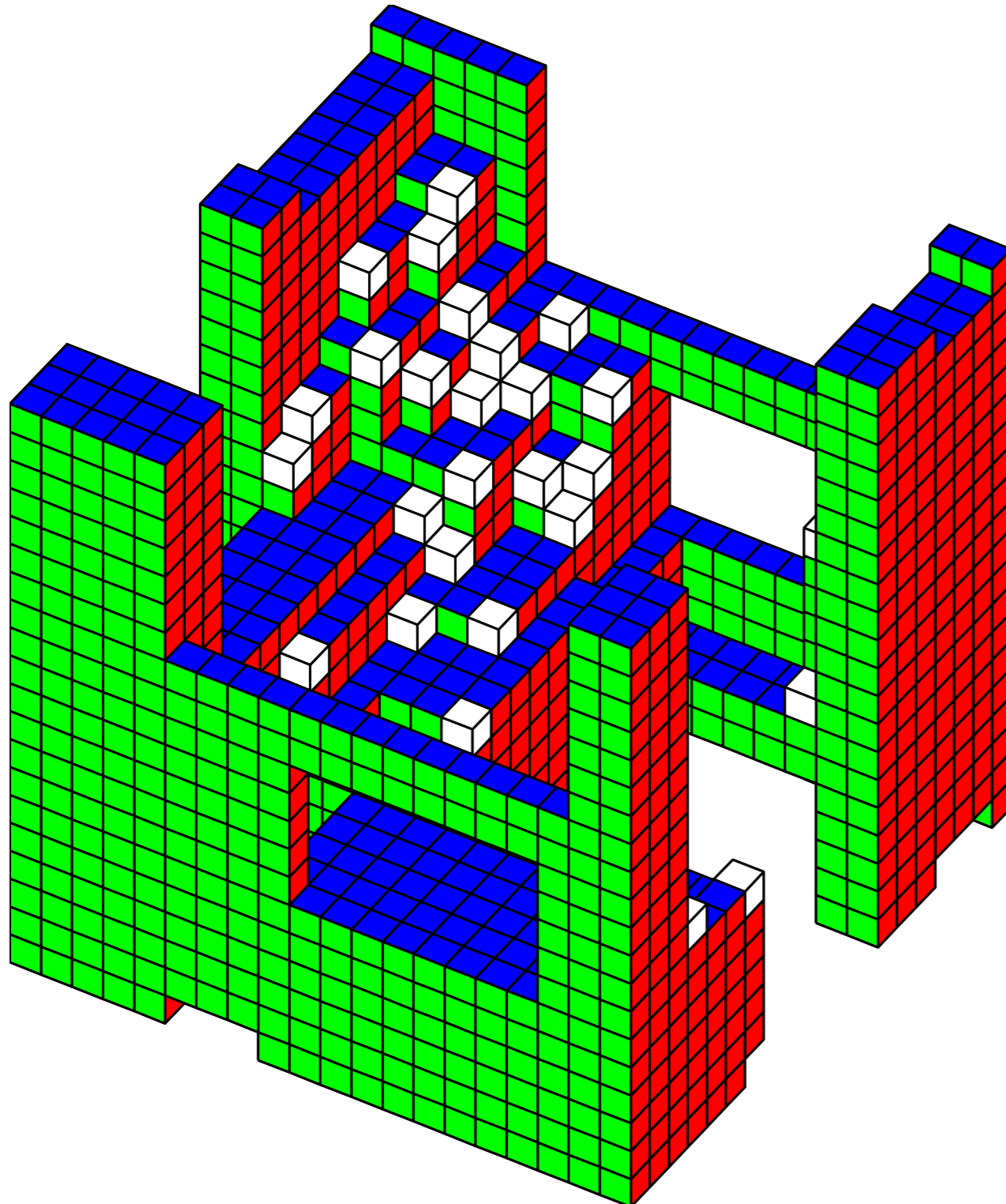
Three Dimensions (Olejarz, Krapvisky, & SR)

Three Dimensions (Olejarz, Krapvisky, & SR)

Basic result: ground state is *never* reached!

Three Dimensions (Olejarz, Krapvisky, & SR)

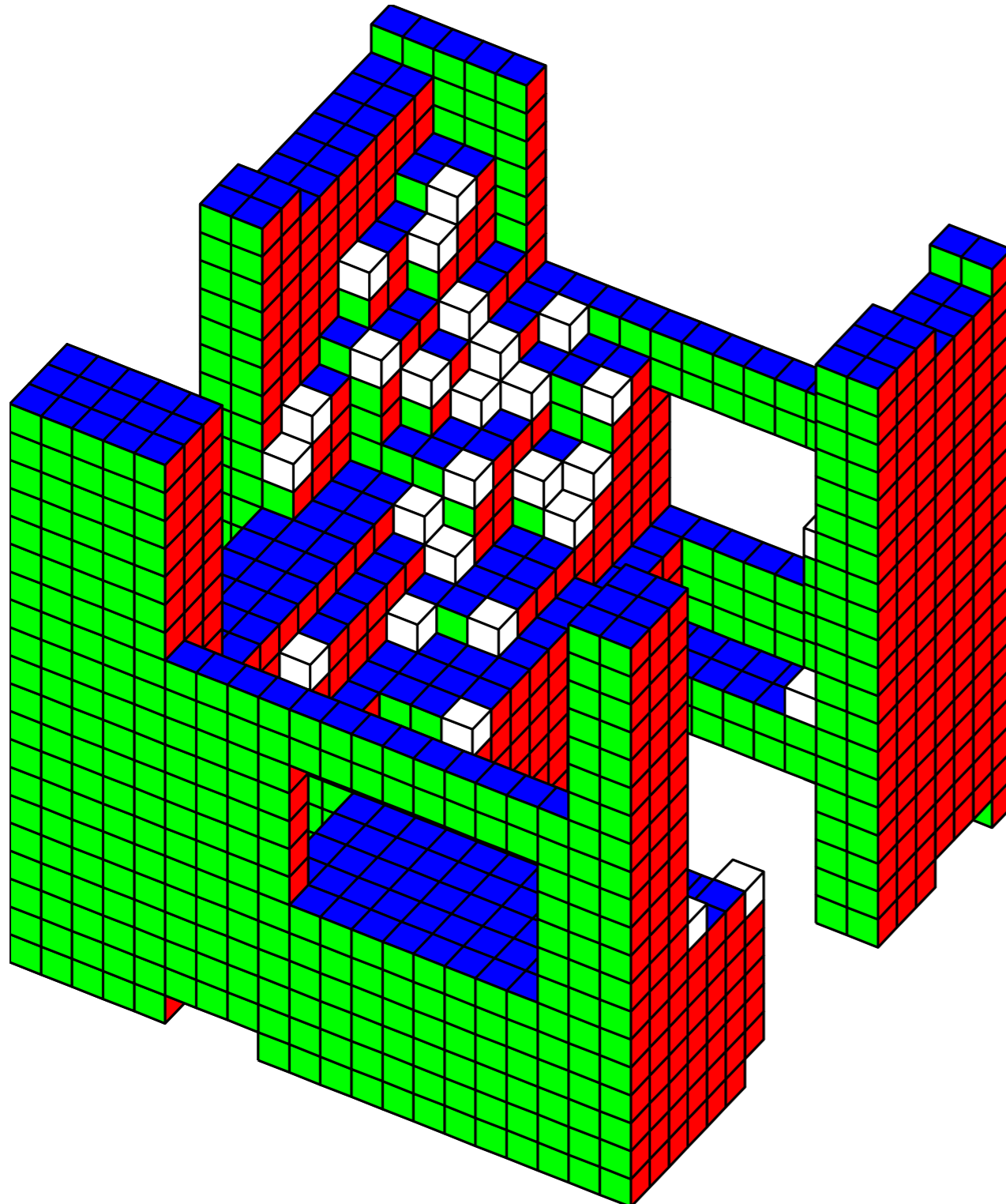
Basic result: ground state is *never* reached!
typical 20x20x20 system



Three Dimensions (Olejarz, Krapvisky, & SR)

Basic result: ground state is *never* reached!
typical 20x20x20 system

Features:



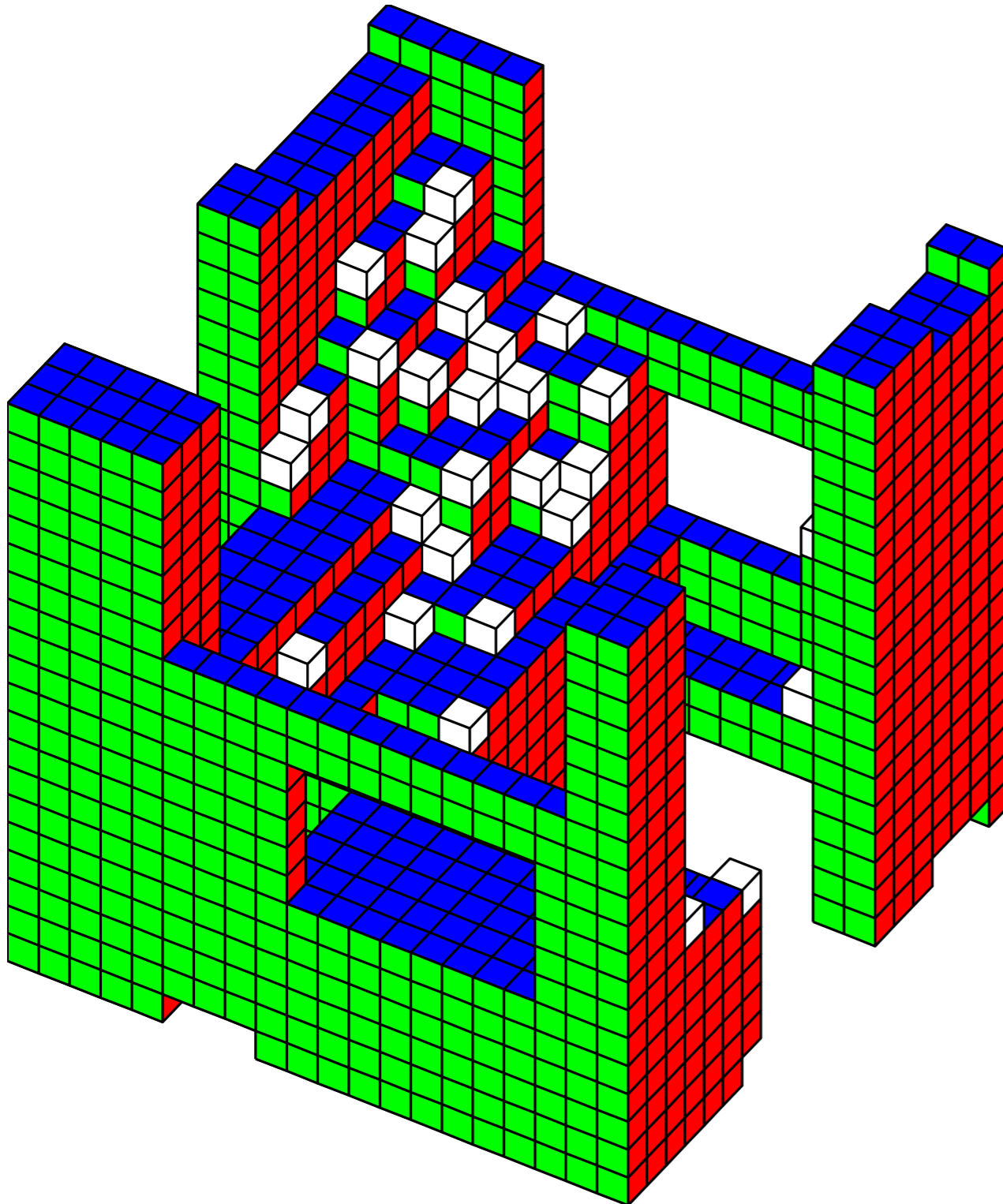
Three Dimensions (Olejarz, Krapvisky, & SR)

Basic result: ground state is *never* reached!

typical 20x20x20 system

Features:

I. Swiss cheesy



Three Dimensions (Olejarz, Krapvisky, & SR)

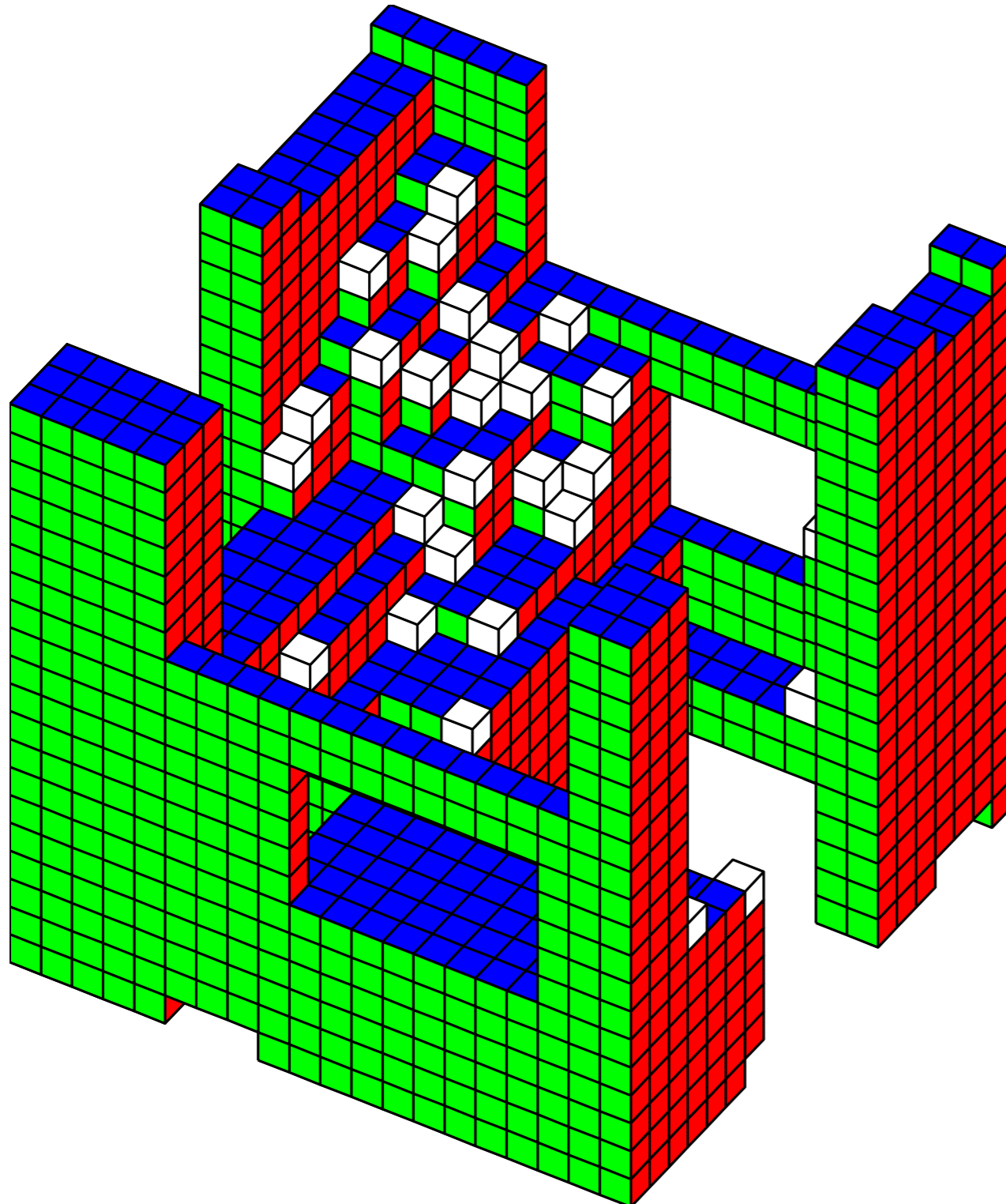
Basic result: ground state is *never* reached!

typical 20x20x20 system

Features:

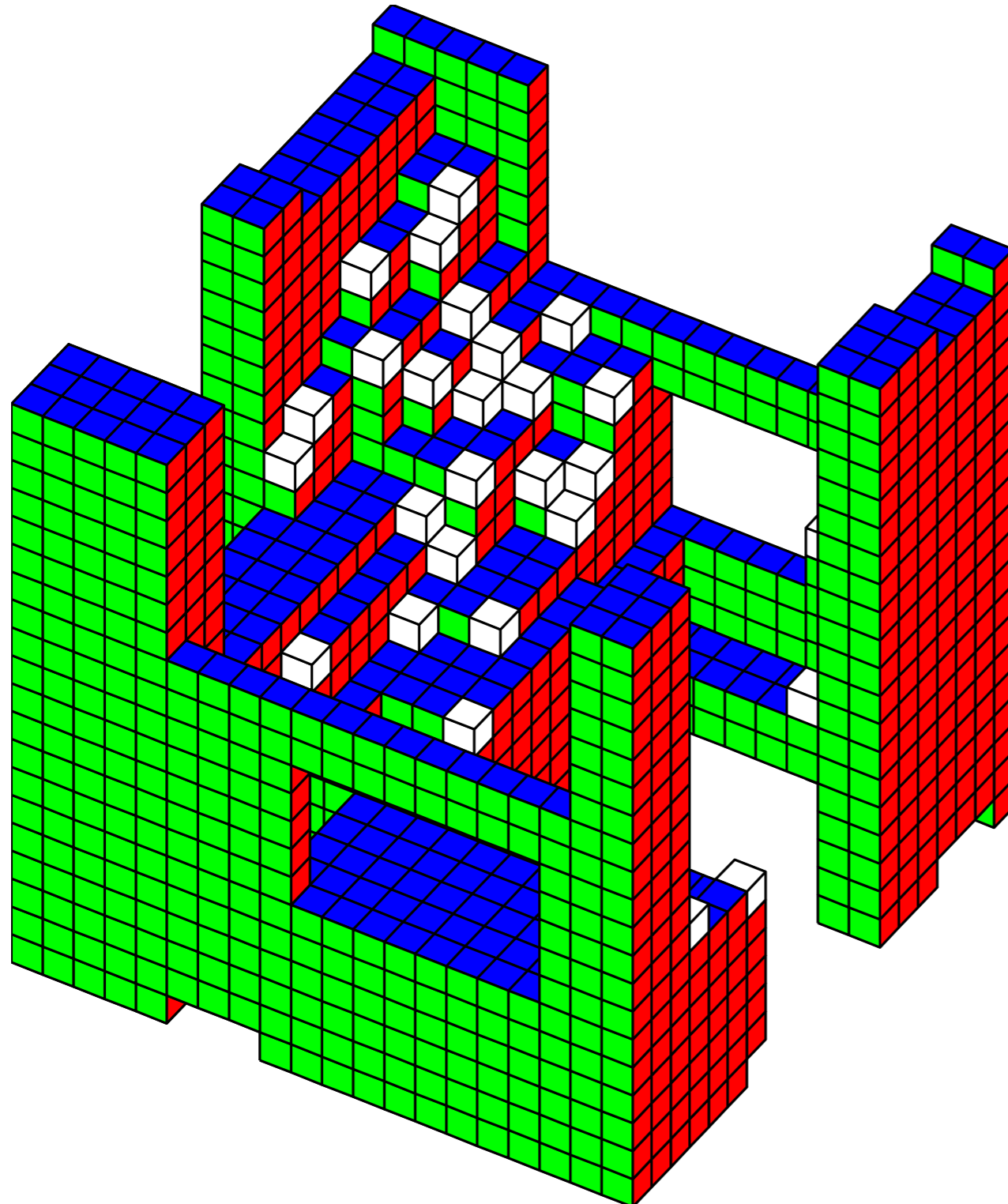
1. Swiss cheesy

2. Zero *average* curvature



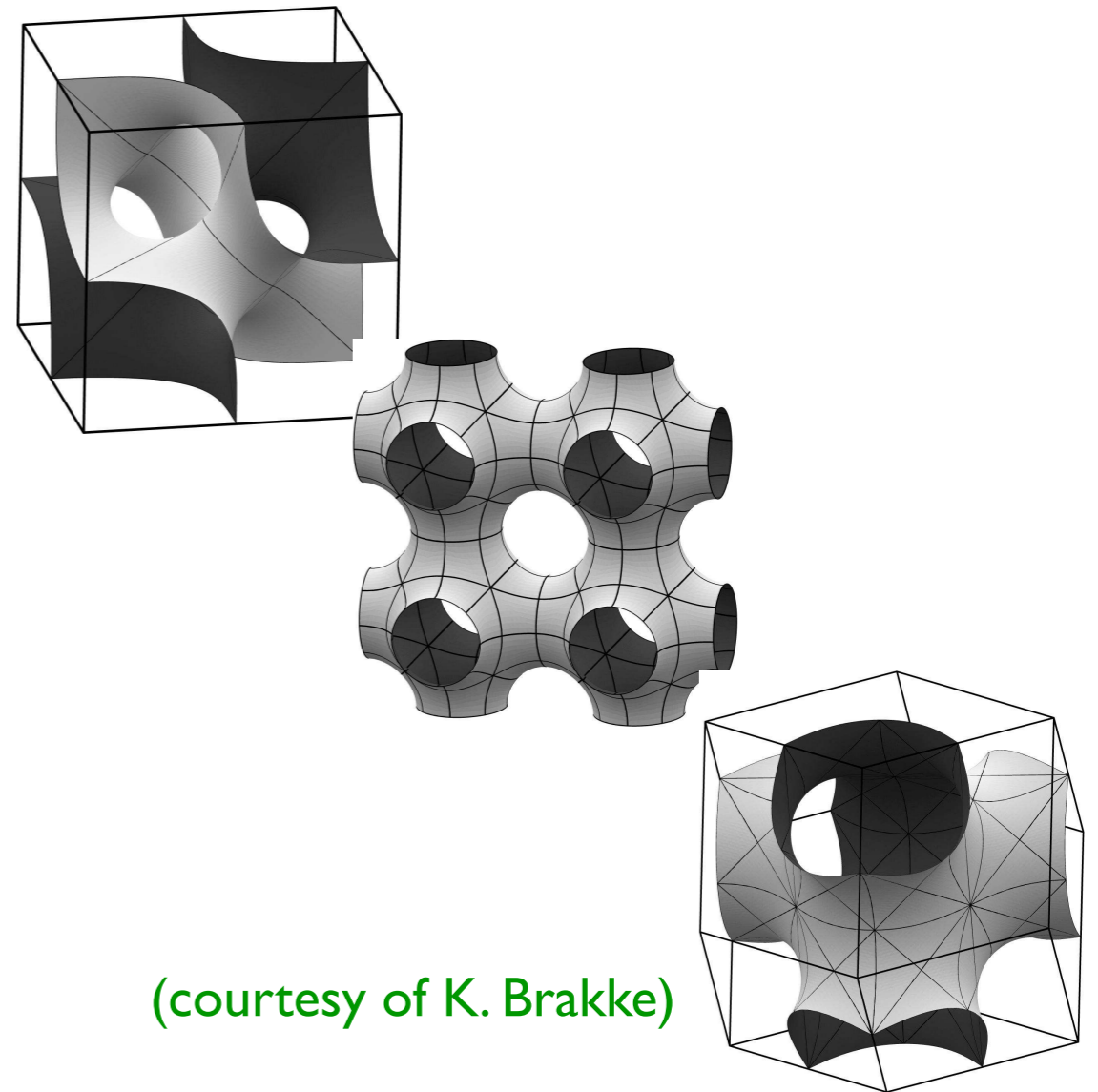
Three Dimensions (Olejarz, Krapvisky, & SR)

Basic result: ground state is *never* reached!
typical 20x20x20 system



Features:

1. Swiss cheesy
2. Zero average curvature



(courtesy of K. Brakke)

Fate of the Kinetic Ising Model

Sid Redner, Physics Department, Boston University, physics.bu.edu/~redner
collaborators: P. L. Krapivsky, V. Spirin, K. Barros, J. Olejarz

Basic question: What is the final state of the Ising-Glauber model @ $T=0$ with symmetric initial conditions?

Expectation:

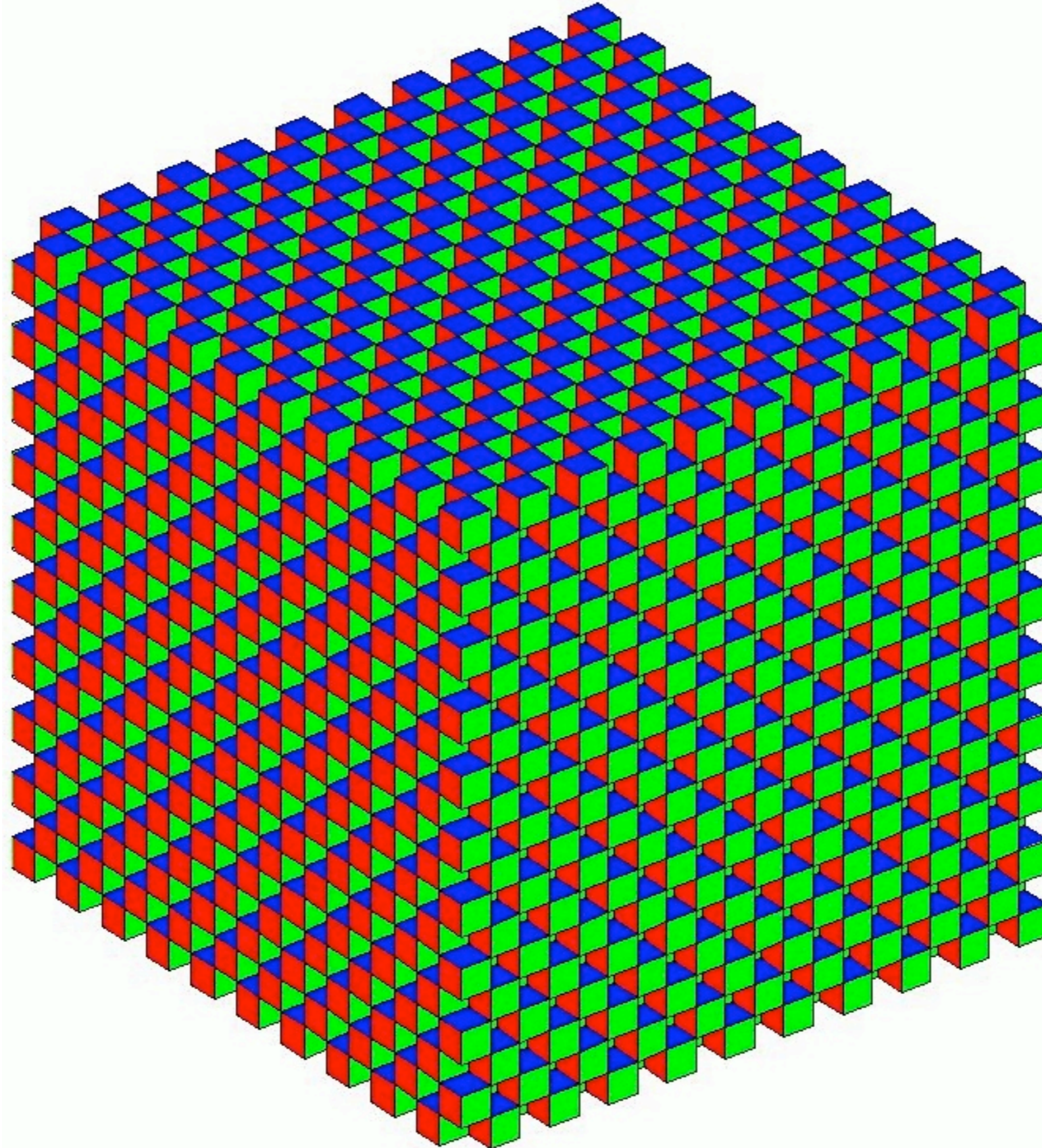
- Ground state is approached as $t \rightarrow \infty$
- Power-law coarsening

The result:

dimension	expectation
1	absolutely correct
2	“sort of” correct
≥ 3	wrong

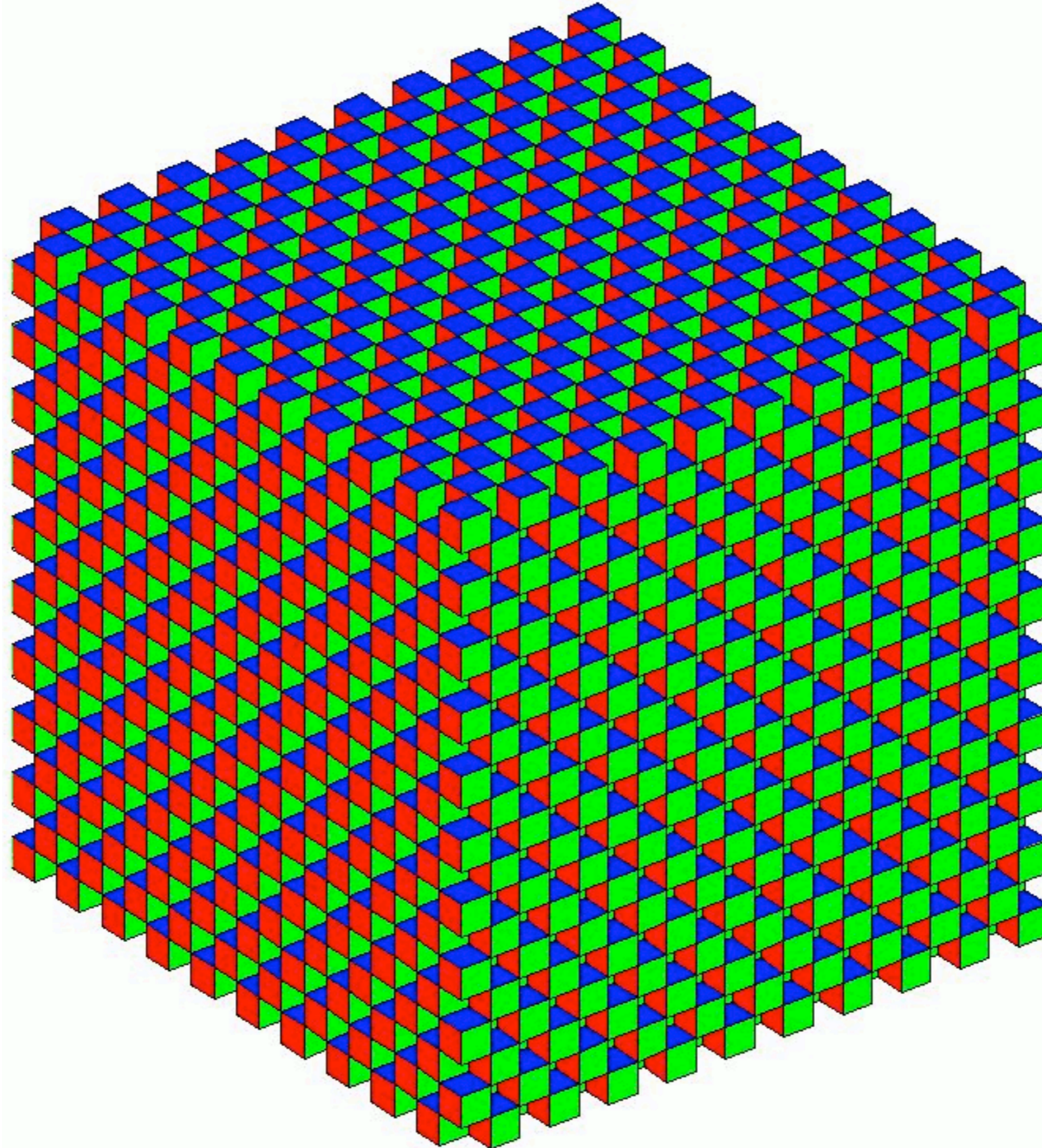
Evolution from Antiferromagnetic State

energy/spin = 6.0000, time = 0.0



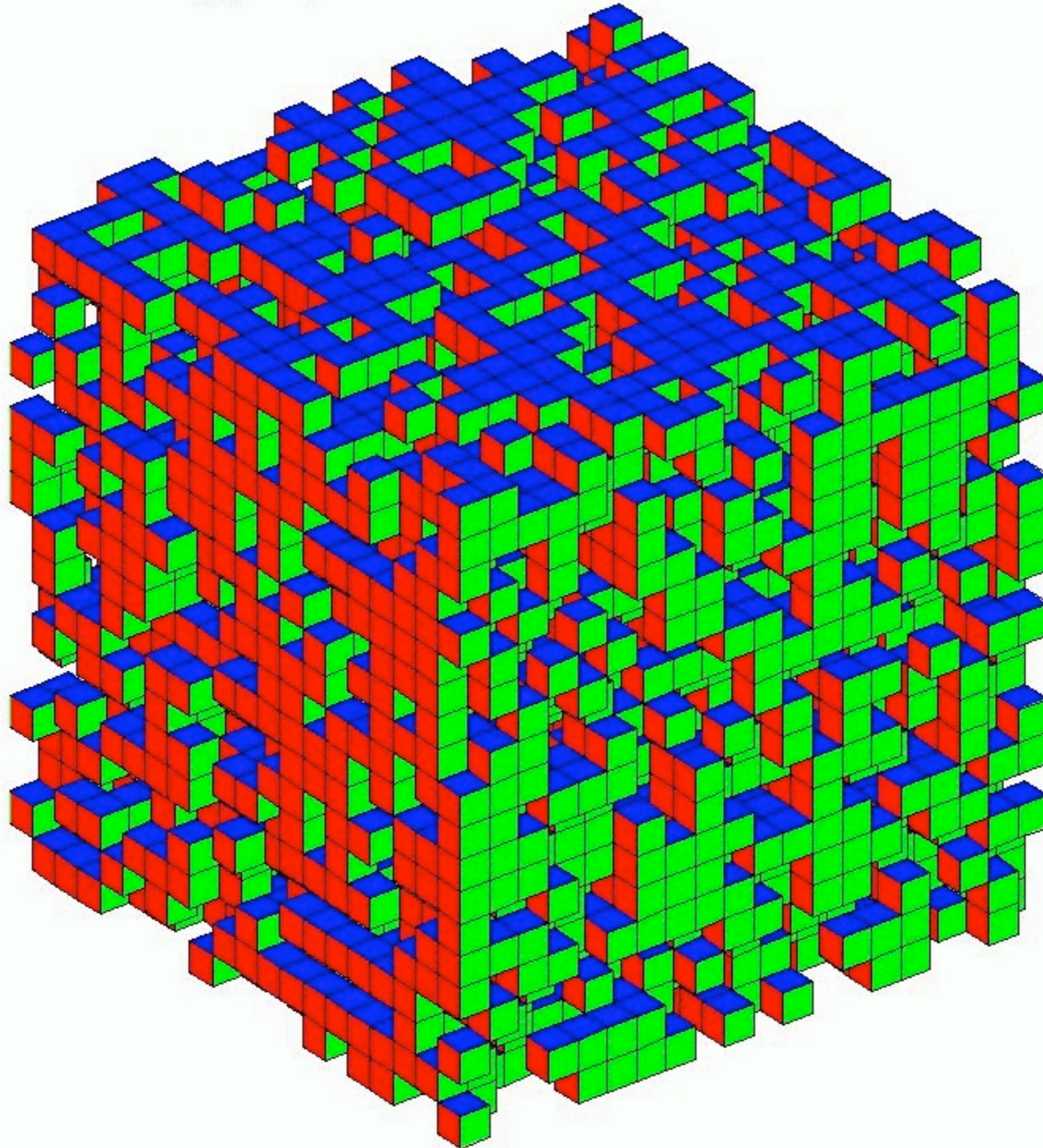
Evolution from Antiferromagnetic State

energy/spin = 6.0000, time = 0.0



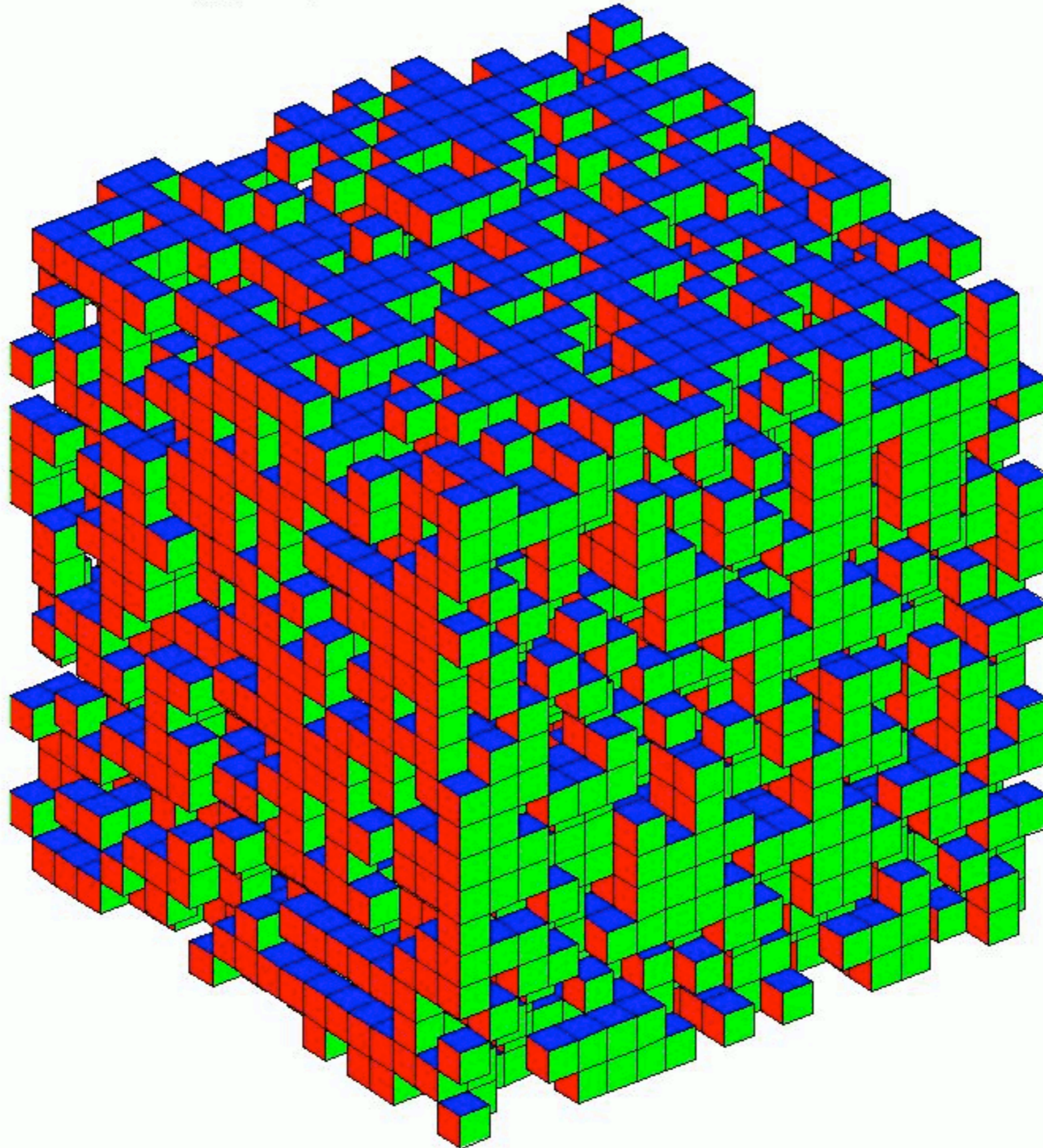
Evolution from Random* State

energy/spin = 3.0010, time = 0.0



Evolution from Random* State

energy/spin = 3.0010, time = 0.0



Three Dimensions (Olejarz, Krapvisky, & SR)

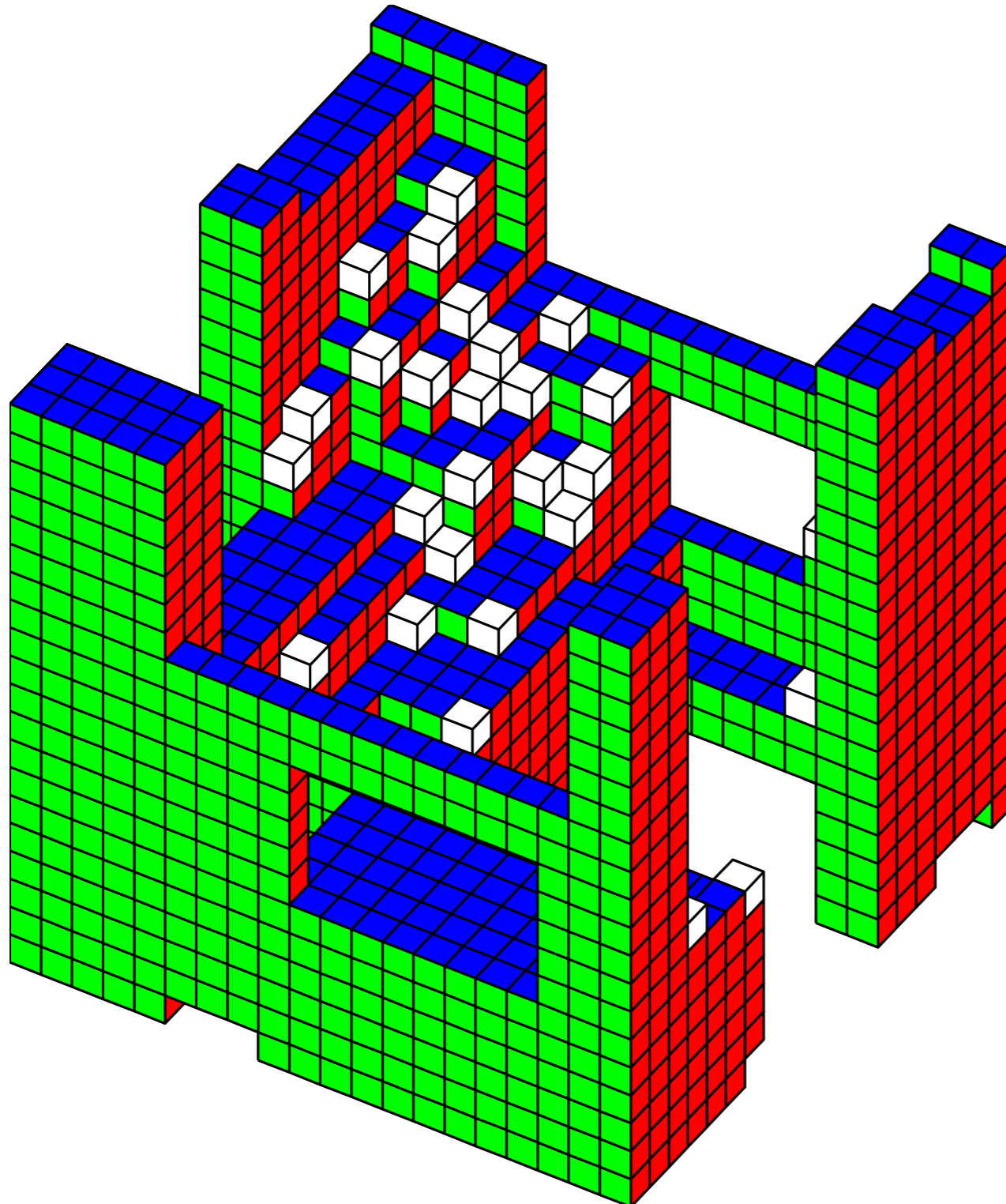
Basic result: ground state is *never* reached!

typical 20x20x20 system

Features:

1. Swiss cheesy

2. Zero *average* curvature



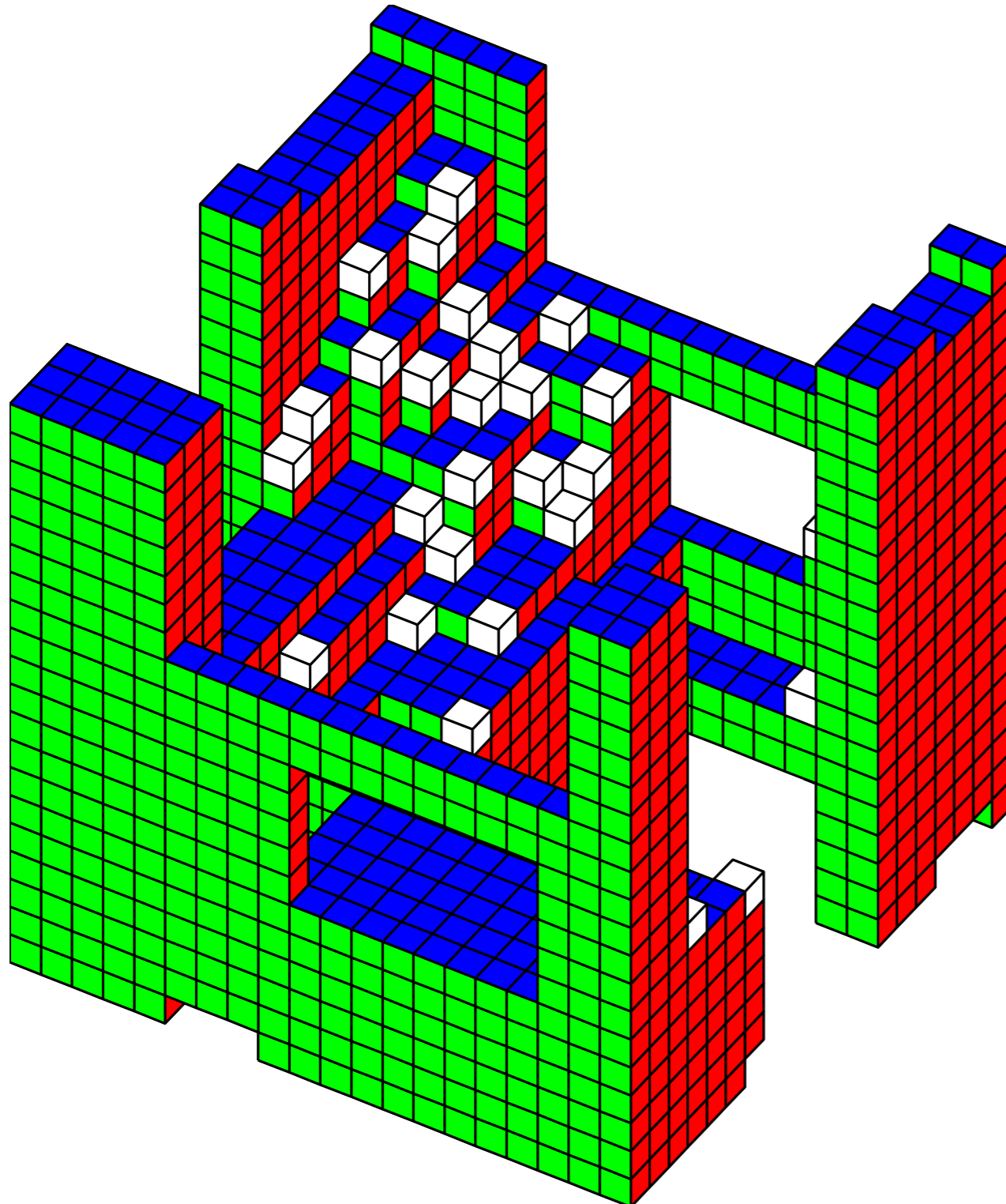
Three Dimensions (Olejarz, Krapvisky, & SR)

Basic result: ground state is *never* reached!

typical 20x20x20 system

Features:

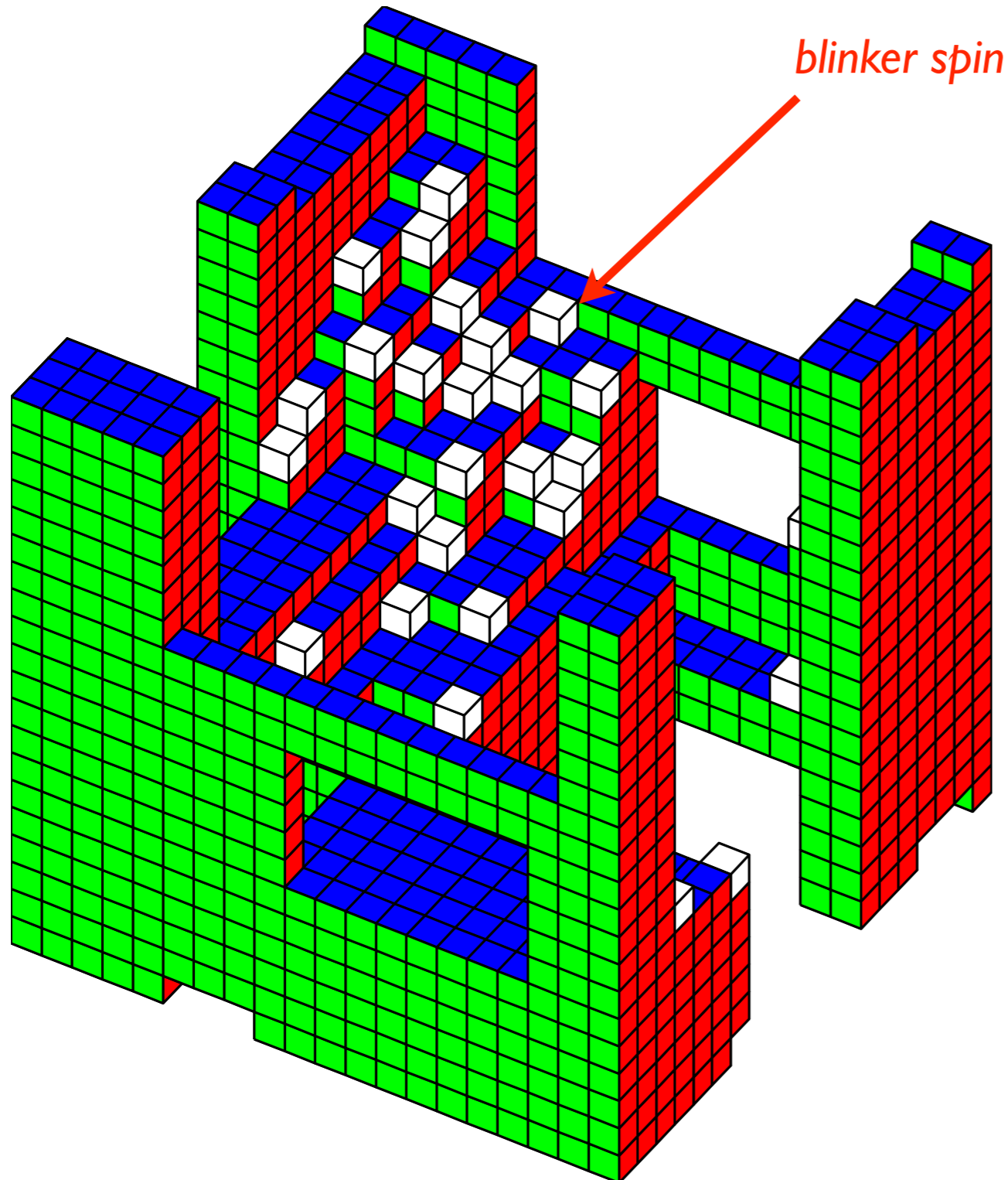
1. Swiss cheesy
2. Zero *average* curvature
3. Non-static



Three Dimensions (Olejarz, Krapvisky, & SR)

Basic result: ground state is *never* reached!

typical 20x20x20 system

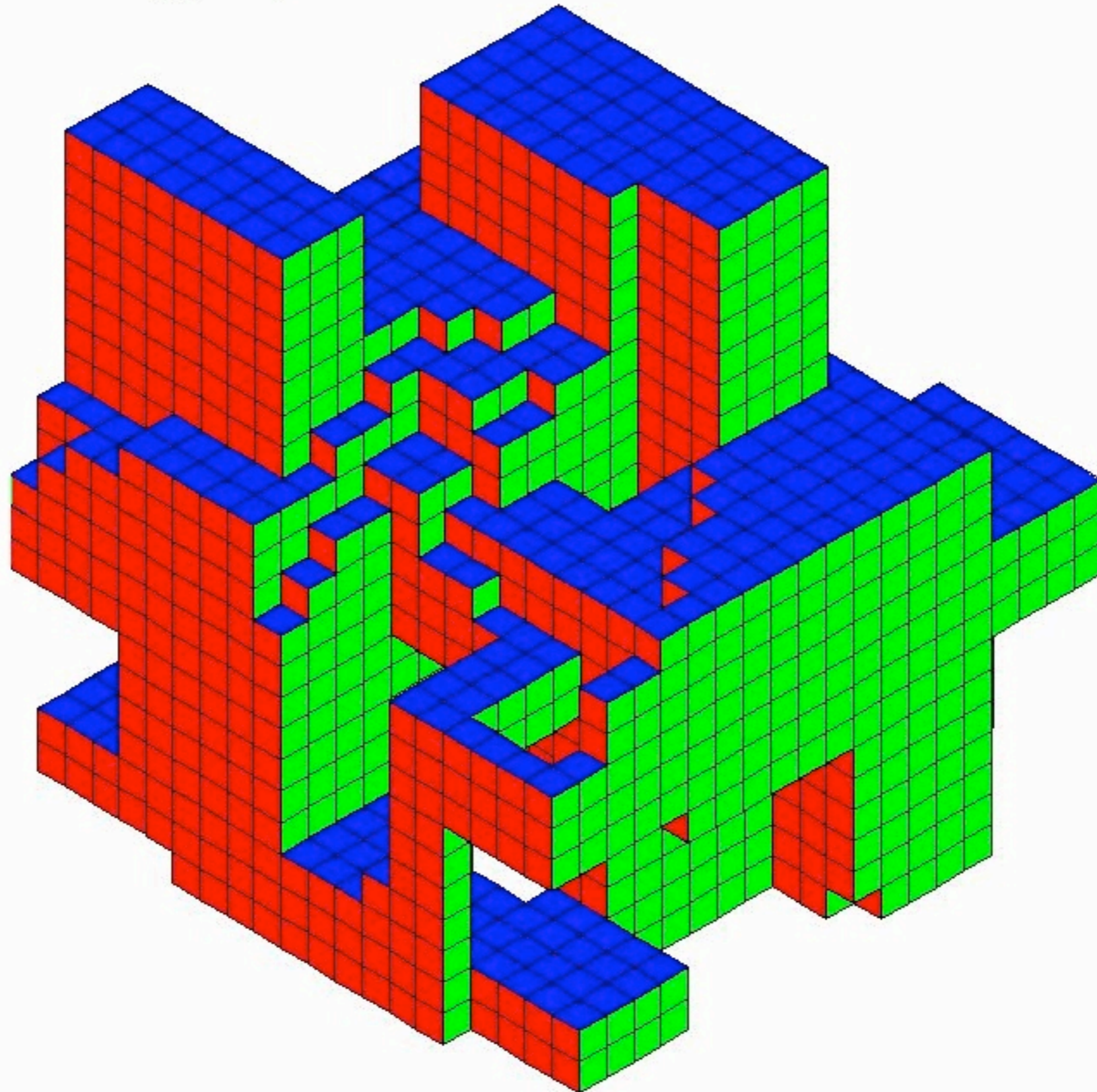


Features:

1. Swiss cheesy
2. Zero *average* curvature
3. Non-static

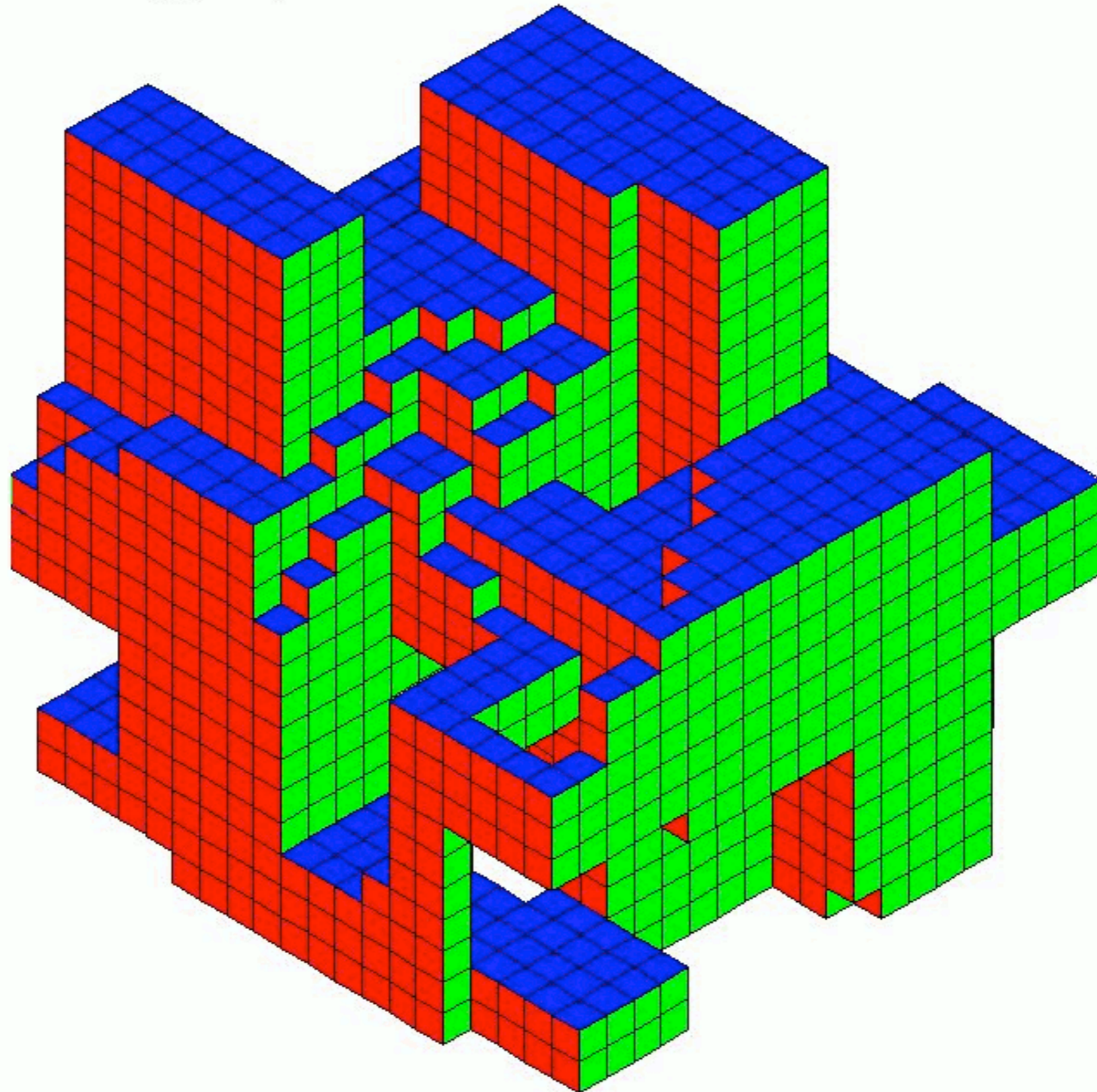
Blinker Evolution in Three Dimensions

energy/spin = 0.5335, time = 942.0

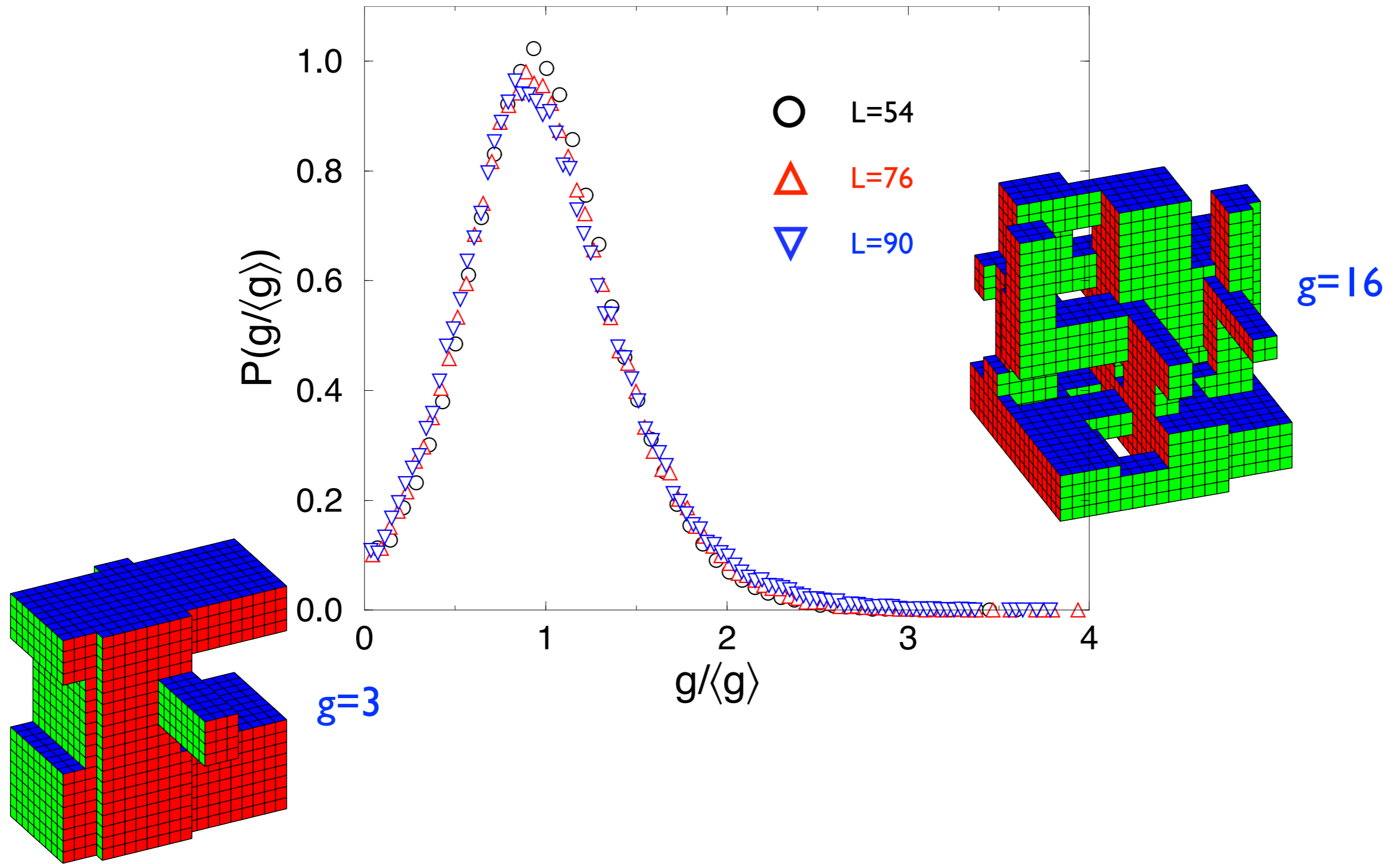


Blinker Evolution in Three Dimensions

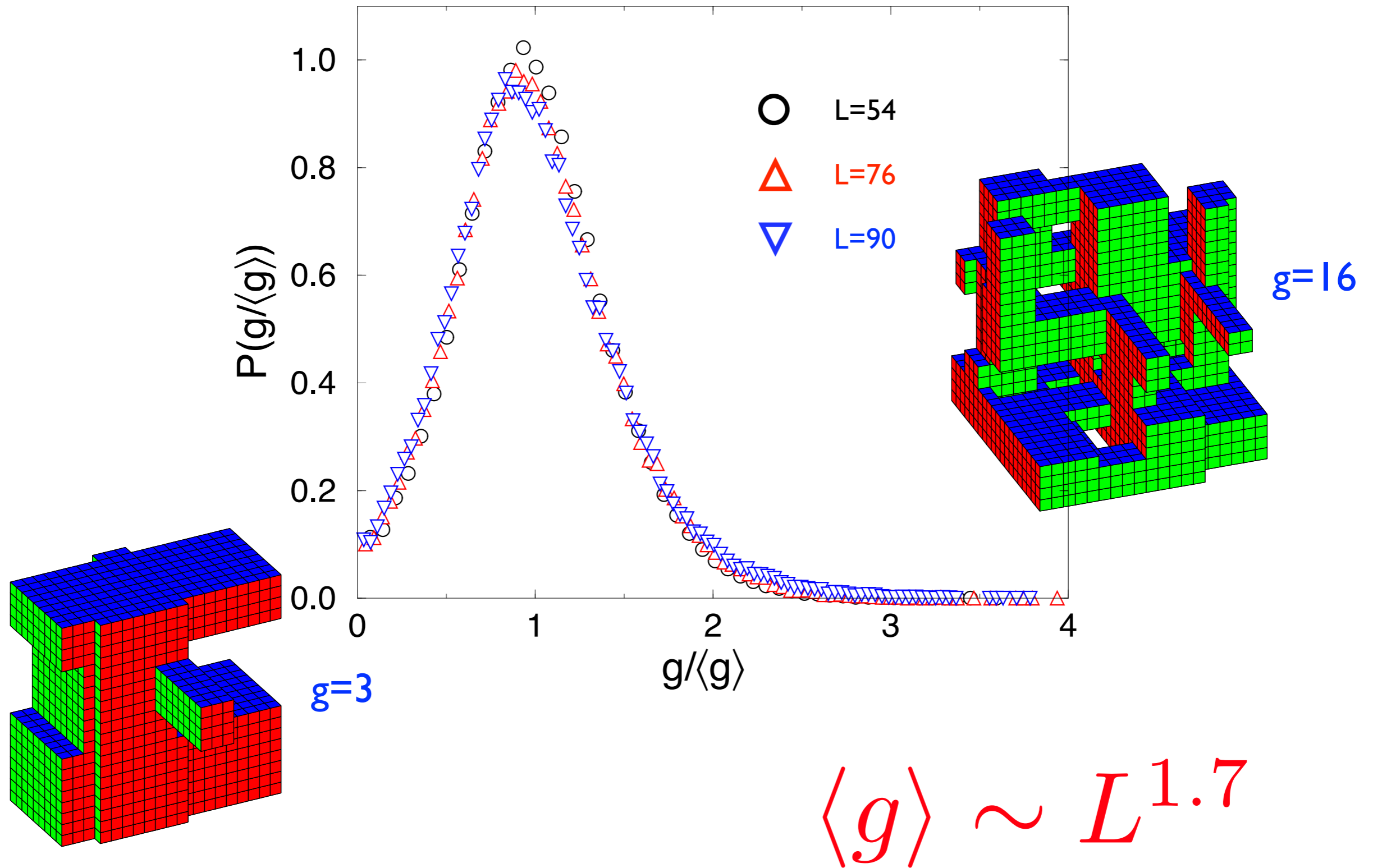
energy/spin = 0.5335, time = 942.0



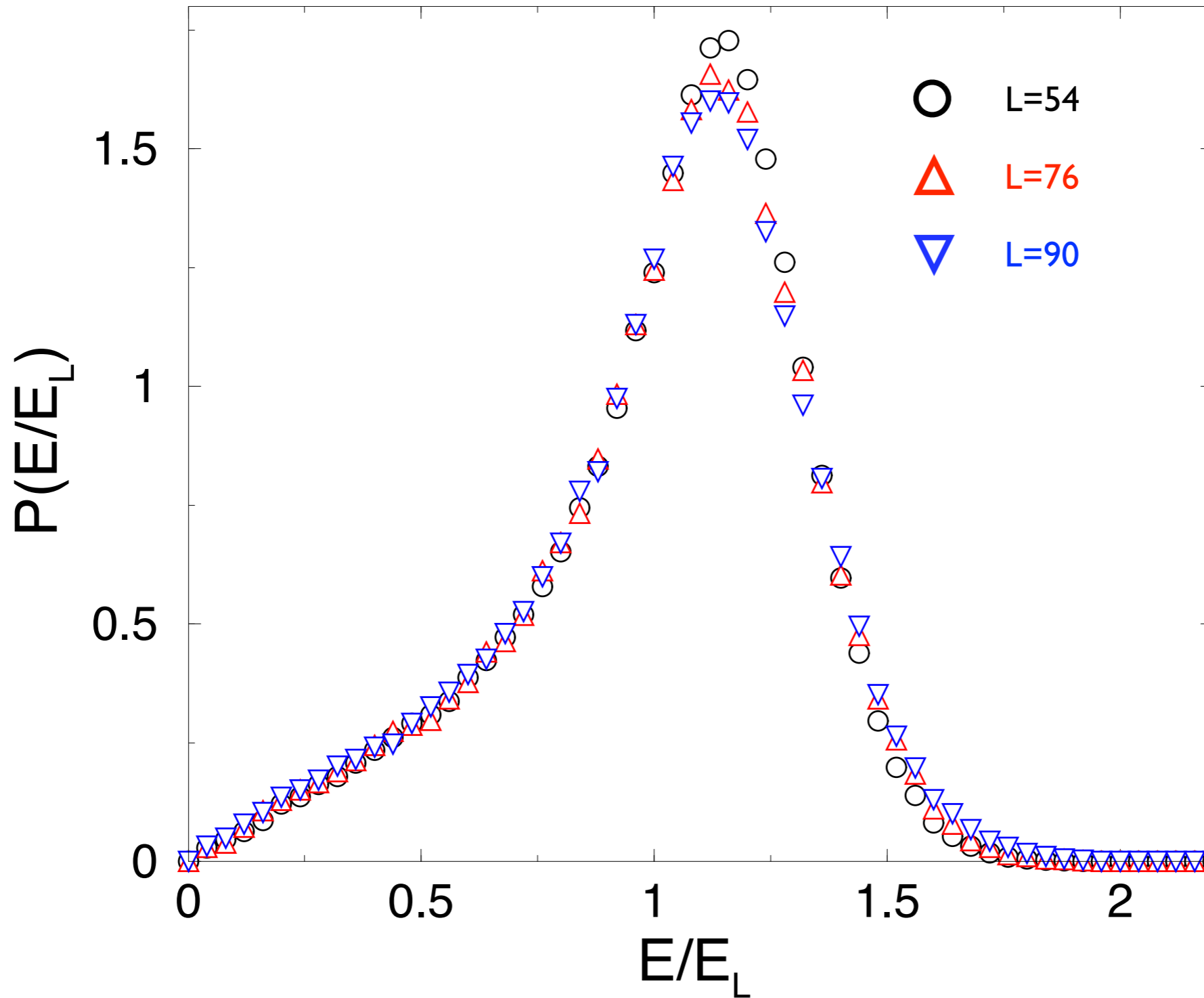
Genus Distribution



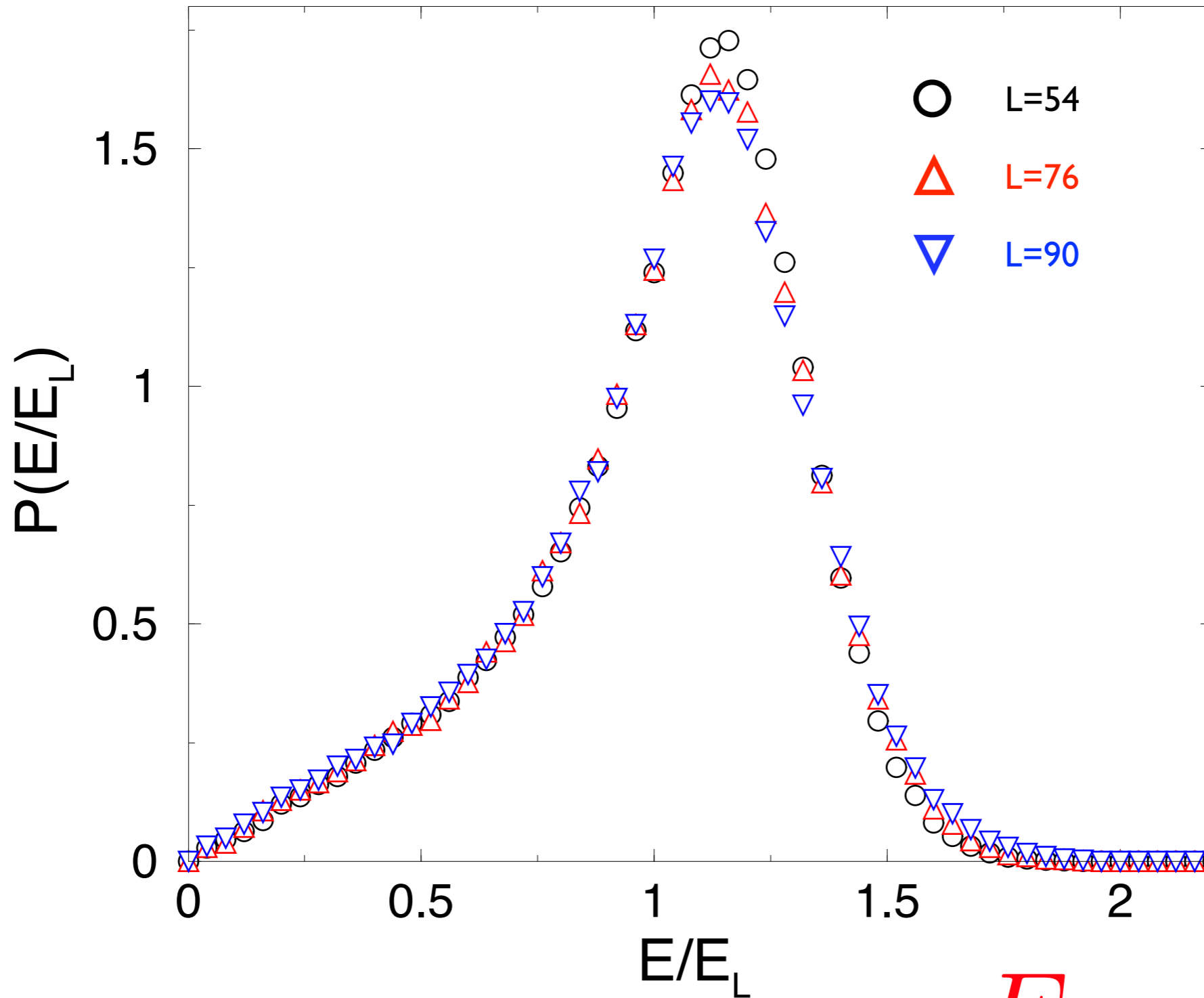
Genus Distribution



Energy Distribution



Energy Distribution



$$E_L \sim L^{-1}$$

Relate Genus and Energy by Topology

$$\chi = 2(1 - g) = \mathcal{V} - \mathcal{E} + \mathcal{F}$$

Euler characteristic genus vertices edges faces

Relate Genus and Energy by Topology

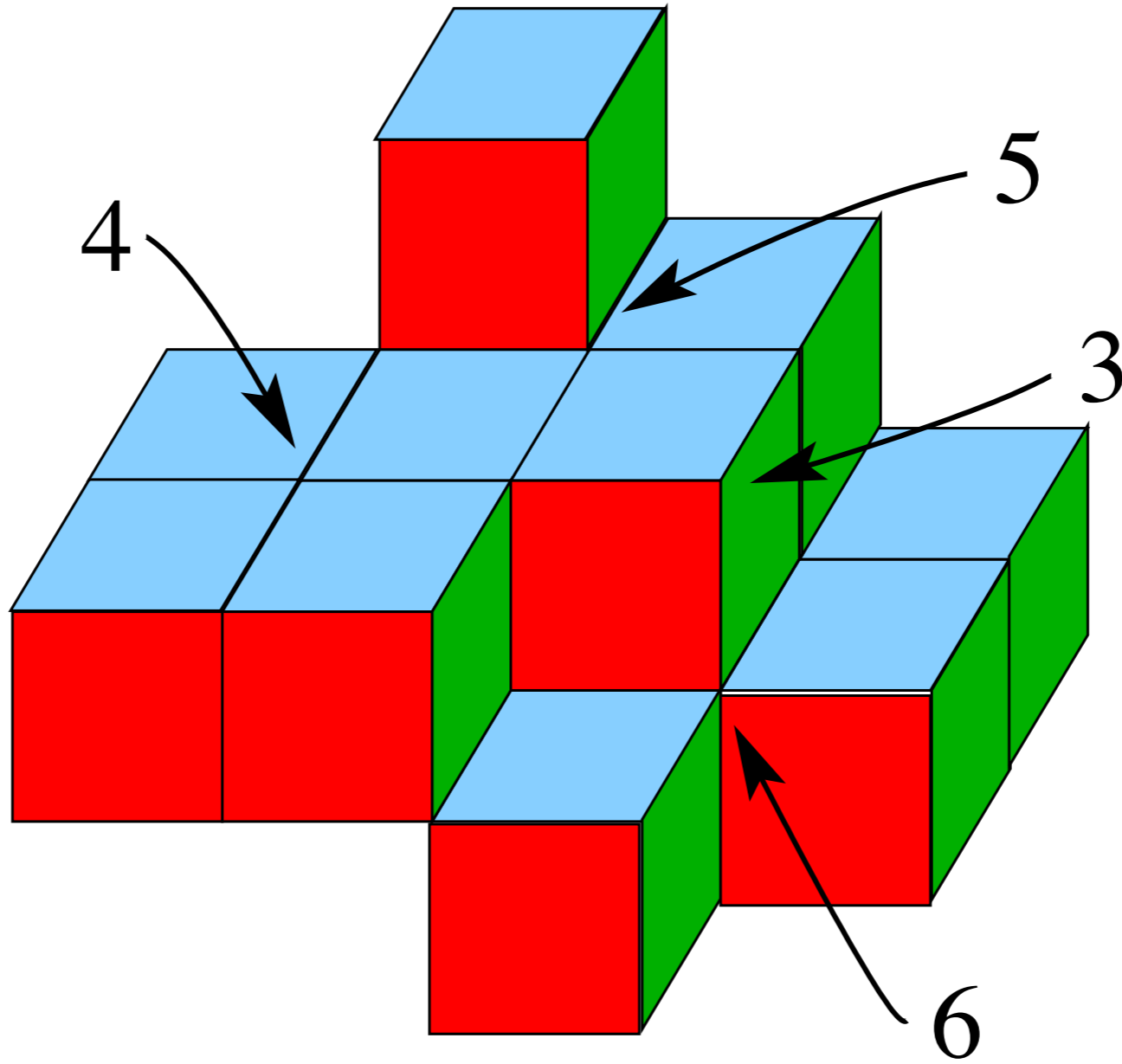
$$\chi = 2(1 - g) = \mathcal{V} - \mathcal{E} + \mathcal{F}$$

Euler characteristic genus vertices edges faces

constraints between \mathcal{V} , \mathcal{E} , \mathcal{F} : $\mathcal{E} = 2\mathcal{F}$ $\frac{\mathcal{E}}{3} \leq \mathcal{V} \leq \frac{2\mathcal{E}}{3}$

$$\mathcal{E} = 2\mathcal{F}$$

$$\frac{\mathcal{E}}{3} \leq \nu \leq \frac{2\mathcal{E}}{3}$$



Relate Genus and Energy by Topology

$$\chi = 2(1 - g) = \mathcal{V} - \mathcal{E} + \mathcal{F}$$

Euler characteristic genus vertices edges faces

constraints between \mathcal{V} , \mathcal{E} , \mathcal{F} : $\mathcal{E} = 2\mathcal{F}$ $\frac{\mathcal{E}}{3} \leq \mathcal{V} \leq \frac{2\mathcal{E}}{3}$

$$\longrightarrow 0 \leq g \leq 1 + \frac{\mathcal{F}}{6}$$

Relate Genus and Energy by Topology

$$\chi = 2(1 - g) = \mathcal{V} - \mathcal{E} + \mathcal{F}$$

Euler characteristic genus vertices edges faces

constraints between \mathcal{V} , \mathcal{E} , \mathcal{F} : $\mathcal{E} = 2\mathcal{F}$ $\frac{\mathcal{E}}{3} \leq \mathcal{V} \leq \frac{2\mathcal{E}}{3}$

$$\longrightarrow 0 \leq g \leq 1 + \frac{\mathcal{F}}{6}$$

if $E_L = \mathcal{F}/L^3 \sim L^{-\epsilon}$ $\longrightarrow \epsilon + \gamma \leq 3$
and $\langle g \rangle \sim L^\gamma$

Relate Genus and Energy by Topology

$$\chi = 2(1 - g) = \mathcal{V} - \mathcal{E} + \mathcal{F}$$

Euler
characteristic
genus
vertices
edges
faces

constraints between \mathcal{V} , \mathcal{E} , \mathcal{F} : $\mathcal{E} = 2\mathcal{F}$ $\frac{\mathcal{E}}{3} \leq \mathcal{V} \leq \frac{2\mathcal{E}}{3}$

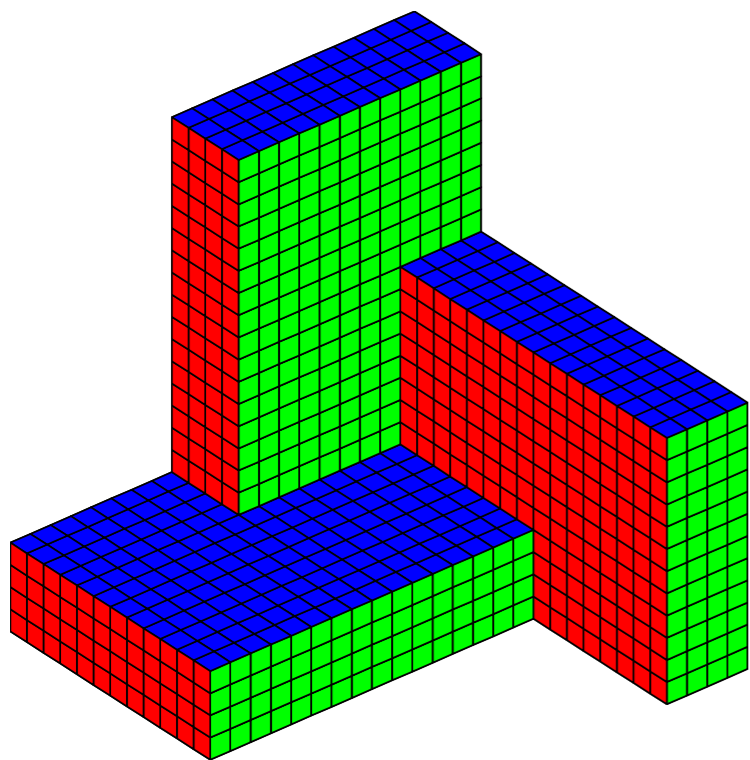
$$\longrightarrow 0 \leq g \leq 1 + \frac{\mathcal{F}}{6}$$

if $E_L = \mathcal{F}/L^3 \sim L^{-\epsilon}$ $\longrightarrow \epsilon + \gamma \leq 3$
 and $\langle g \rangle \sim L^\gamma$ 1 1.7 simulations

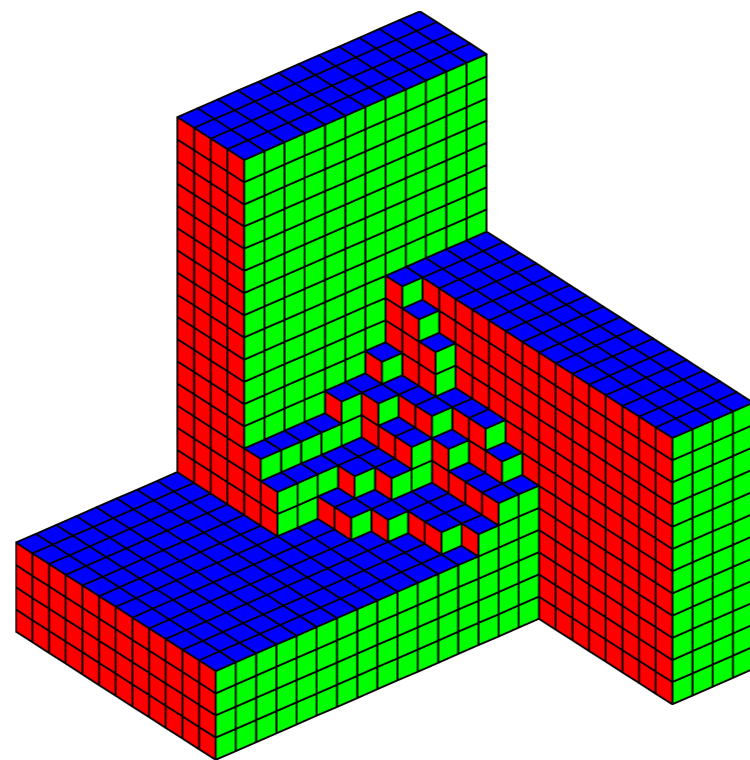
Slow Relaxation of Blinkers

Slow Relaxation of Blinkers

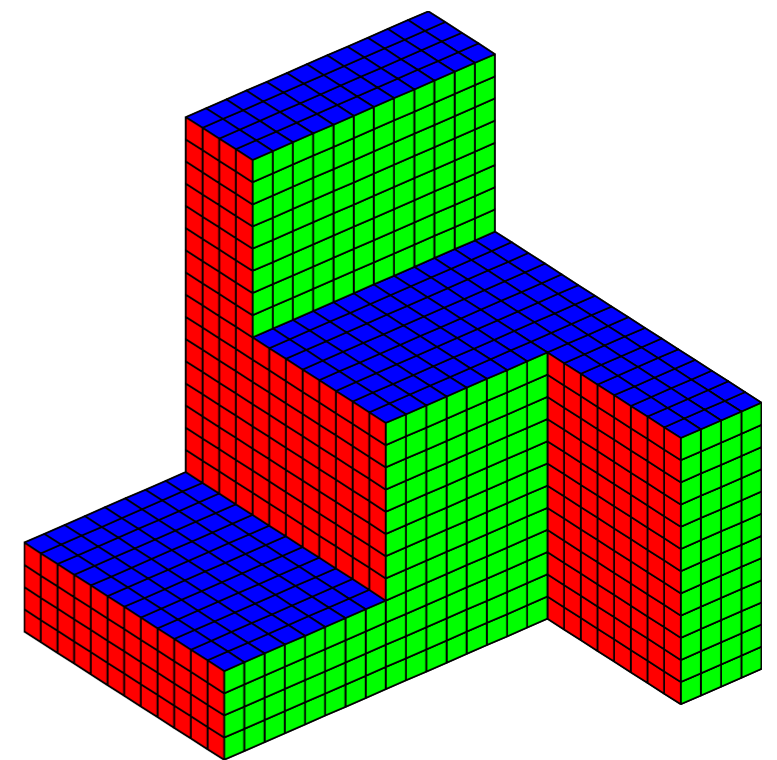
synthetic blinker configuration



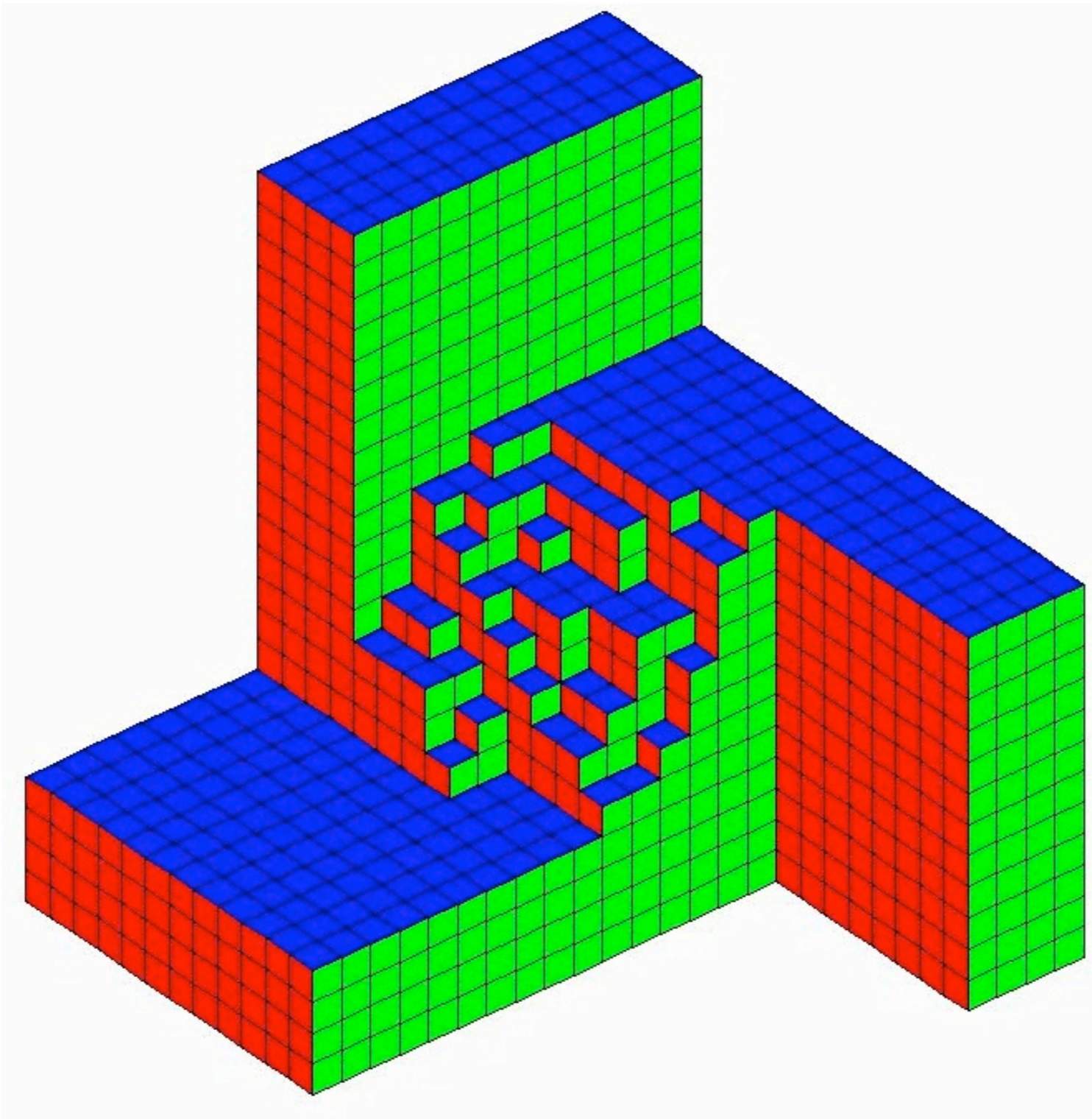
deflated

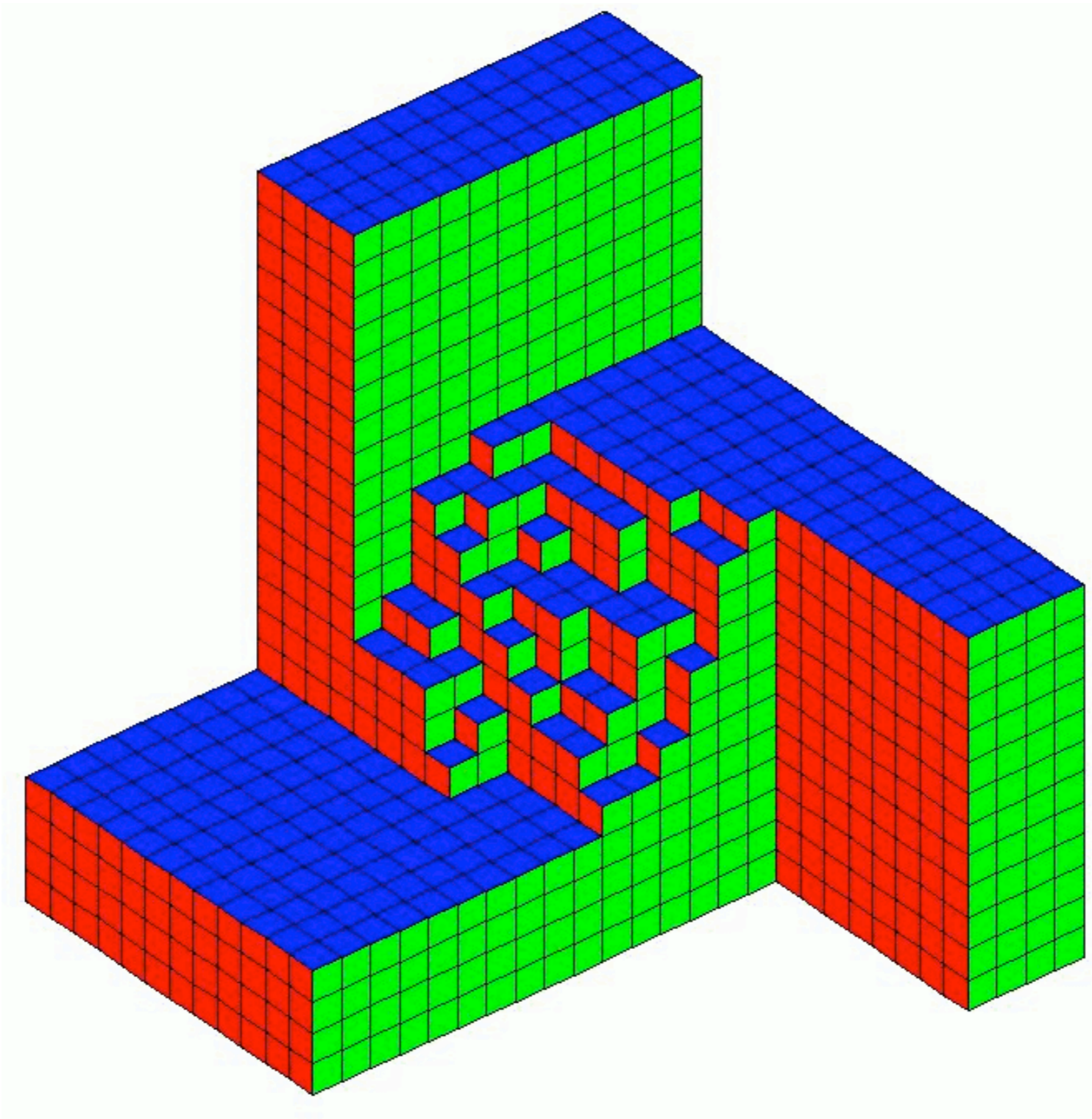


intermediate

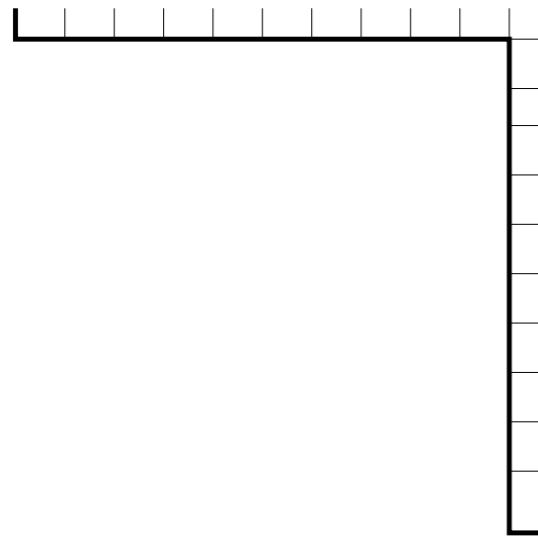


inflated

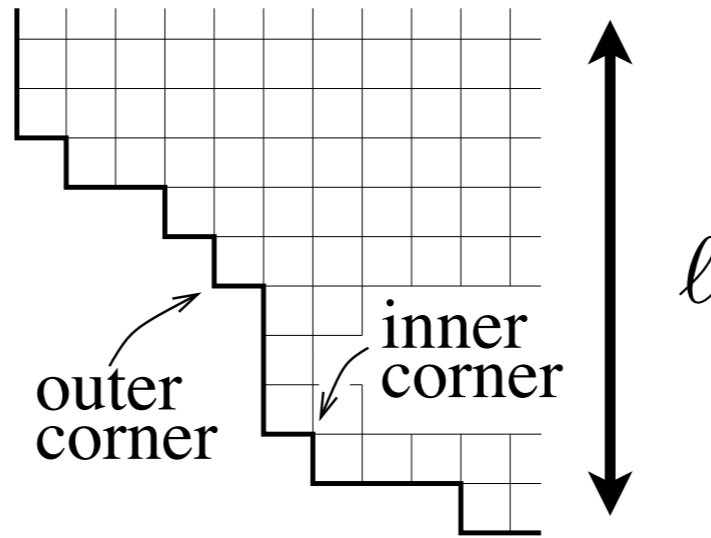




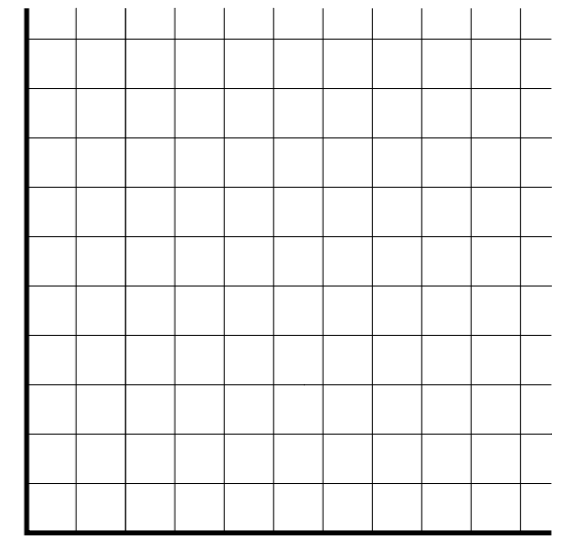
2d analog:



deflated



intermediate



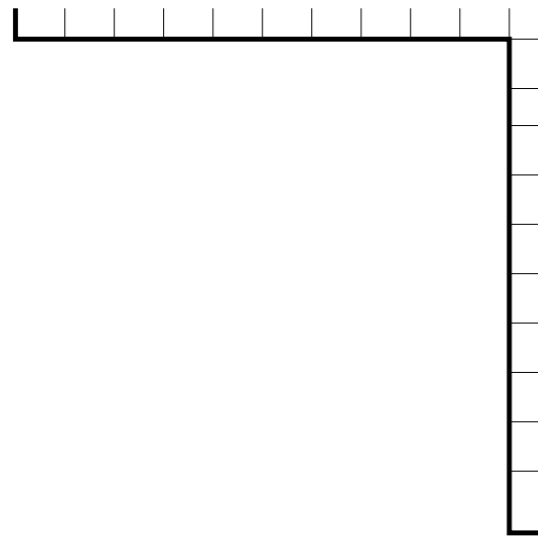
inflated

N_+ outer corners

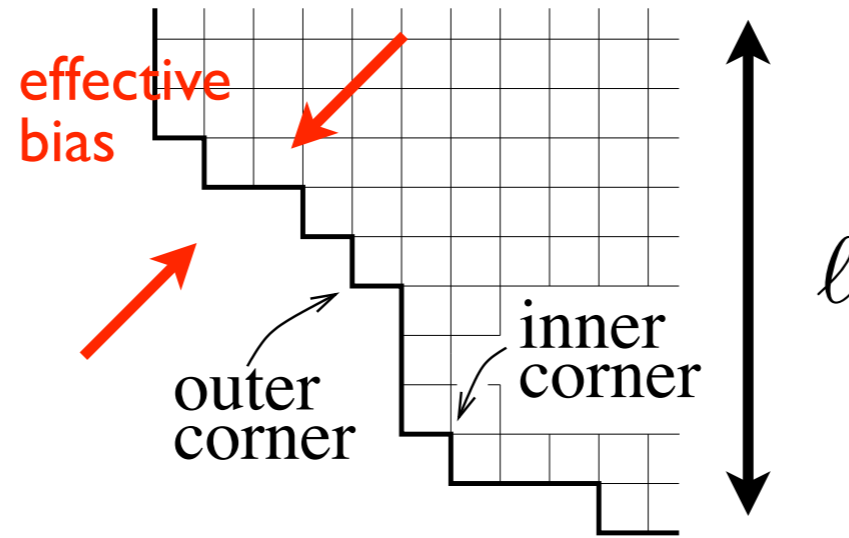
N_- inner corners

$$N_+ - N_- = 1$$

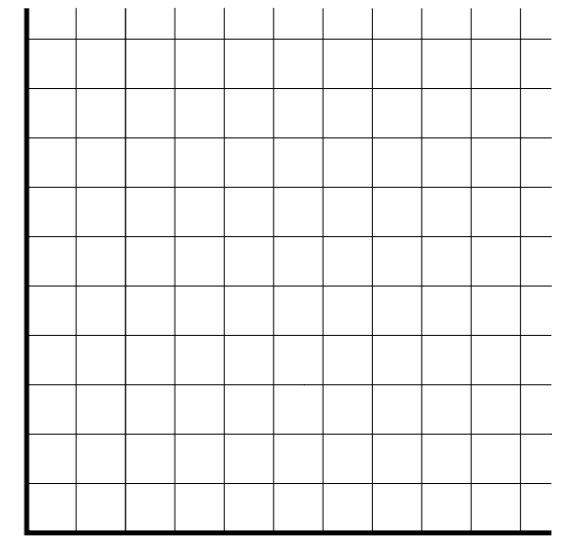
2d analog:



deflated



intermediate

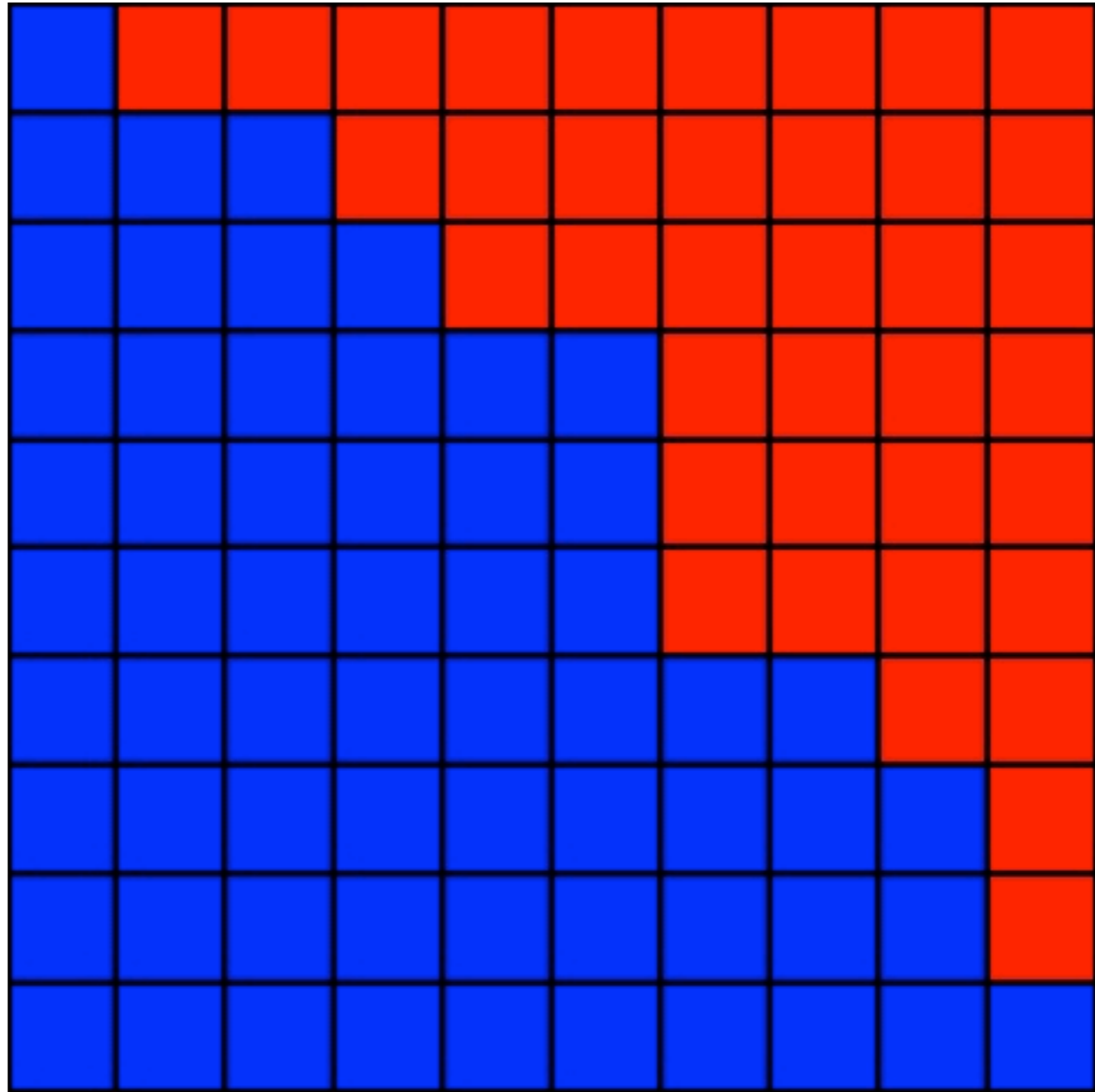


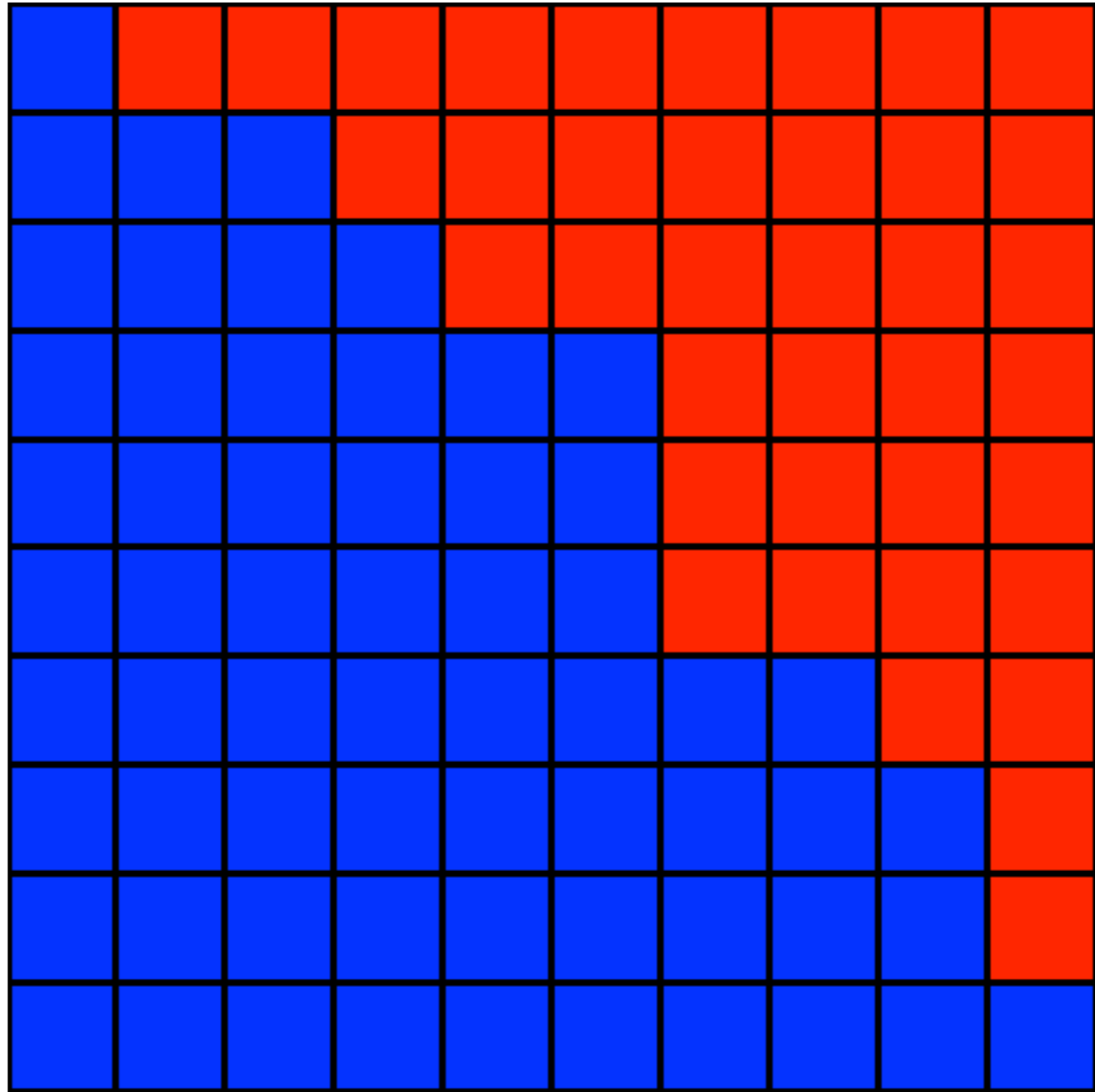
inflated

N_+ outer corners

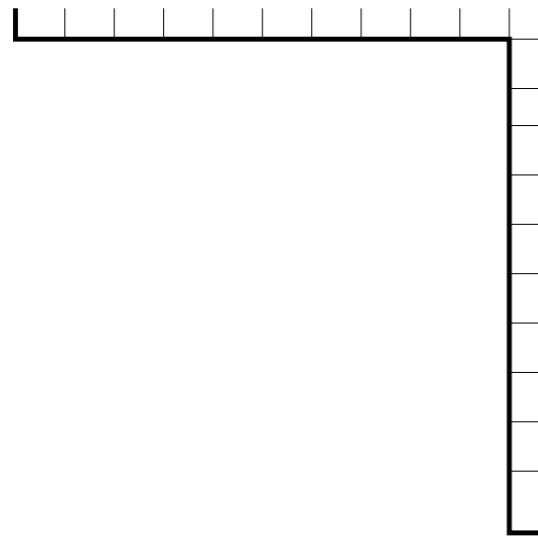
N_- inner corners

$$N_+ - N_- = 1$$

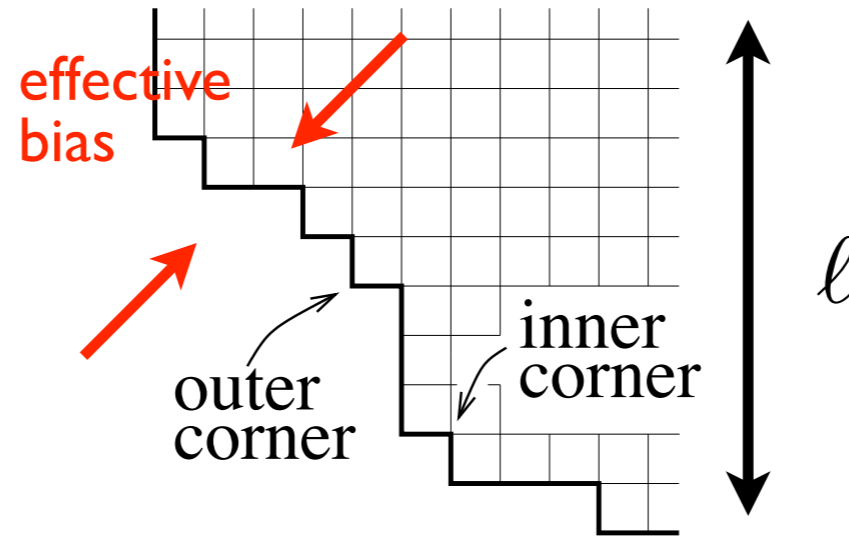




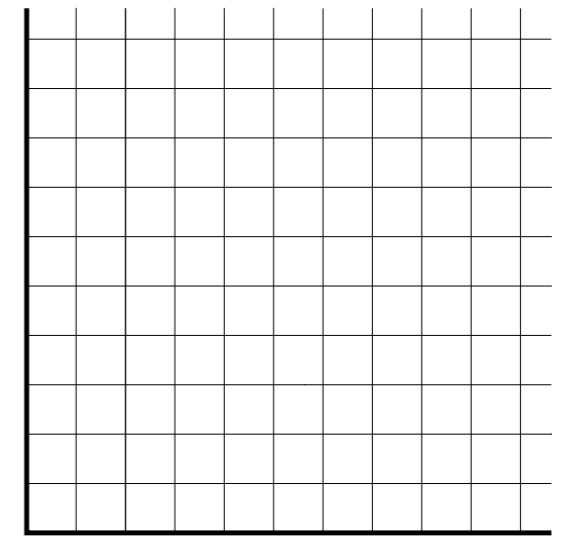
2d analog:



deflated



intermediate



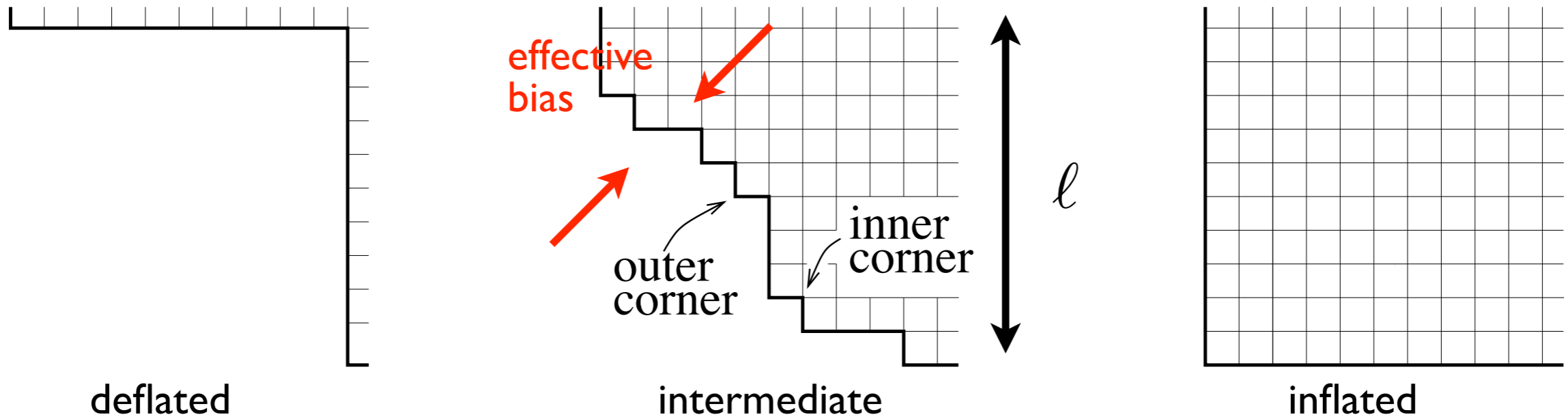
inflated

N_+ outer corners

N_- inner corners

$$N_+ - N_- = 1$$

2d analog:



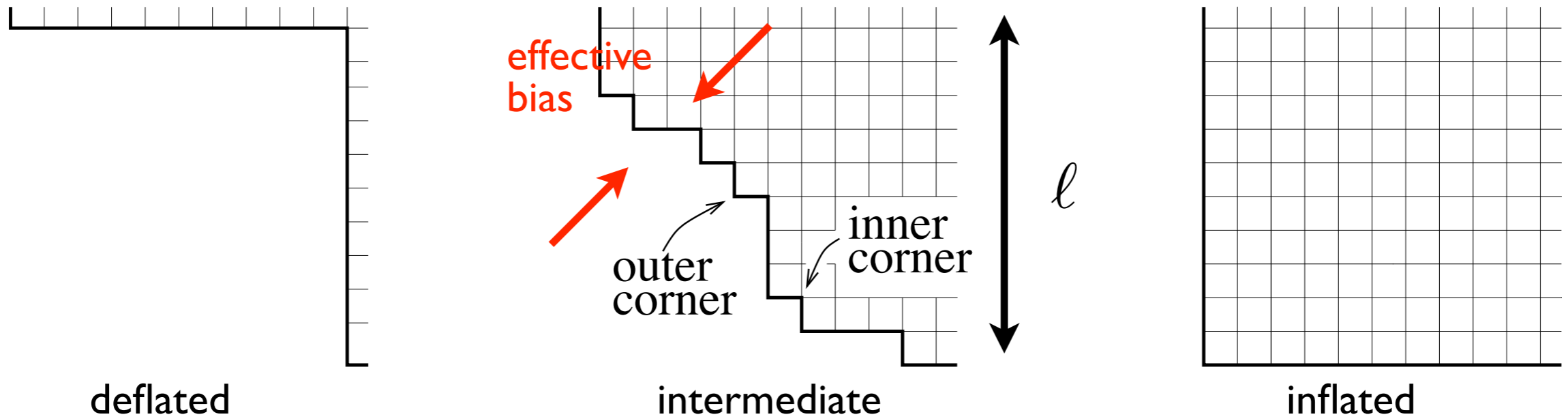
N_+ outer corners
 N_- inner corners

$$N_+ - N_- = 1$$

$$u = \frac{\Delta A}{\Delta t} = -1$$

$$D = \frac{(\Delta A)^2}{\Delta t} \sim N_{\pm} \sim l$$

2d analog:



N_+ outer corners
 N_- inner corners

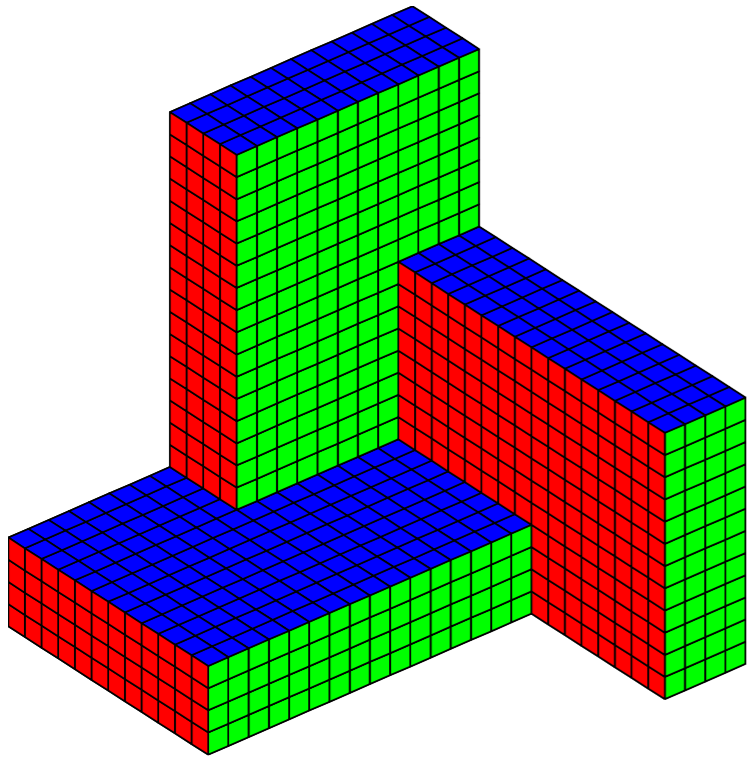
$$N_+ - N_- = 1$$

$$u = \frac{\Delta A}{\Delta t} = -1$$

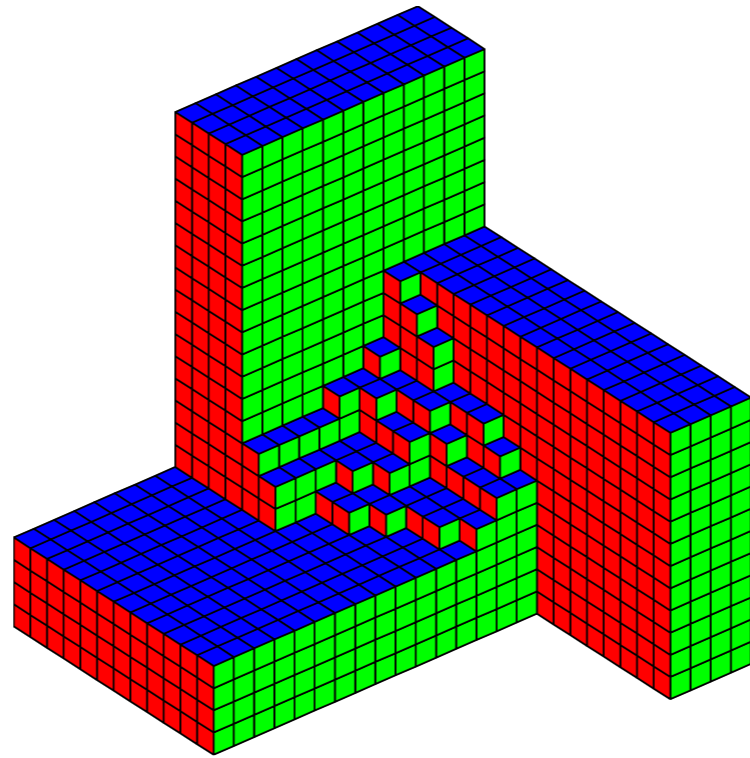
$$\tau \sim e^{|u|\ell^2/D} \sim e^\ell$$

$$D = \frac{(\Delta A)^2}{\Delta t} \sim N_\pm \sim \ell$$

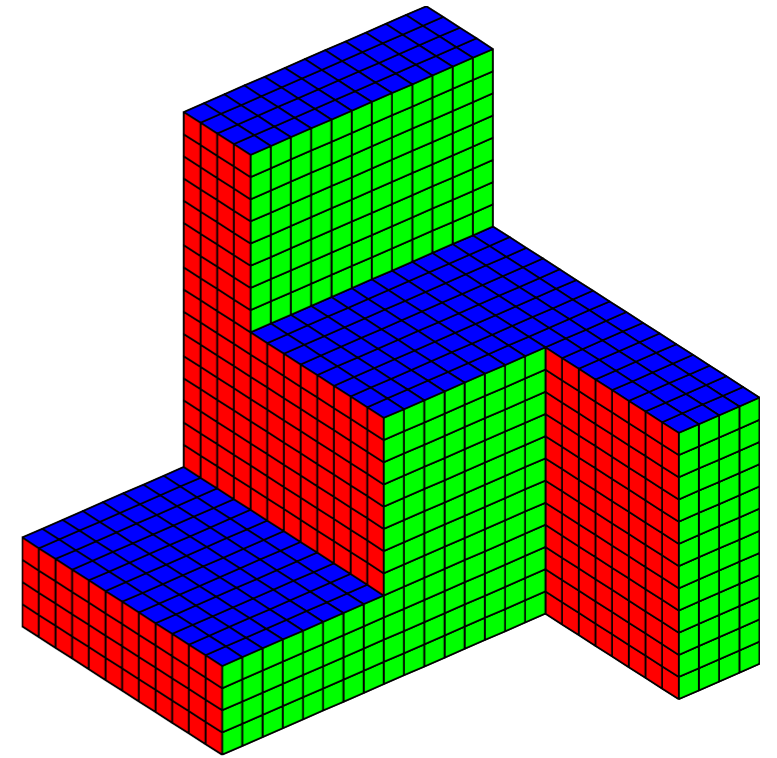
Slow Blinker Relaxation in 3d



deflated



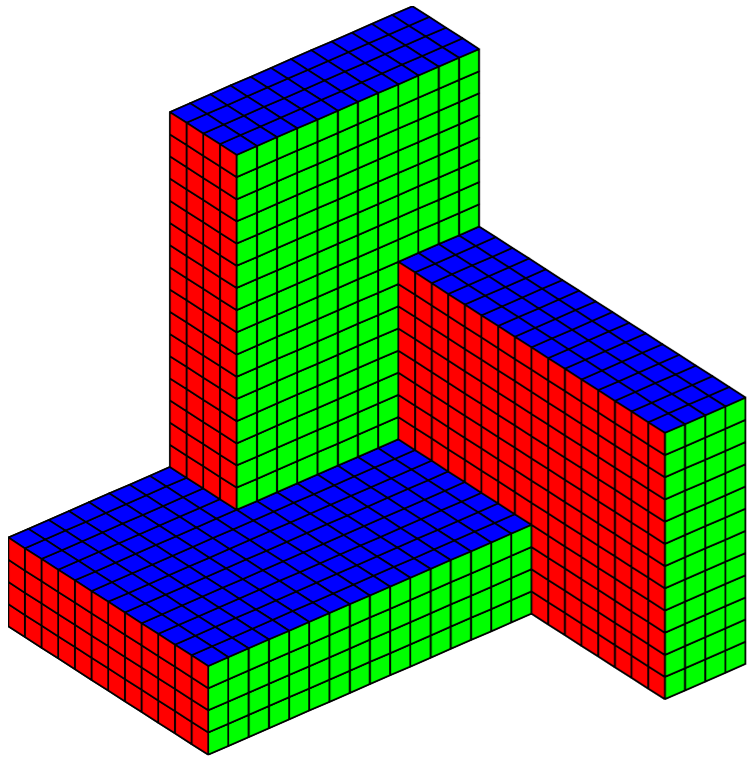
intermediate



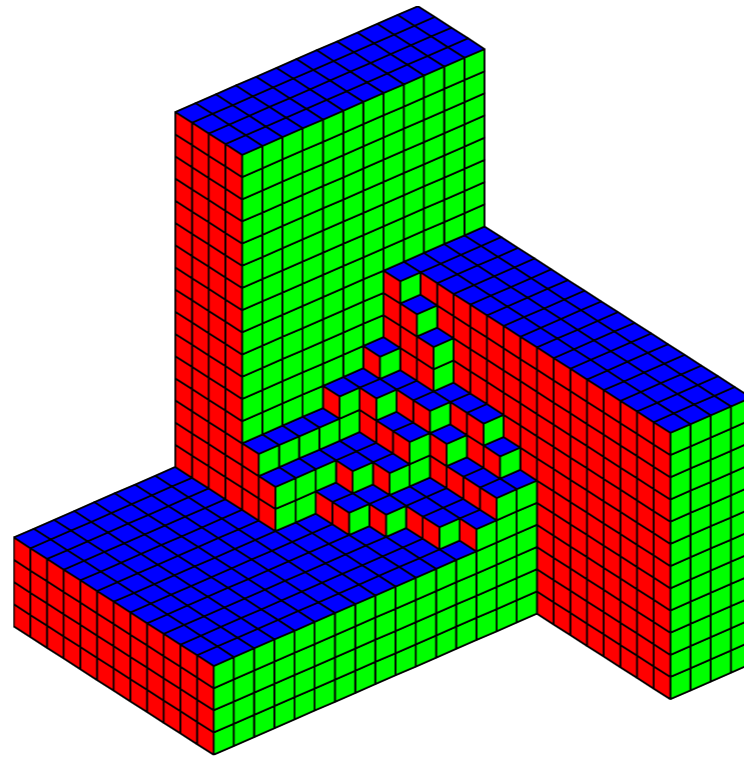
inflated

$$N_{\pm} \sim l^2$$
$$N_{+} - N_{-} \sim l$$

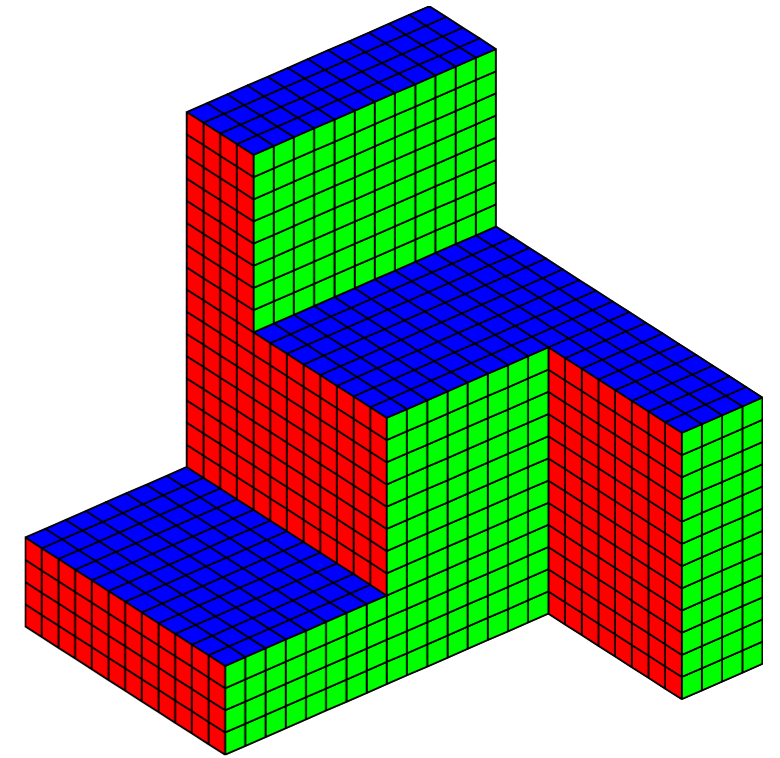
Slow Blinker Relaxation in 3d



deflated



intermediate



inflated

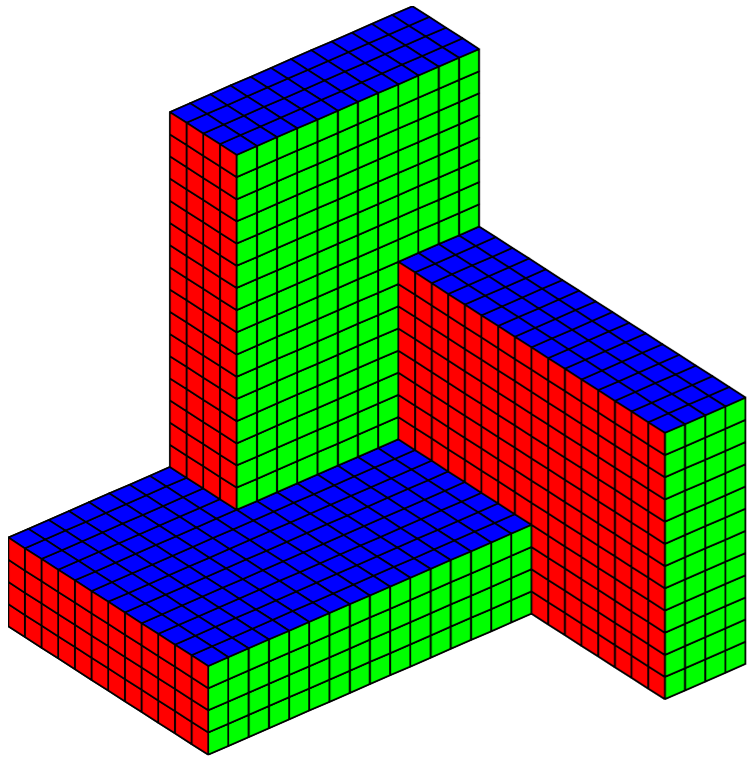
$$u = \frac{\Delta A}{\Delta t} = -\ell$$

$$D = \frac{(\Delta A)^2}{\Delta t} \sim N_{\pm} \sim \ell^2$$

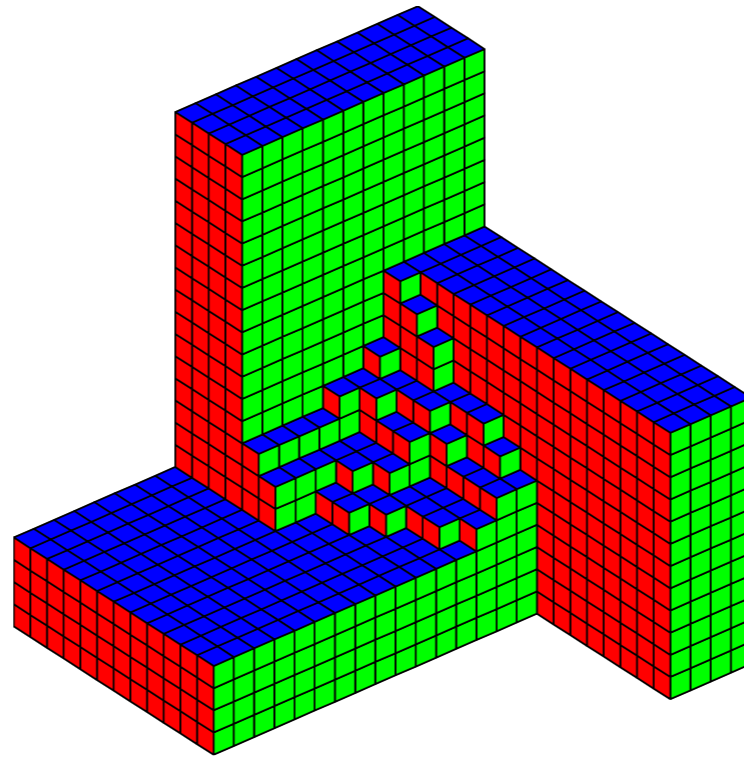
$$N_{\pm} \sim \ell^2$$

$$N_{+} - N_{-} \sim \ell$$

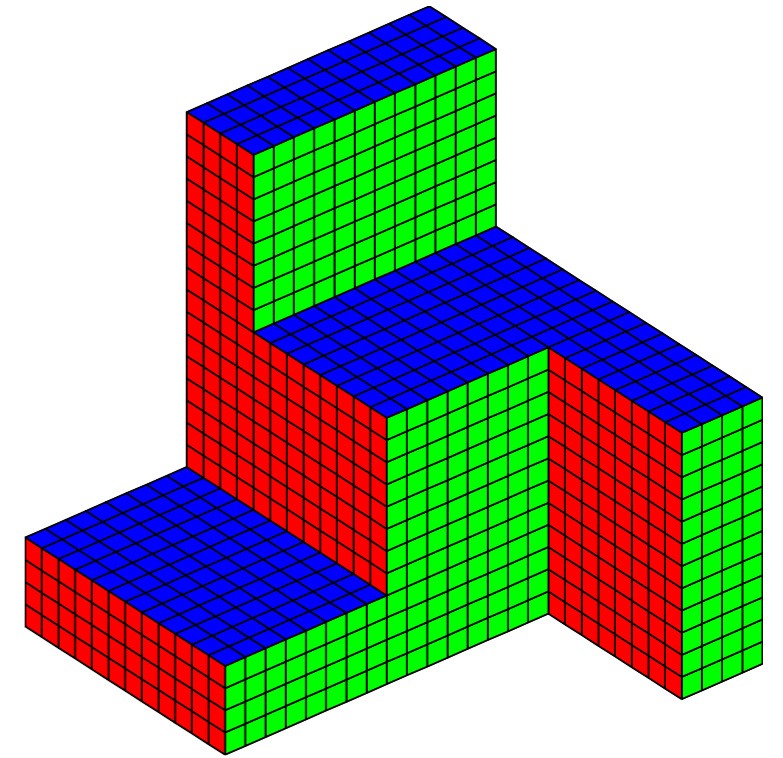
Slow Blinker Relaxation in 3d



deflated



intermediate



inflated

$$u = \frac{\Delta A}{\Delta t} = -l$$

$$D = \frac{(\Delta A)^2}{\Delta t} \sim N_{\pm} \sim l^2$$

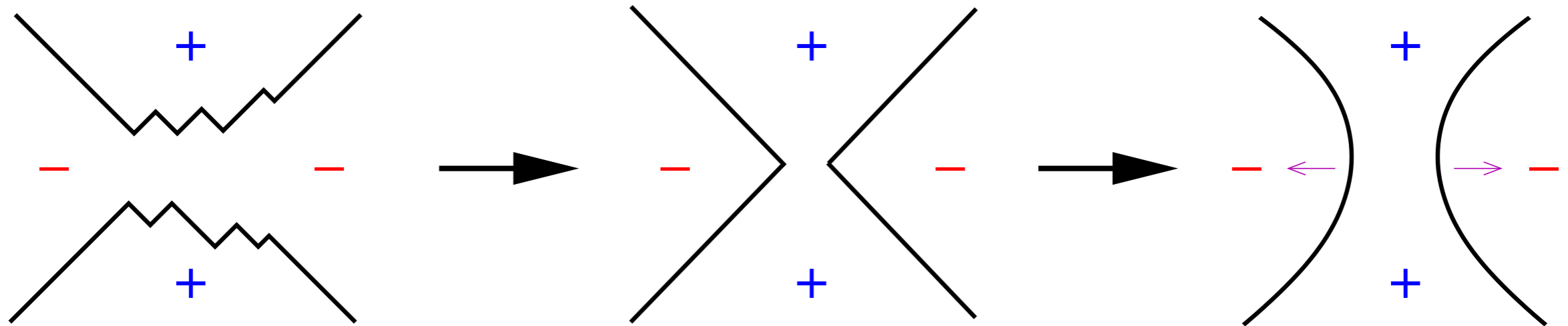
$$N_{\pm} \sim l^2$$

$$N_{+} - N_{-} \sim l$$

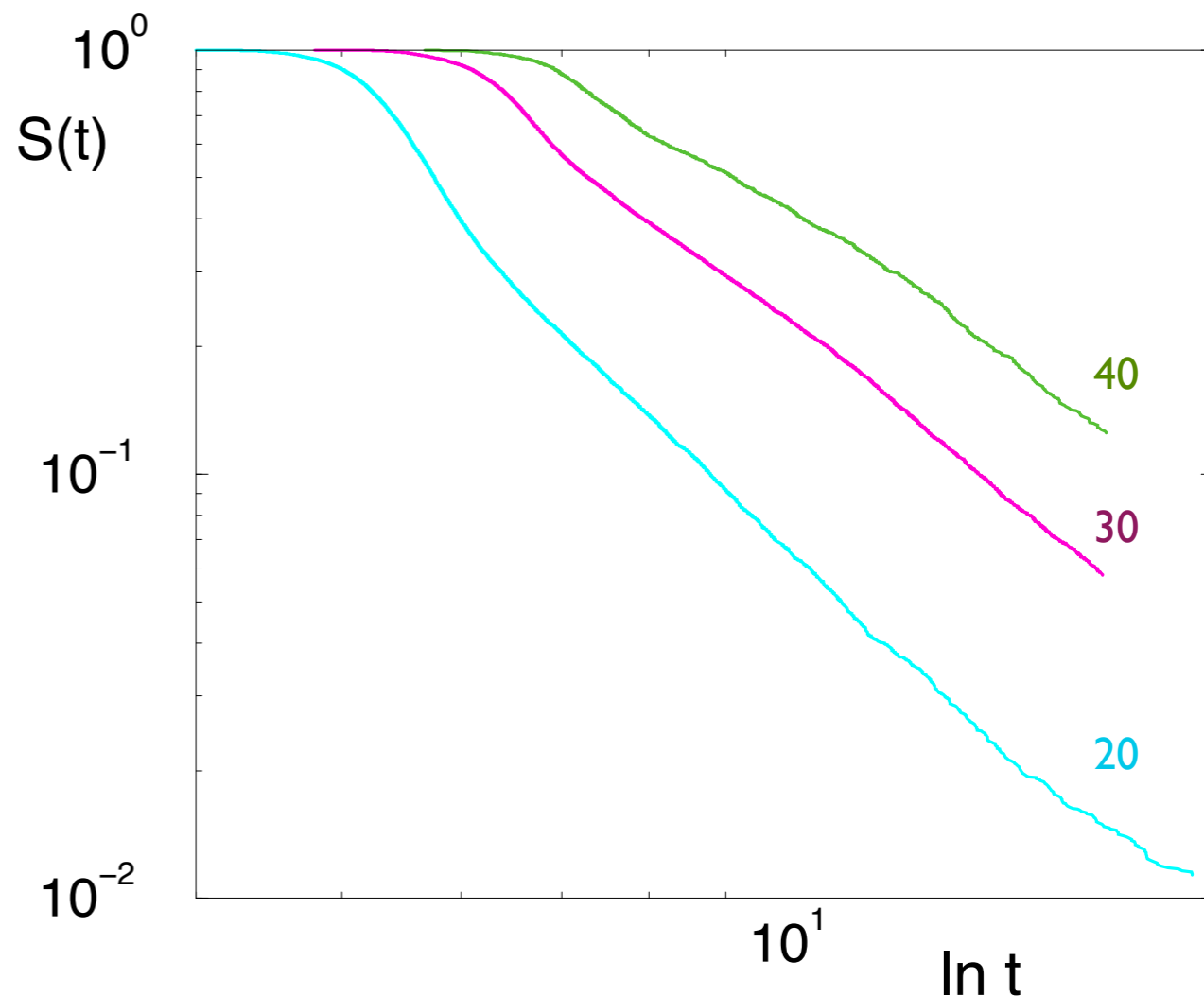
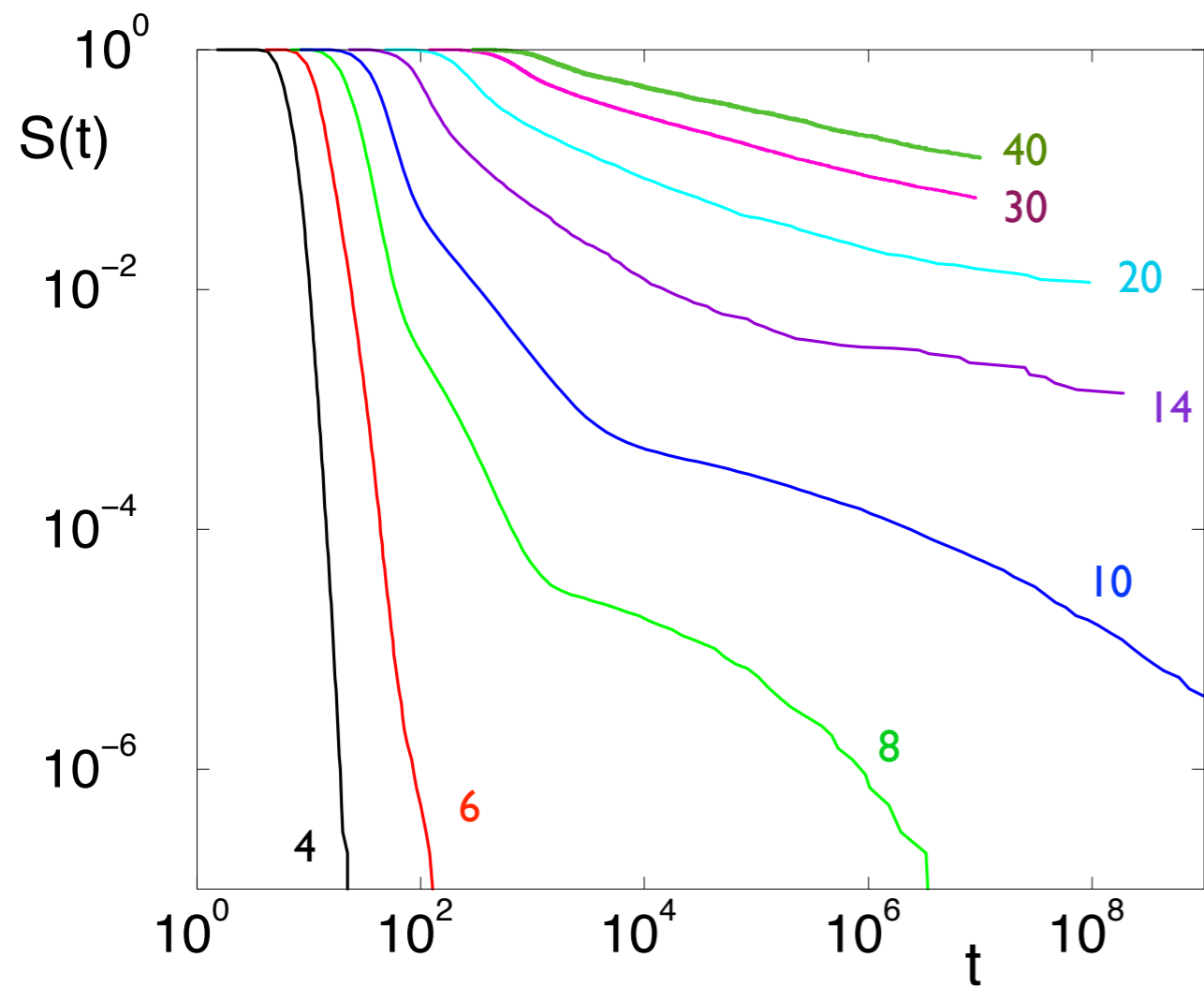
$$\tau \sim e|u|l^3/D \sim e l^2$$

Merging of 2 Inflated Blinkers

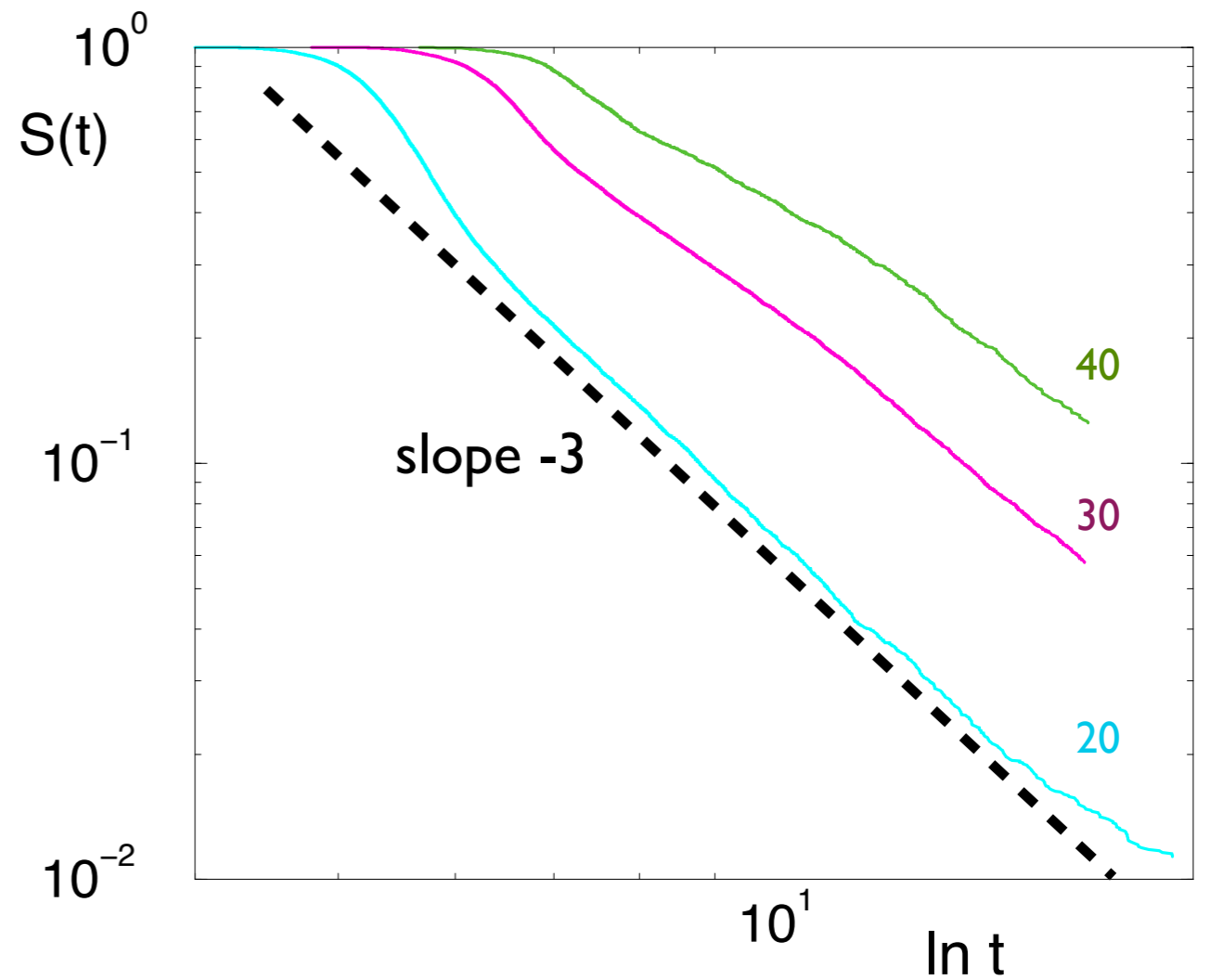
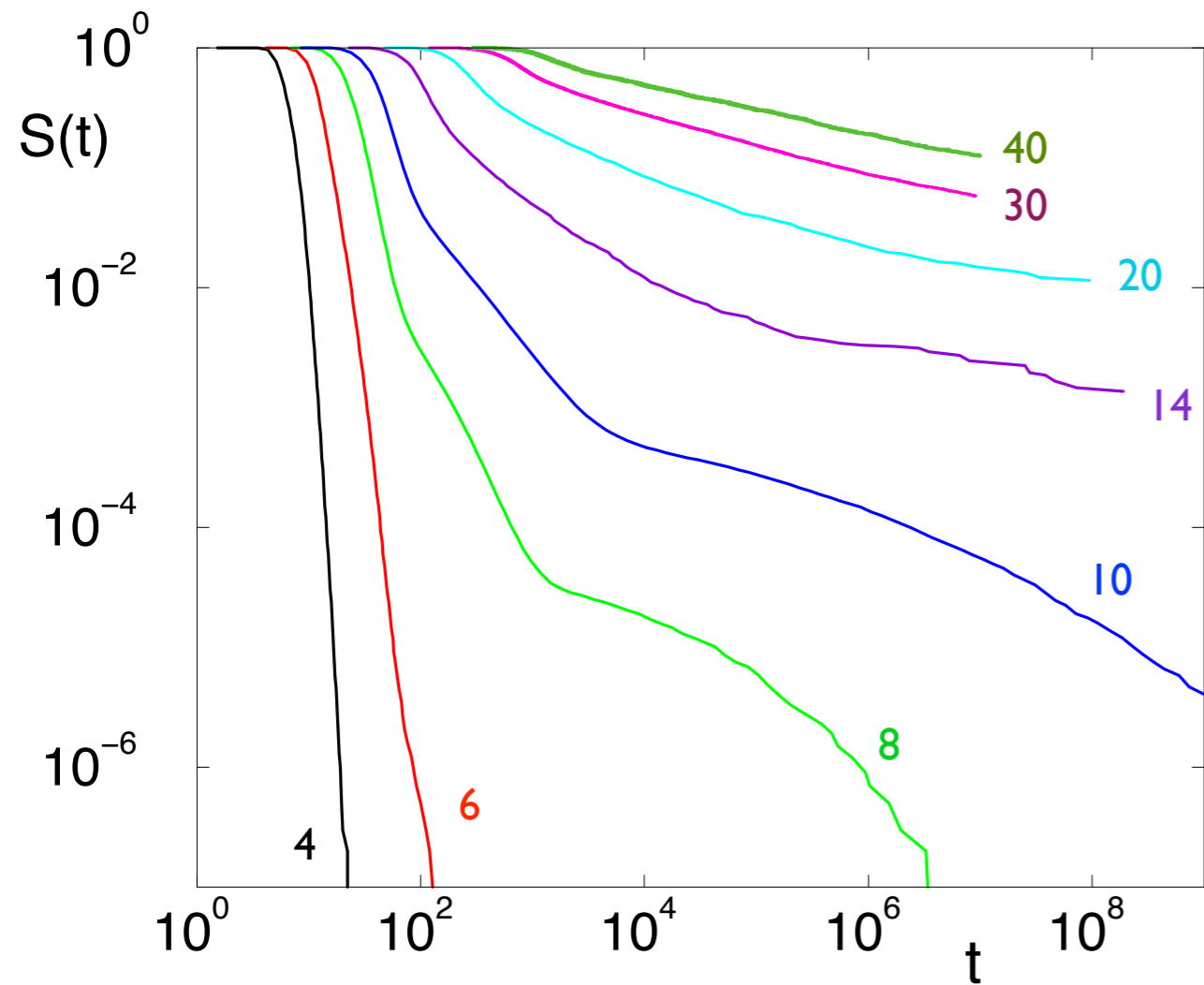
controls the long-time relaxation



Slow Relaxation in 3d



Slow Relaxation in 3d



$$S(t) \sim (\ln t)^{-3} ?$$

Summary & Open Problems

Summary & Open Problems

$d=1$: *almost, but not quite, completely soluble*

final state: the ground state

completion time: L^2

domain length distribution still unsolved

Summary & Open Problems

d=1: *almost, but not quite, completely soluble*

final state: the ground state

completion time: L^2

domain length distribution still unsolved

d=2: **ground & stripe metastable minima**

final state: *usually* the ground state

connection to percolation crossing probabilities

completion time: usually L^2 , sometimes $L^{3.5}$

finite temperature

corner geometry

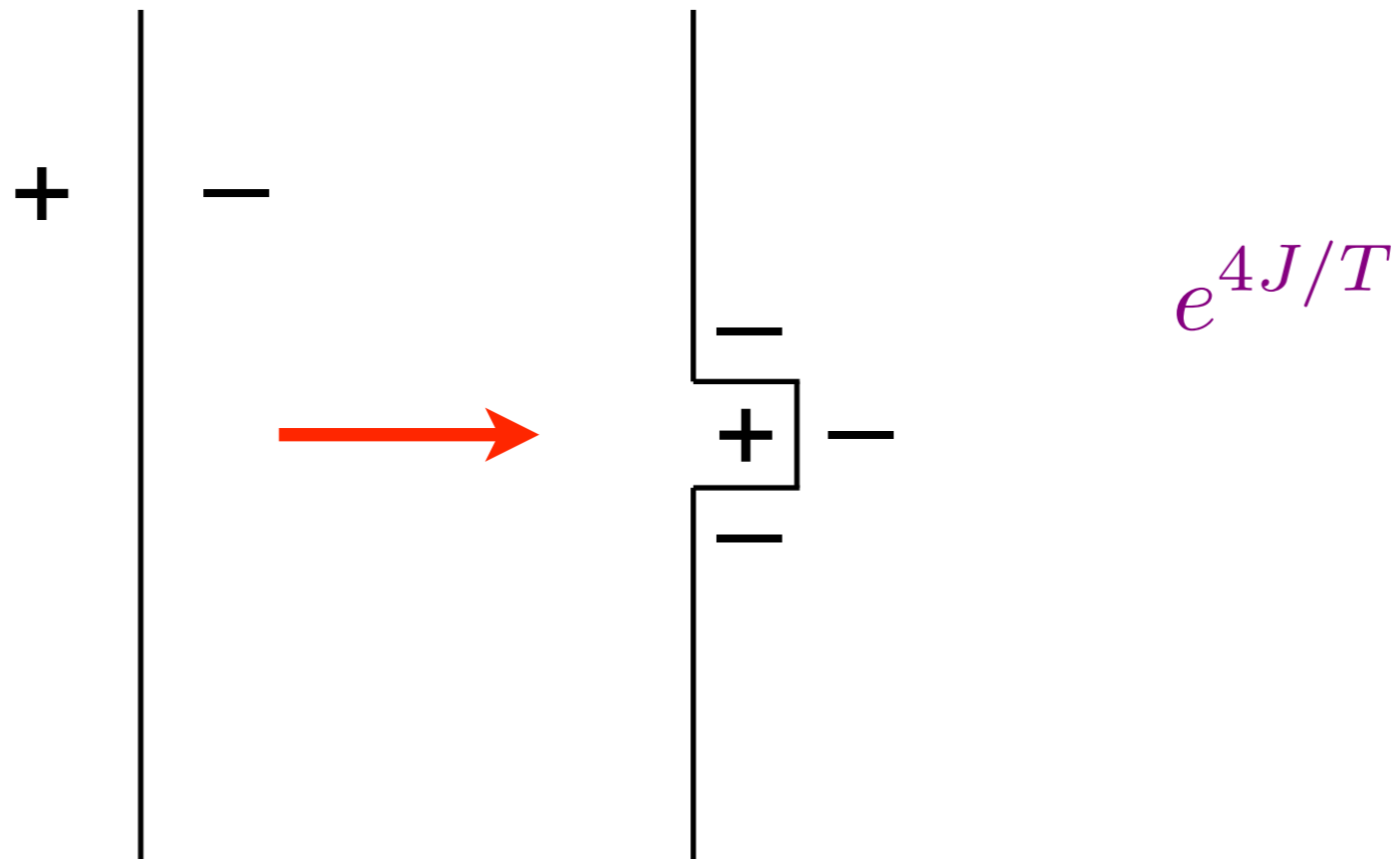
Finite Temperature

lifetime of stripe state:

Finite Temperature

lifetime of stripe state:

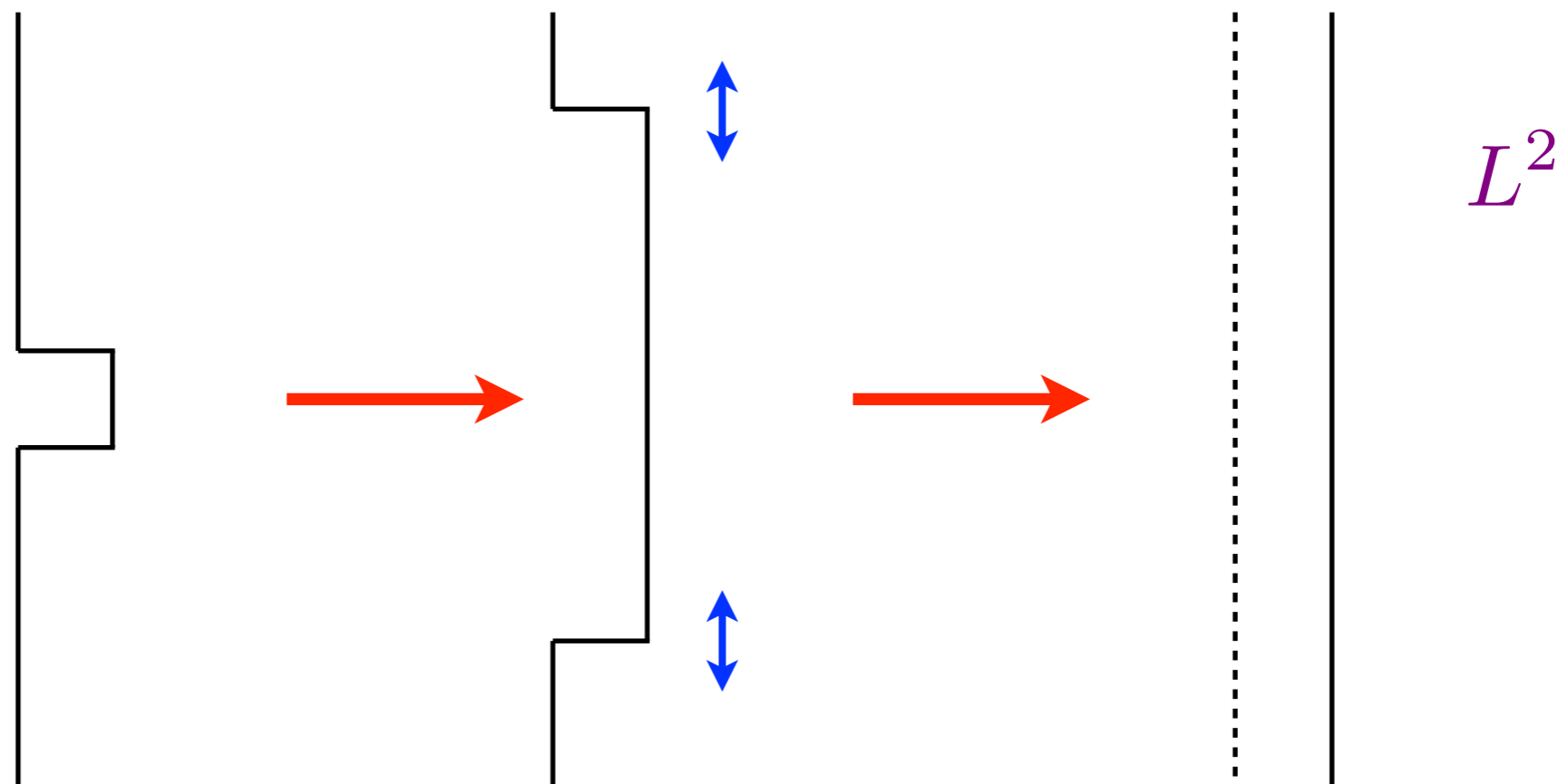
I. defect nucleation



Finite Temperature

lifetime of stripe state:

1. defect nucleation
2. defect propagation, stripe translates by l



Finite Temperature

lifetime of stripe state:

1. defect nucleation
2. defect propagation, stripe translates by l
3. annihilation of two stripes after time L^2

Finite Temperature

lifetime of stripe state:

1. defect nucleation
2. defect propagation, stripe translates by l
3. annihilation of two stripes after time L^2

stripe state lifetime: $\tau \simeq L^4 e^{4J/T}$

Finite Temperature

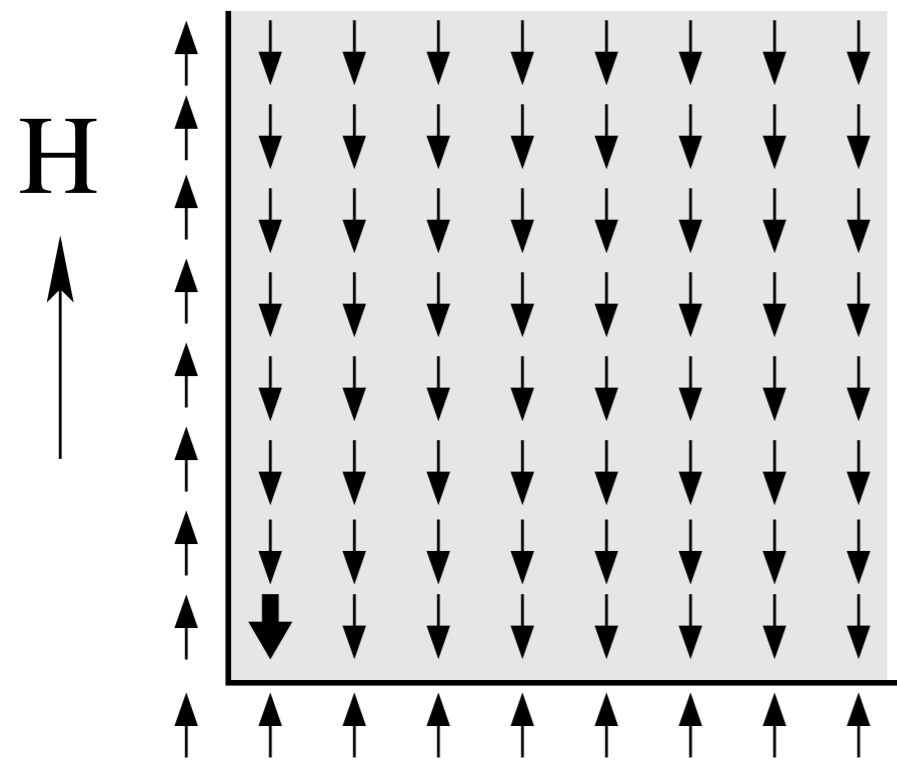
lifetime of stripe state:

1. defect nucleation
2. defect propagation, stripe translates by l
3. annihilation of two stripes after time L^2

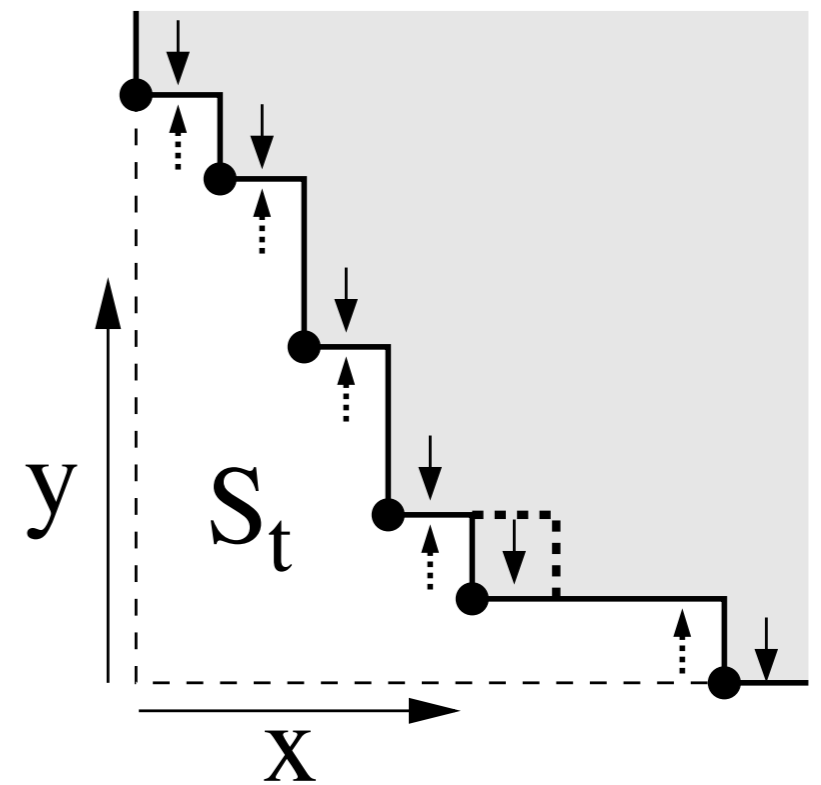
stripe state lifetime: $\tau \simeq L^4 e^{4J/T}$

Open: what is the optimal cooling schedule to ensure that ground state is reached?

Corner Geometry (driven interface)

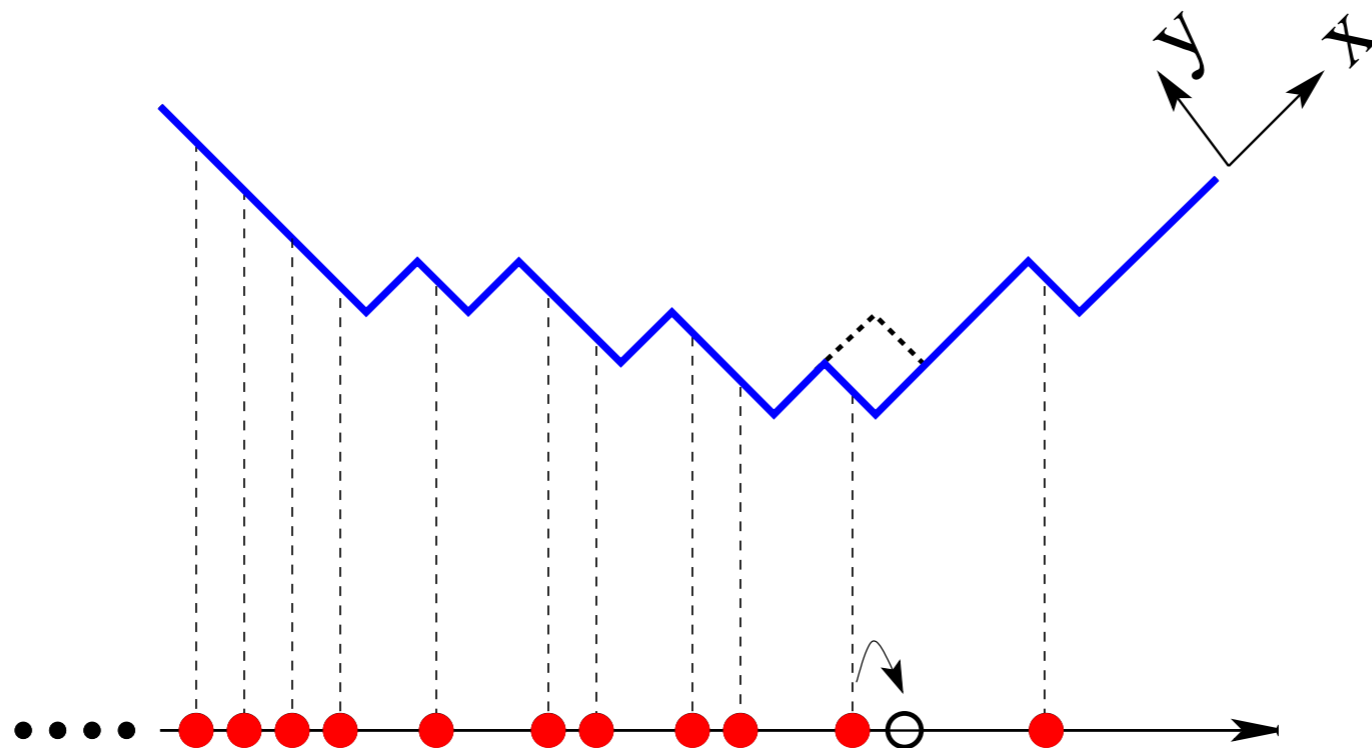


(a)

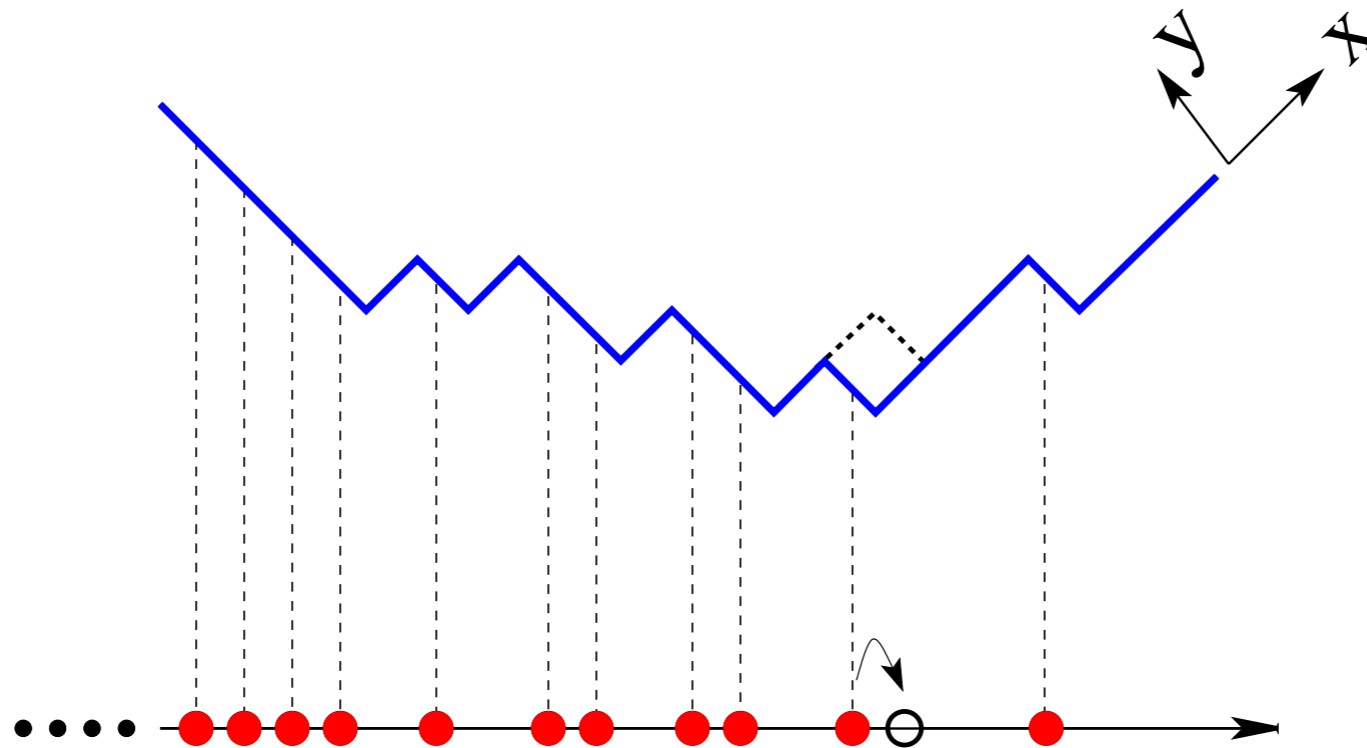


(b)

ASEP Correspondence



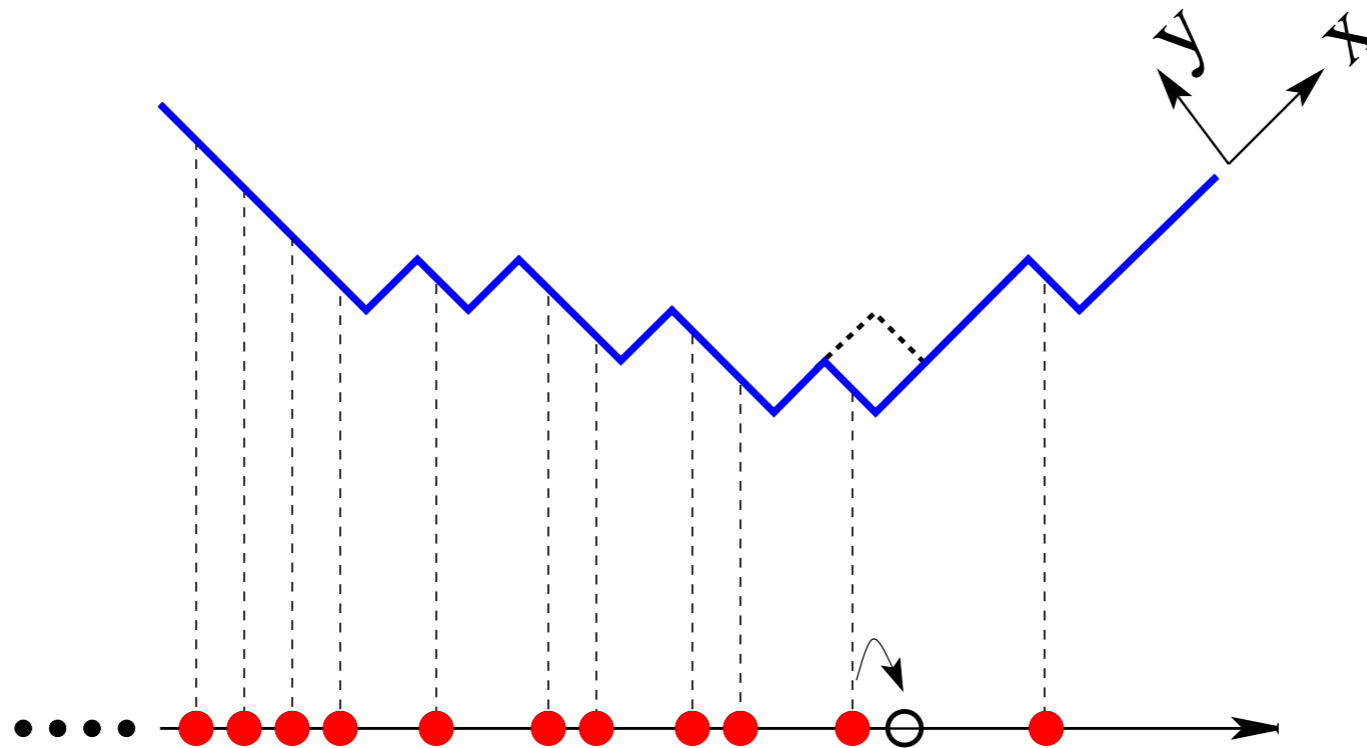
ASEP Correspondence



particle equation
of motion:

$$\frac{\partial n}{\partial t} + \frac{\partial [n(1-n)]}{\partial z} = \frac{\partial^2 n}{\partial z^2} = 0$$

ASEP Correspondence



particle equation
of motion:

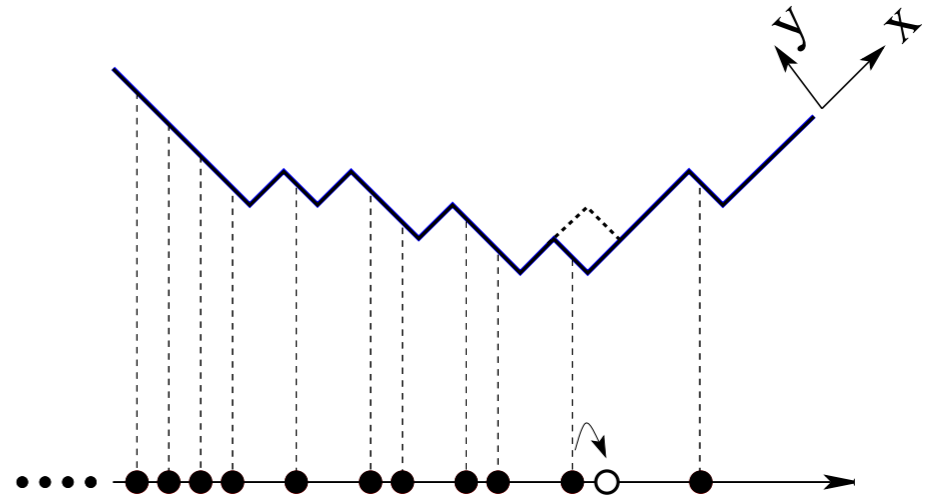
$$\frac{\partial n}{\partial t} + \frac{\partial [n(1-n)]}{\partial z} = \frac{\partial^2 n}{\partial z^2} = 0$$

solution for
step IC:

$$n(z, t) = \begin{cases} 1 & z < -t \\ \frac{1}{2} \left(1 - \frac{z}{t}\right) & |z| < t \\ 0 & z > t \end{cases}$$

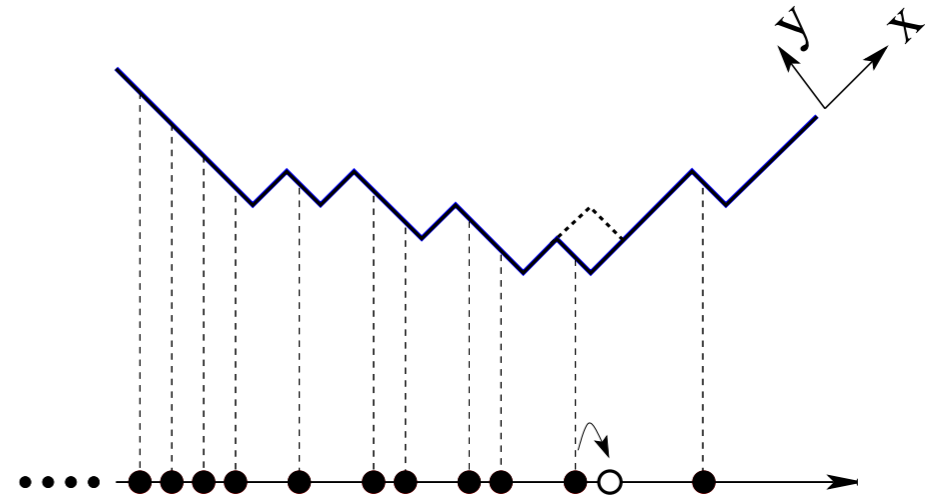
Mapping to *Driven* Ising Interface

$$n(z) = -y'(x) \quad z = x - y$$



Mapping to *Driven* Ising Interface

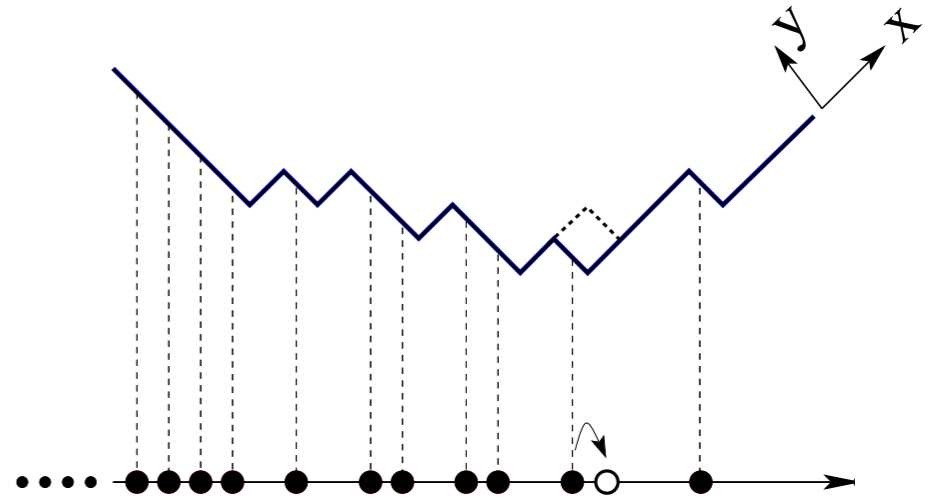
$$n(z) = -y'(x) \quad z = x - y$$



$$y(x, t) = \int_{x-y}^{\infty} n(z, t) dz = \int_{\max(x-y, -t)}^t n(z, t) dz$$

Mapping to *Driven Ising Interface*

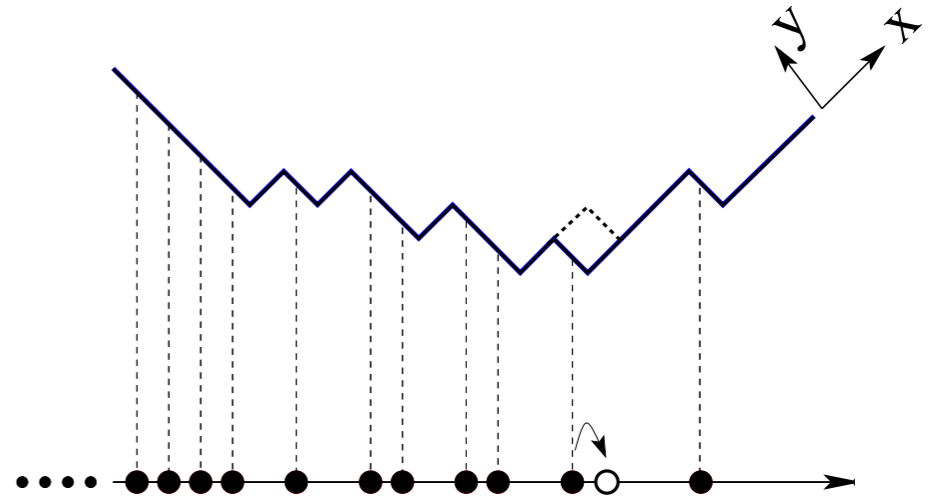
$$n(z) = -y'(x) \quad z = x - y$$



$$y(x, t) = \int_{x-y}^{\infty} n(z, t) dz = \int_{\max(x-y, -t)}^t n(z, t) dz$$

$$y = \begin{cases} \frac{1}{2} \left[\frac{t}{2} - (x - y) + \frac{1}{2t} (x - y)^2 \right] & |x - y| < t \\ 0 & x - y > t \end{cases}$$

Mapping to *Driven Ising Interface*



$$n(z) = -y'(x) \quad z = x - y$$

$$y(x, t) = \int_{x-y}^{\infty} n(z, t) dz = \int_{\max(x-y, -t)}^t n(z, t) dz$$

$$y = \begin{cases} \frac{1}{2} \left[\frac{t}{2} - (x - y) + \frac{1}{2t} (x - y)^2 \right] & |x - y| < t \\ 0 & x - y > t \end{cases}$$

$$\sqrt{x} + \sqrt{y} = \sqrt{t} \quad 0 < x, y < t \quad (\text{Rost, 1981})$$

Undriven Ising Interface

Undriven Ising Interface

particle eqn
of motion:

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial z^2}$$

Undriven Ising Interface

particle eqn
of motion:

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial z^2}$$

solution for
step IC:

$$n(z, t) = \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{4t}}^{\infty} e^{-w^2} dw = \frac{1}{2} \operatorname{erfc} \left(\frac{z}{\sqrt{4t}} \right)$$

Undriven Ising Interface

particle eqn
of motion:

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial z^2}$$

solution for
step IC:

$$n(z, t) = \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{4t}}^{\infty} e^{-w^2} dw = \frac{1}{2} \operatorname{erfc} \left(\frac{z}{\sqrt{4t}} \right)$$

→ implicit form for interface in x, y, t

Undriven Ising Interface

particle eqn
of motion:

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial z^2}$$

solution for
step IC:

$$n(z, t) = \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{4t}}^{\infty} e^{-w^2} dw = \frac{1}{2} \operatorname{erfc} \left(\frac{z}{\sqrt{4t}} \right)$$

→ implicit form for interface in x, y, t

Note: *equilibrium* Ising interface related to
partitions of the integers

Summary & Open Problems

d=1: *almost, but not quite, completely soluble*

final state: the ground state

completion time: L^2

domain length distribution still unsolved

d=2: **ground & stripe metastable minima**

final state: *usually* the ground state

connection to percolation crossing probabilities

completion time: usually L^2 , sometimes $L^{3.5}$

finite temperature

corner geometry

Summary & Open Problems

d=1: *almost, but not quite, completely soluble*

final state: the ground state

completion time: L^2

domain length distribution still unsolved

d=2: **ground & stripe metastable minima**

final state: *usually* the ground state

connection to percolation crossing probabilities

completion time: usually L^2 , sometimes $L^{3.5}$

finite temperature

corner geometry

d \geq 3: **rich state space structure**

topologically complex final state

topological connection between energy & genus

perpetually blinking spins

ultra-slow relaxation *whose functional form is unknown*

finite temperature

corner geometry

Crass Commercialism

Aimed at graduate students, this book explores some of the core phenomena in non-equilibrium statistical physics. It focuses on the development and application of theoretical methods to help students develop their problem-solving skills.

The book begins with microscopic transport processes: diffusion, collision-driven phenomena, and exclusion. It then presents the kinetics of aggregation, fragmentation, and adsorption, where basic phenomenology and solution techniques are emphasized. The following chapters cover kinetic spin systems, by developing both a discrete and a continuum formulation, the role of disorder in non-equilibrium processes, and hysteresis from the non-equilibrium perspective. The concluding chapters address population dynamics, chemical reactions, and a kinetic perspective on complex networks. The book contains more than 200 exercises to test students' understanding of the subject. A link to a website hosted by the authors, containing an up-to-date list of errata and solutions to some of the exercises, can be found at www.cambridge.org/9780521851039.

Pavel L. Krapivsky is Research Associate Professor of Physics at Boston University. His current research interests are in strongly interacting many-particle systems and their applications to kinetic spin systems, networks, and biological phenomena.

Sidney Redner is a Professor of Physics at Boston University. His current research interests are in non-equilibrium statistical physics and its applications to reactions, networks, social systems, biological phenomena, and first-passage processes.

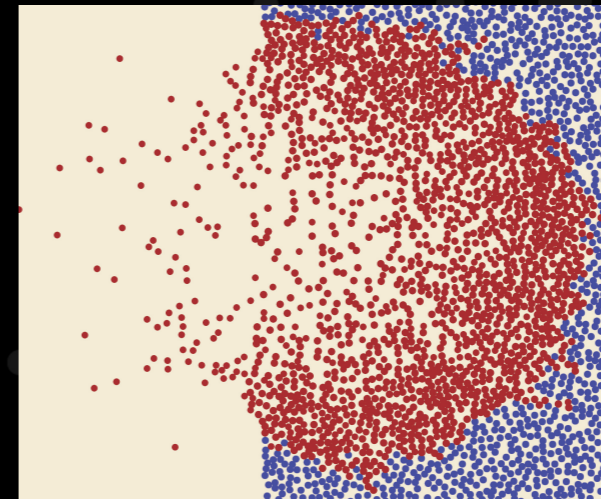
Eli Ben-Naim is a member of the Theoretical Division and an affiliate of the Center for Nonlinear Studies at Los Alamos National Laboratory. He conducts research in statistical, nonlinear, and soft condensed-matter physics, including the collective dynamics of interacting particle and granular systems.

Cover illustration: Snapshot of a collision cascade in a perfectly elastic, initially stationary hard-sphere gas in two dimensions due to a single incident particle. Shown are the cloud of moving particles (red) and the stationary particles (blue) that have not yet experienced any collisions. Figure courtesy of Tibor Antal.

Krapivsky
Redner
Ben-Naim

A Kinetic View of STATISTICAL PHYSICS

A Kinetic View of STATISTICAL PHYSICS



Pavel L. Krapivsky
Sidney Redner
Eli Ben-Naim

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-0-521-85103-9



9 780521 851039

CAMBRIDGE

published
Dec. 2010

1. Aperitifs
2. Diffusion
3. Collisions
4. Exclusion
5. Aggregation

6. Fragmentation
7. Adsorption
8. Spin Dynamics
9. Coarsening
10. Disorder

11. Hysteresis
12. Population Dynamics
13. Diffusion Reactions
14. Complex Networks

> 200 problems