Structure of Preferential Attachment Networks

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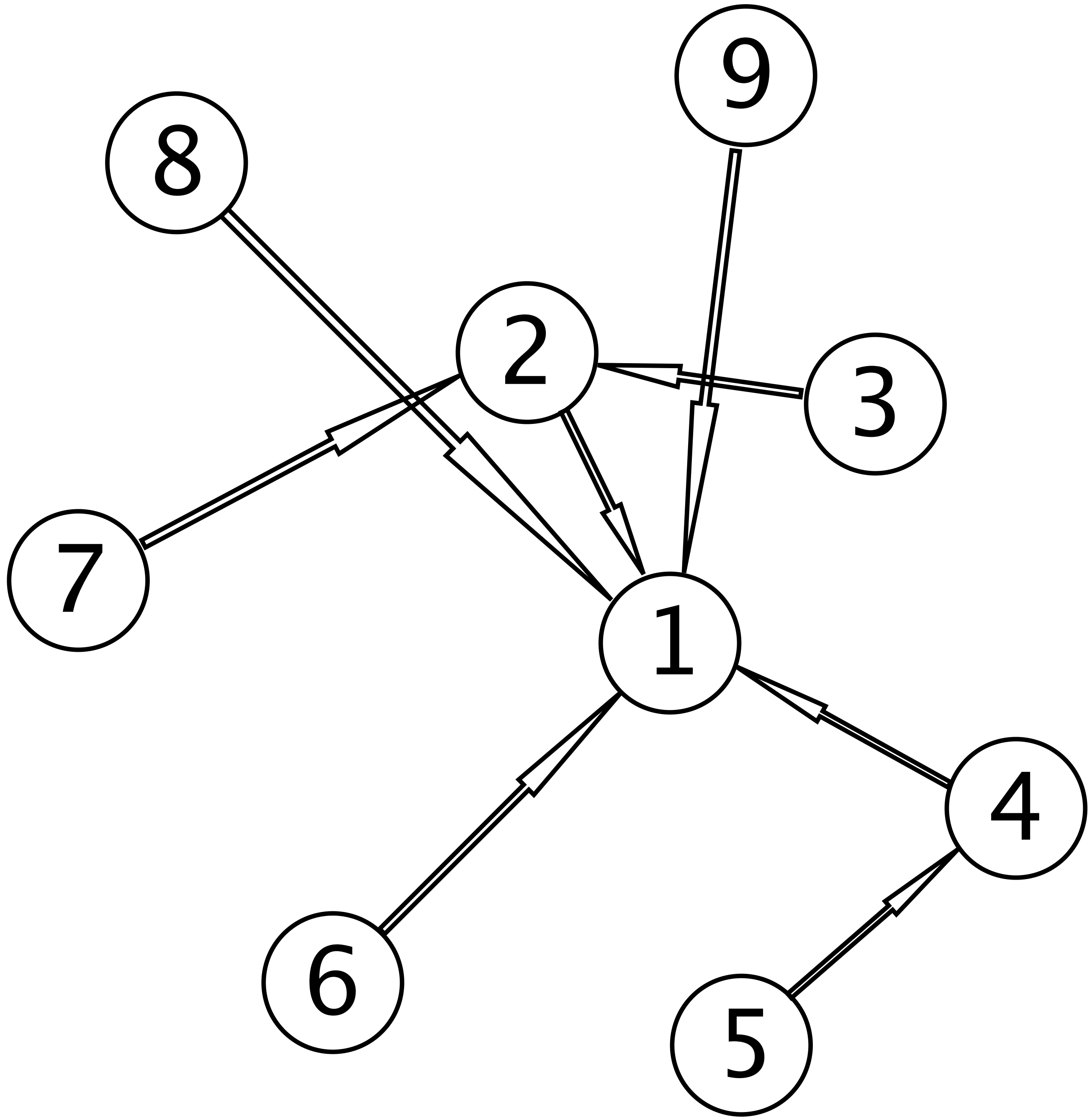
Outline

- universal and non-universal degree distributions
- fitness, redirection & copying
- do the rich really get richer?
- finiteness & fluctuations

Outlook

Preferential Attachment Network

Yule (1926), Simon (1955)
Barabasi & Albert (1999)
Dorogovtsev, Mendes et al.
KRL, KR



1. Introduce nodes one at a time
2. Attach to one earlier node with k links at rate A_k

Master Equation Approach

Basic observable: N_k, *average* number of nodes with k links
the degree distribution

Master Equation:

	attach to node of degree k-1	attach to node of degree k	create node of degree 1

$$\frac{dN_k}{dN} = \frac{A_{k-1}N_{k-1}}{A} - \frac{A_k N_k}{A} + \delta_{k,1} \qquad A = \text{total rate}$$

Attachment Rate: $A_k \sim k^\gamma$

Total Rate: $A = A(N) = \sum_{j=1}^{\infty} A_j N_j = \sum_{j=1}^{\infty} j^\gamma N_j \equiv M_\gamma(N)$

Moment equations:

$$\dot{M}_0 \equiv \sum_j \dot{N}_j = 1; \quad \dot{M}_1 \equiv \sum_j j\dot{N}_j = 2$$

These suggest: $A(N) = \sum_j j^\gamma N_j \propto \mu(\gamma)N \quad \text{for } 0 \leq \gamma \leq 1$

$$N_k(N) \equiv N\, n_k$$

Converts rate eqns. to linear recursions

$$\frac{dN_k}{dN} = \frac{A_{k-1}N_{k-1}}{A} - \frac{A_k N_k}{A} + \delta_{k,1}$$

$$\Rightarrow \quad n_k = \frac{A_{k-1}n_{k-1} - A_k n_k}{\mu} + \delta_{k,1}$$

Formal solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$ $\quad n_k = \frac{A_{k-1} n_{k-1} - A_k n_k}{\mu} + \delta_{k,1}$

Asymptotics for $A_k \sim k^{\gamma}$

$$n_k \sim \begin{cases} k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \leq \gamma < 1 \end{cases}$$

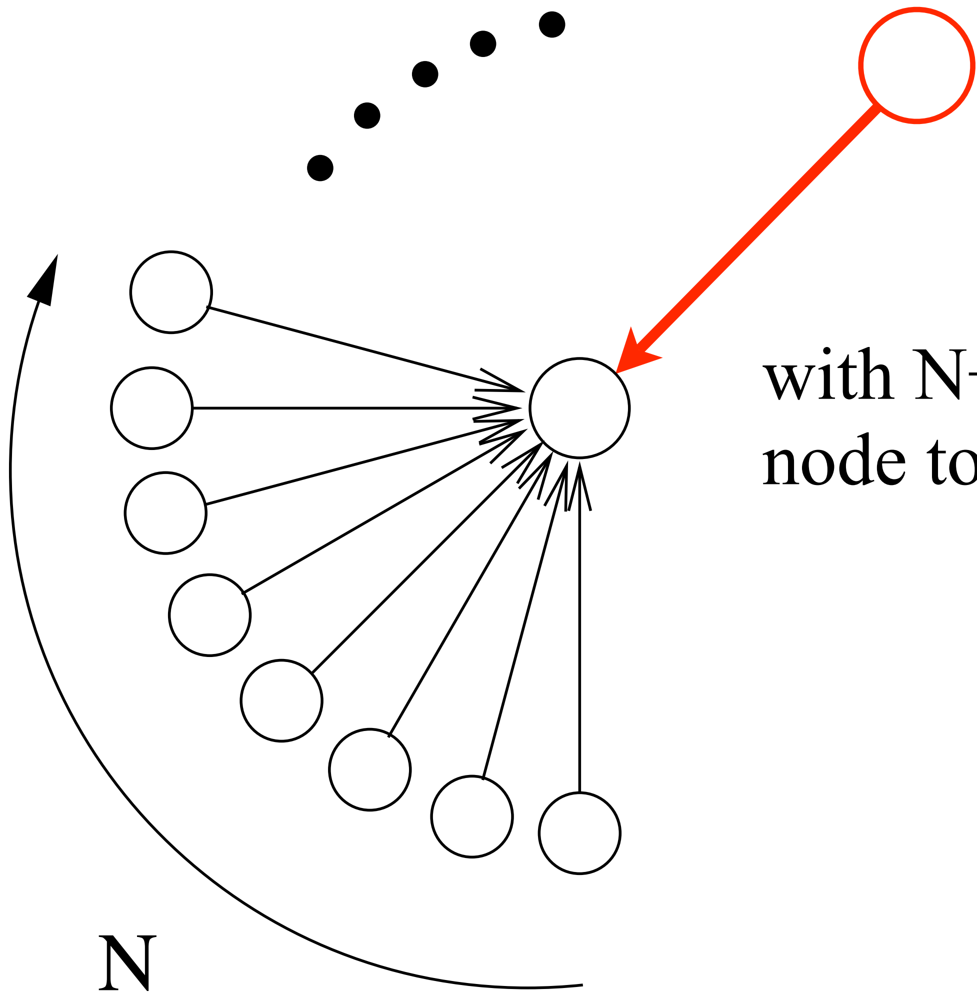
universal, generic

Formal solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

Asymptotics for $A_k \sim k^\gamma$

$$n_k \sim \begin{cases} k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \le \gamma < 1 \\ \\ \text{“bible”} & \gamma > 2 \end{cases}$$

Creating a "Bible"



with N+1 total nodes: attach new node to bible with probability $\frac{N^\gamma}{N + N^\gamma}$

bible probability: $$\mathcal{P} = \prod_{N=1}^{\infty} \frac{1}{1 + N^{1-\gamma}} = \begin{cases} \text{zero} & \gamma \leq 2 \\ \text{non-zero} & \gamma > 2 \end{cases}$$

Formal solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

Asymptotics for $A_k \sim k^\gamma$

$$n_k \sim \begin{cases} k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \le \gamma < 1 \\ \\ \text{“best seller”} & 1 < \gamma \le 2 \\ \text{“bible”} & \gamma > 2 \end{cases}$$

"Best-Seller" Phase: $1<\gamma\leq 2$

fact: $A(N) = \sum_k k^\gamma N_k \sim N^\gamma$

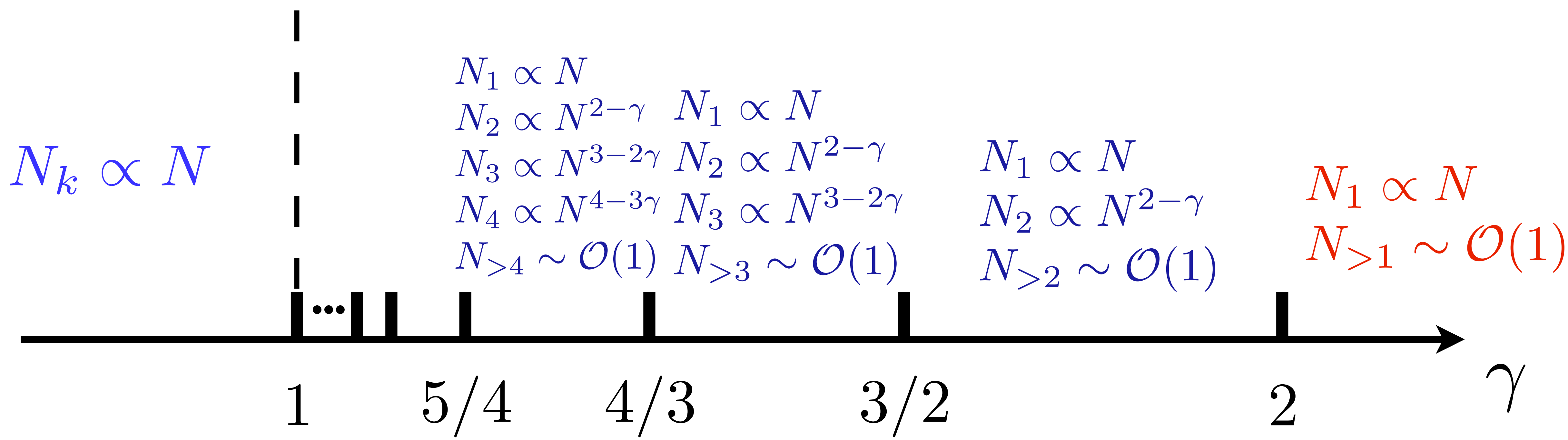
$$\dot{N}_1 = 1 - \frac{N_1}{A} \quad \rightarrow N_1 \propto N$$

$$\dot{N}_2 = \frac{N_1 - 2^\gamma N_2}{N^\gamma} \quad \propto N^{1-\gamma} \rightarrow N_2 \propto N^{2-\gamma}$$

$$\dot{N}_3 = \frac{2^\gamma N_2 - 3^\gamma N_3}{N^\gamma} \quad \propto N^{2-2\gamma} \rightarrow N_3 \propto N^{3-2\gamma}$$

$$\longrightarrow N_k \propto N^{k-(k-1)\gamma}$$

for positive exponent



Formal solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

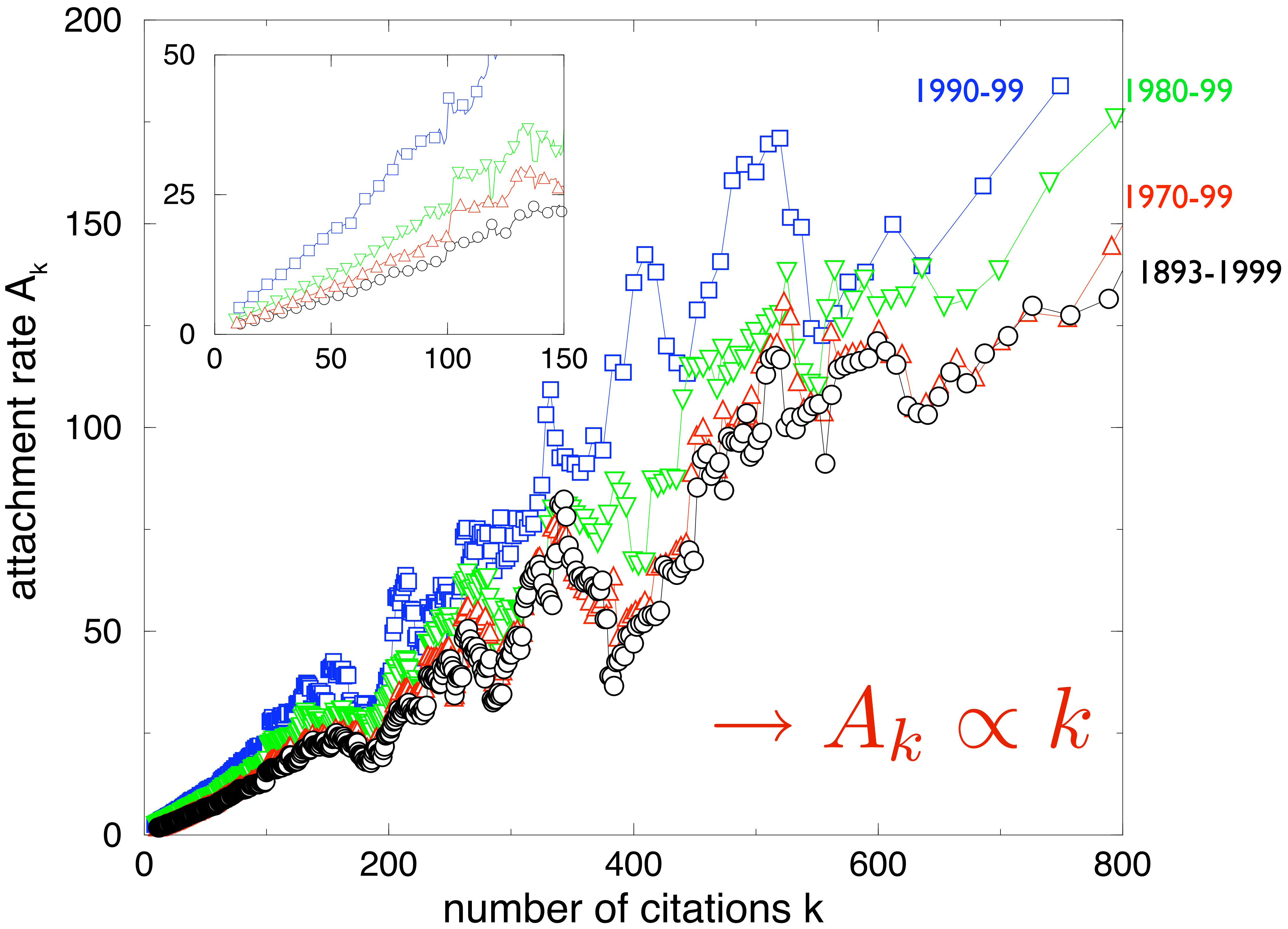
Asymptotics for $A_k \sim k^\gamma$

$$n_k \sim \begin{cases} k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \le \gamma < 1 \\ k^{-\nu},\ \nu > 2 \quad \textbf{!!!} & \gamma = 1 \\ \text{“best seller”} & 1 < \gamma \le 2 \\ \text{“bible”} & \gamma > 2 \end{cases}$$

Special case, $A_k = k$: $n_k = \frac{4}{k(k+1)(k+2)}$

Why Care About *Linear* Preferential Attachment?

Distribution of Citations



Why Worry About Linear Preferential Attachment?

consider $A_k = \begin{cases} \alpha & k = 1 \\ k & k \geq 2 \end{cases}$

substitute into $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_k}\right)^{-1}$ with $\mu = \sum_j A_j n_j$

leads to $\sum_{k=1}^{\infty} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1} = 1$

$$n_k \sim k^{-\nu} \qquad \nu = 1 + \mu = \frac{3 + \sqrt{1 + 8\alpha}}{2}$$

Intrinsic Node Fitness

Bianconi & Barabasi (2000)

- each node has fitness η chosen from $p_0(\eta)$

Master equation:

$$\frac{dN_k(\eta)}{dN} = \frac{A_{k-1}(\eta)N_{k-1}(\eta) - A_k(\eta)N_k(\eta)}{A} + p_0(\eta)\delta_{k1}$$

$$A = \int d\eta \sum_k A_k(\eta)N_k(\eta)$$

Solution:

$$n_k(\eta) = p_0(\eta)\frac{\mu}{A_k(\eta)}\prod_{j=1}^{k}\left(1 + \frac{\mu}{A_j(\eta)}\right)^{-1}$$

For $A_k(\eta) = \eta k$:

$$n_k(\eta) = \frac{\mu\, p_0(\eta)}{\eta}\,\frac{\Gamma(k)\,\Gamma\left(1 + \frac{\mu}{\eta}\right)}{\Gamma\left(k + 1 + \frac{\mu}{\eta}\right)} \to k^{-(1+\mu/\eta)}$$

Determine μ:

$$A = \mu N = \int d\eta \sum_{k\geq 1} A_k(\eta)\, N_k(\eta) \quad \to 1 = \int d\eta\, p_0(\eta)\left(\frac{\mu}{\eta} - 1\right)^{-1}$$

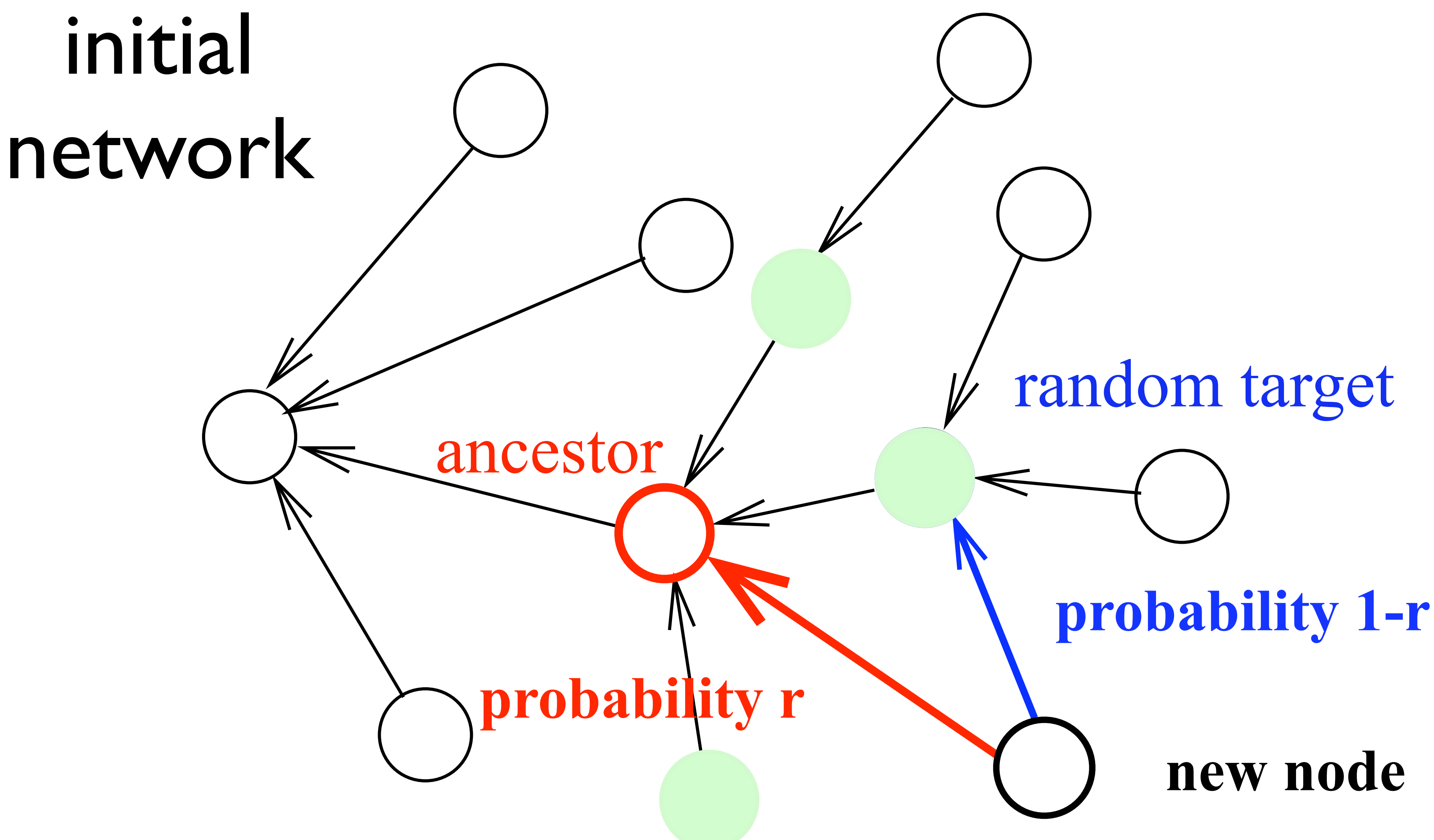
arbitrary charm: BE condensation

bounded charm: $n_k \sim k^{-(1+\mu/\eta_{\max})}(\ln k)^{\omega}$

A Reason to Like Linear Attachment:

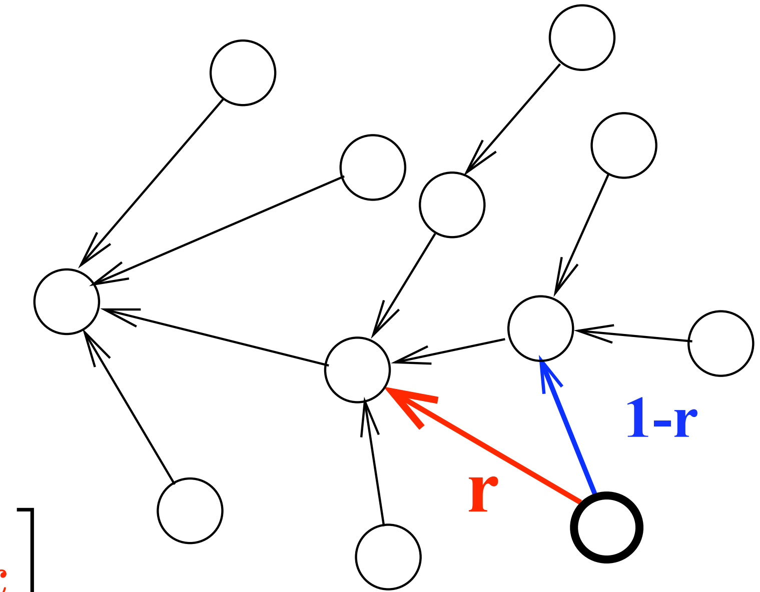
Random Attachment+Redirection

Kleinberg et al (1999)



attachment rate to ancestor node:
∝ number of upstream neighbors

Master Equation:



$$\frac{dN_k}{dN} = \frac{1-r}{M_0}\left[N_{k-1} - N_k\right] + \delta_{k,1} + \frac{r}{M_0}\left[(k-2)N_{k-1} - (k-1)N_k\right]$$

$$= \frac{r}{M_0}\left\{\left[(k-1) + \frac{1}{r} - 2\right]N_{k-1} - \left[k + \frac{1}{r} - 2\right]N_k\right\} + \delta_{k,1}$$

⇨ *shifted* linear attachment rate:

$$A_k = k + \left(\frac{1}{r} - 2\right) \equiv k + \lambda$$

O(1) rule → *global* (shifted) linear preferential attachment!

Solution for Shifted Linear Attachment:

substitute $A_k = k + \lambda$ into

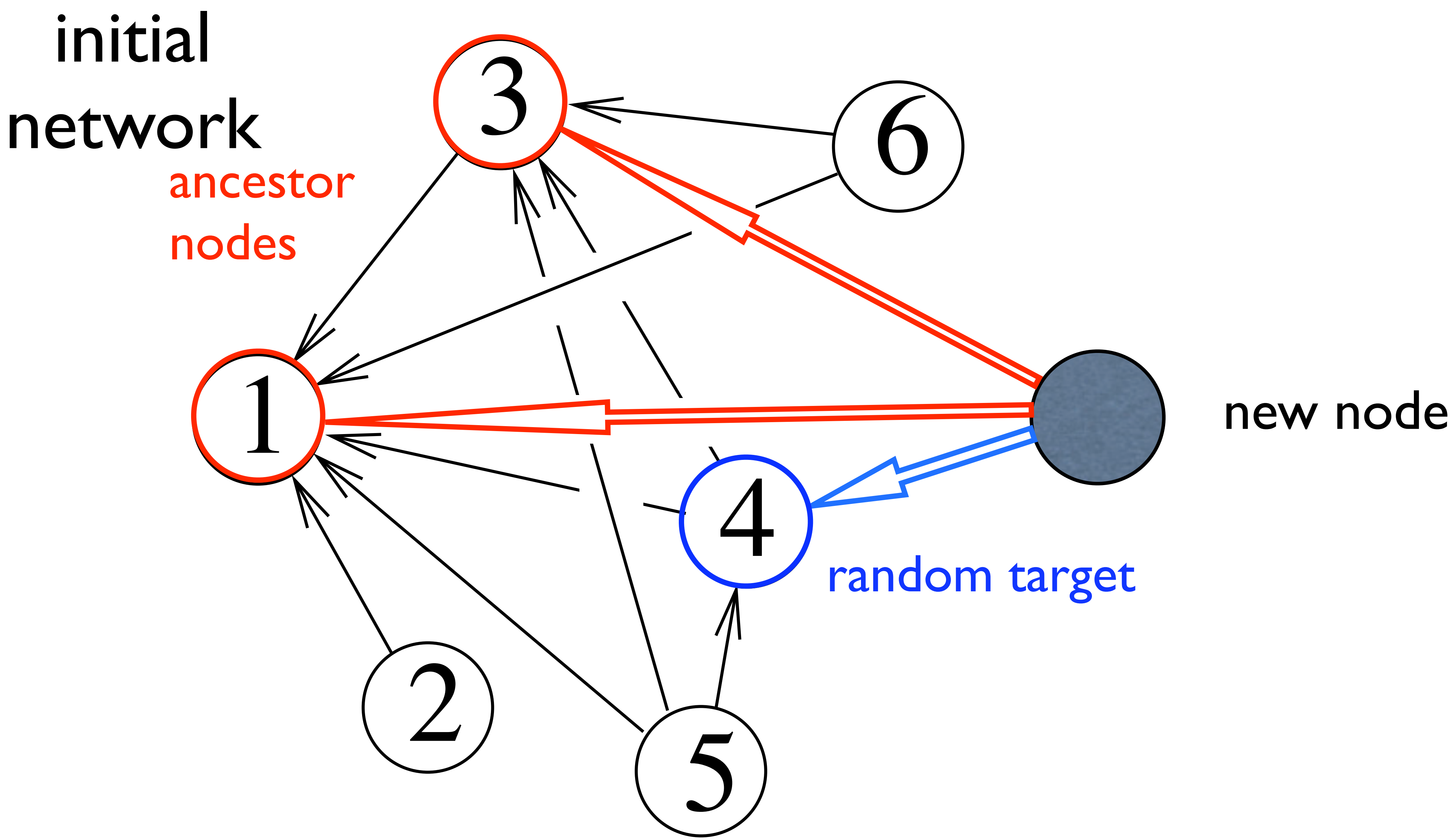
$$n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$$

gives

$$\begin{aligned} n_k &= (2+\lambda)\,\frac{\Gamma(3+2\lambda)}{\Gamma(1+\lambda)}\,\frac{\Gamma(k+\lambda)}{\Gamma(k+3+2\lambda)} \\ &\sim k^{-(3+\lambda)} \qquad (-1 < \lambda < \infty) \end{aligned}$$

Random Attachment + Copying

(a lazy person's approach to references)

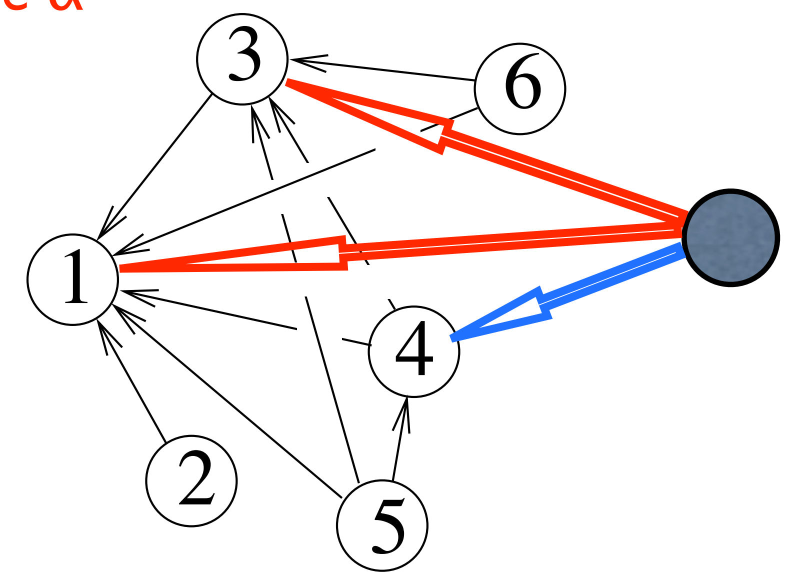


Mean Number of Links L(N)

Evolution equation:

ancestors of node α

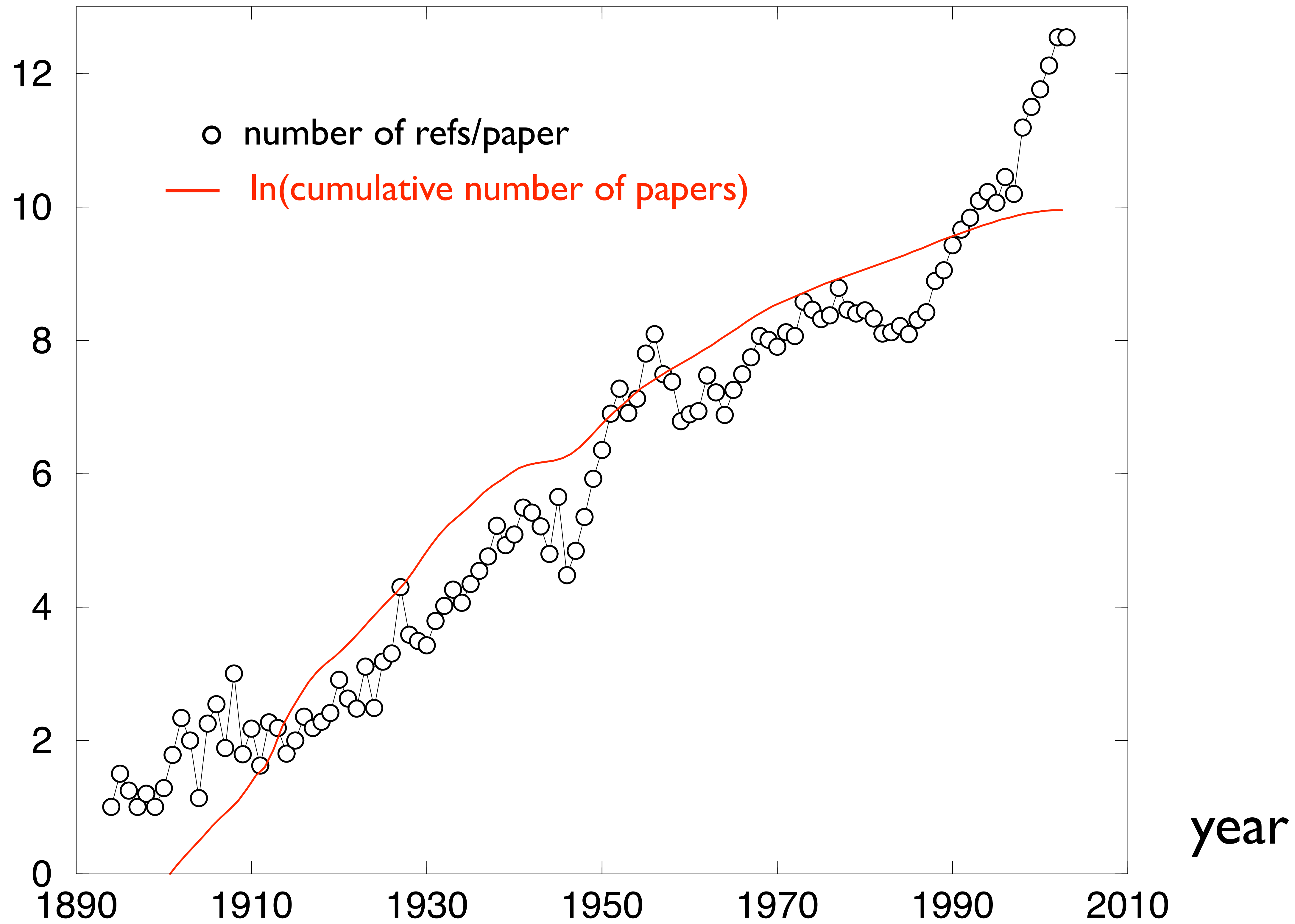
$$L(N+1) = L(N) + \frac{1}{N}\sum_{\alpha}(1+j_\alpha)$$
$$= L(N) + 1 + \frac{L(N)}{N}$$



Solution:

$$L(N) = N(H_N - 1)$$
$$= N\ln N - N(1-\gamma) + \frac{1}{2} - \frac{1}{12N} + \dots$$

Comparison with Phys Rev Reference Data



How Do The Rich Get Richer?

P(k,N)≡ prob. 1st node has degree k in a network of N links*

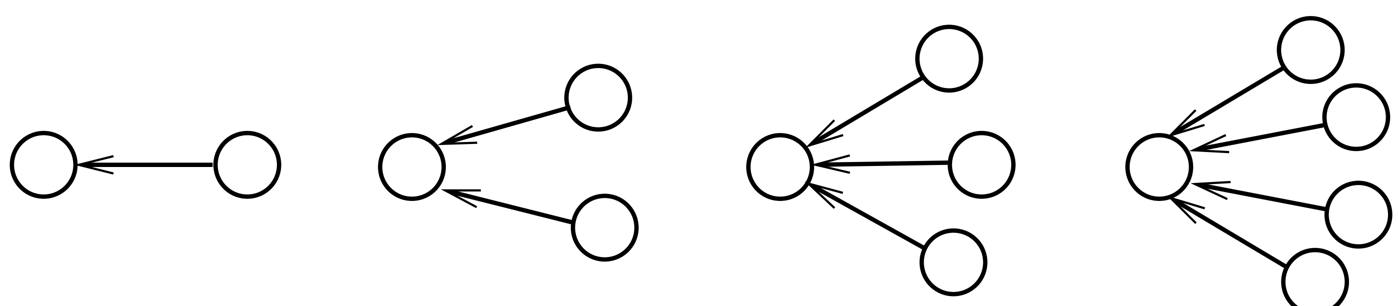
Master equation for linear preferential attachment:

attach to 1st node — attach to later node

$$P(k, N+1) = \frac{k-1}{2N}\, P(k-1, N) + \frac{2N-k}{2N}\, P(k, N)$$

Mean degree of 1st node: $\langle k \rangle_{N+1} = \langle k \rangle_N \left(1 + \frac{1}{2N}\right)$

Solution: $\langle k \rangle_N = \Lambda \frac{\Gamma\left(N+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(N)} \sim \frac{\Lambda}{\sqrt{\pi}} N^{1/2}$



Λ=2 Λ=8/3 Λ=16/5 Λ=128/35

initial "wealth" matters!

Master equation for random attachment:

$$P(k, N+1) = \frac{1}{N} P(k-1, N) + \frac{N-1}{N} P(k, N)$$

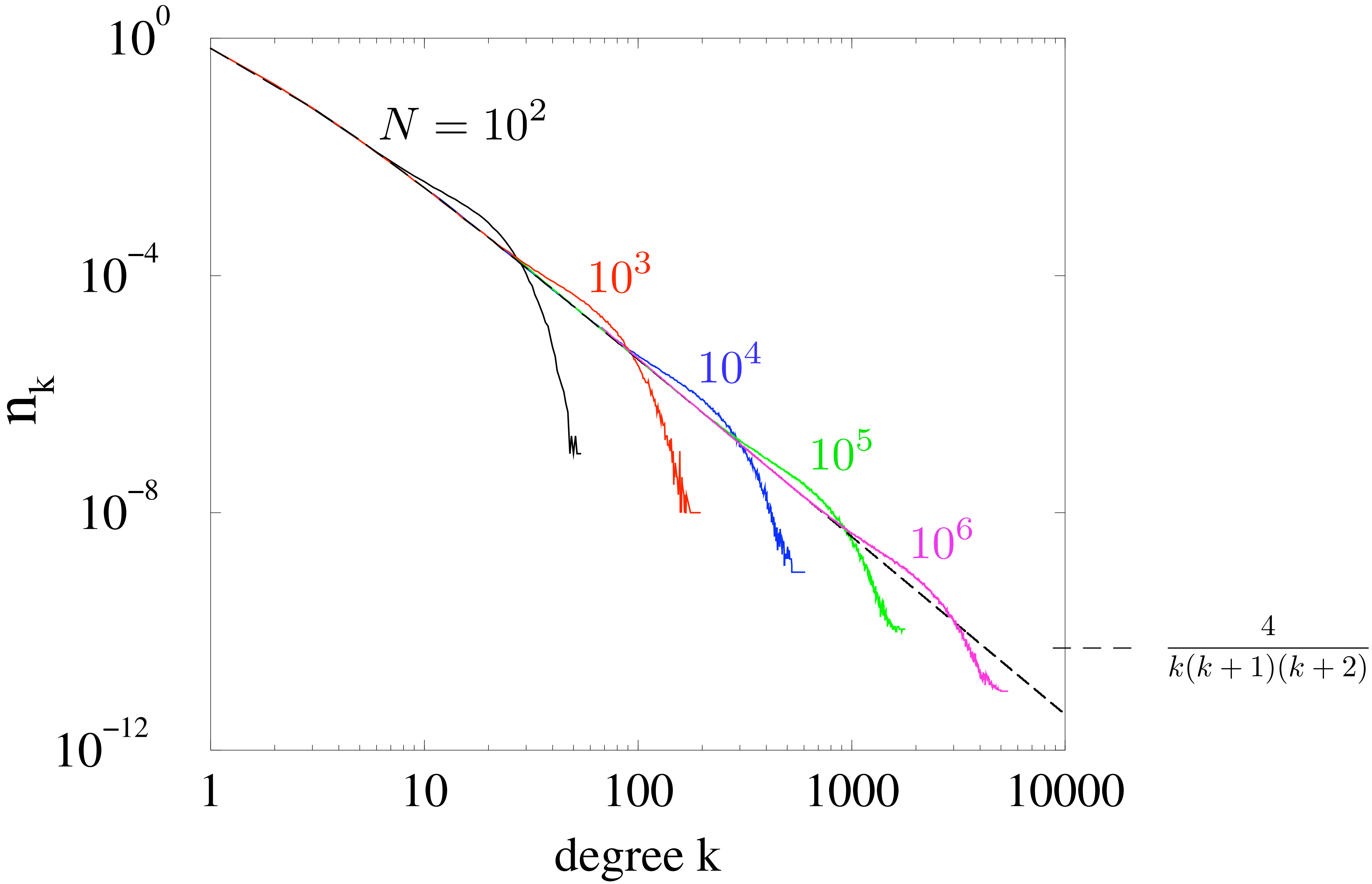
Mean degree of 1st node: $\langle k \rangle_{N+1} = \langle k \rangle_N + \frac{1}{N}$

Solution: $\langle k \rangle_N = H_N + \Lambda \sim \ln N + \Lambda$

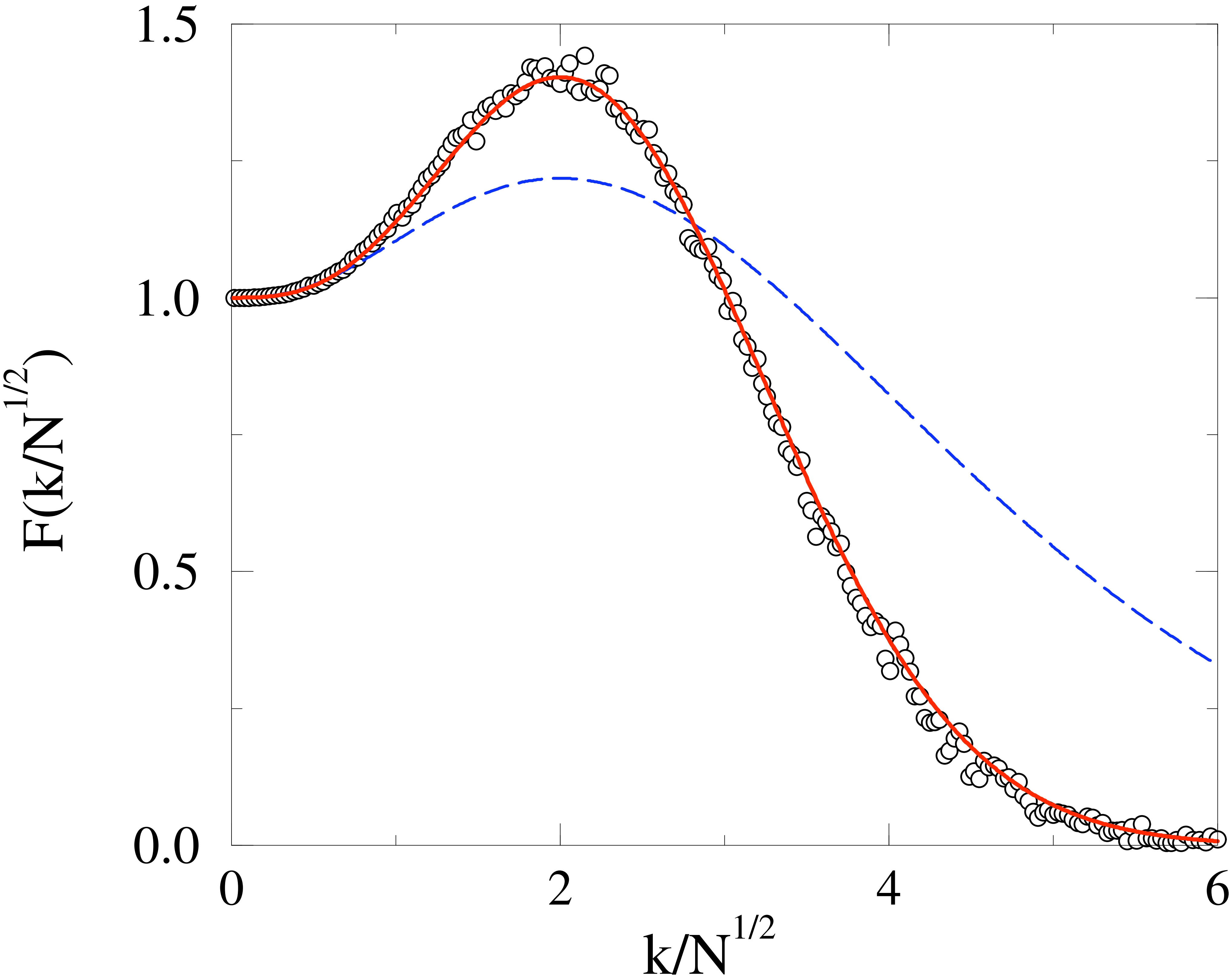
initial "wealth" doesn't really matter

Degree Distribution of a *Finite* Network

$$N_k(N) \sim N\, n_k\, \underbrace{F(k/k_{\max})}_{\text{sensitive to IC}} \qquad k_{\max} \propto N^{1/2}$$



The Scaling Function



The Scaling Function (continuum) (& discrete)

substitute $\mathcal{N}(N,z) = \sum_{k=1}^{\infty} N_k(N)\, z^k$ in $\frac{dN_k(N)}{dN} = \frac{(k-1)N_{k-1}(N) - kN_k(N)}{2N} + \delta_{k,1}$

leads to $\left[2N\frac{\partial}{\partial N} + z(1-z)\frac{\partial}{\partial z}\right]\mathcal{N}(N,z) = 2Nz$

solution:

$$\mathcal{N}(N,z) = (3-2z^{-1})N + 2(z^{-1}-1)\sqrt{N} + \frac{1-(z^{-1}-1)\sqrt{N}}{1+(z^{-1}-1)\sqrt{N}}$$
$$- 2(z^{-1}-1)^2 N \ln\left(1 - z + \frac{z}{\sqrt{N}}\right)$$

extract scaling function:

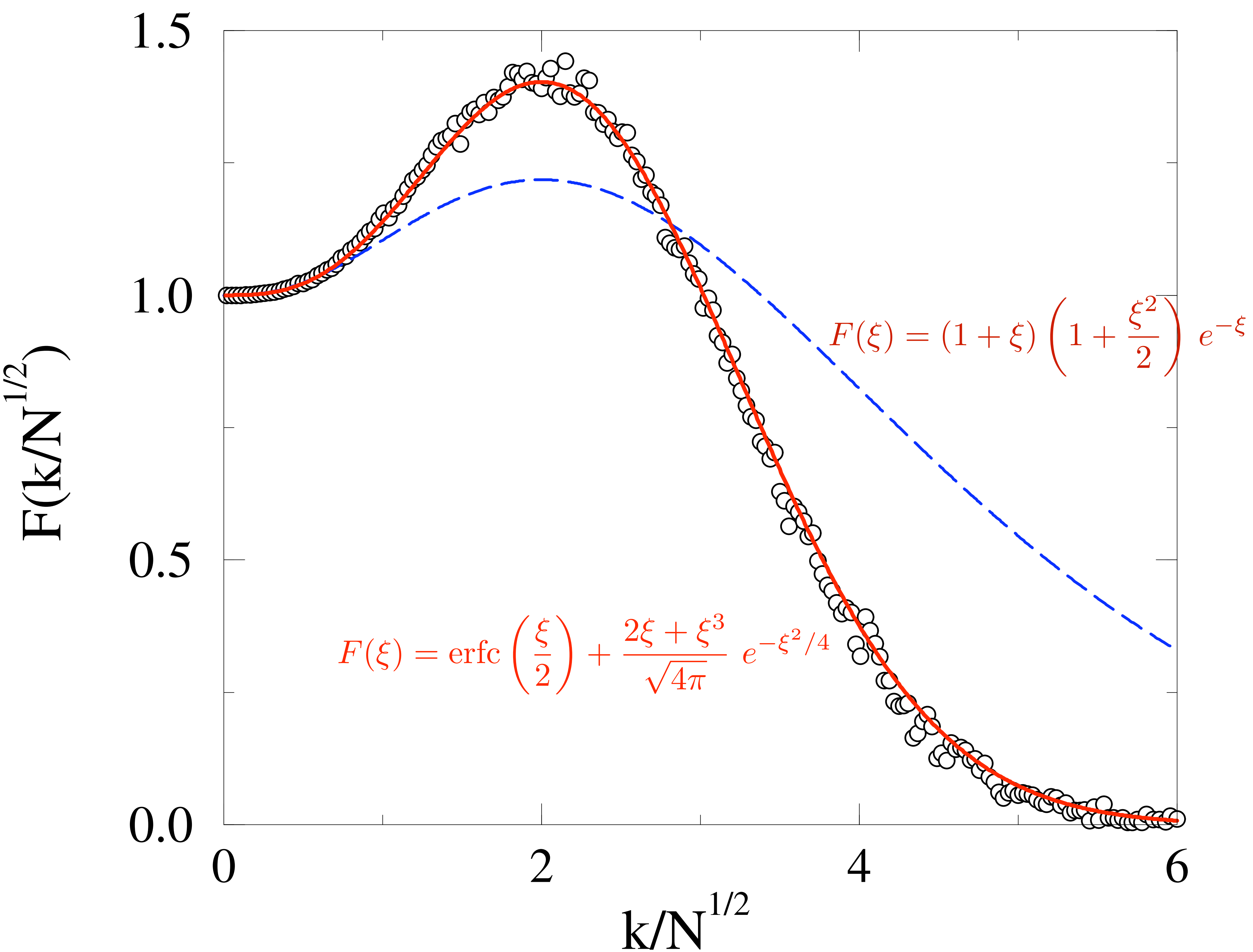
$$F(\xi) = (1+\xi)\left(1+\frac{\xi^2}{2}\right) e^{-\xi}$$

exact discrete scaling function:

$$F(\xi) = \mathrm{erfc}\left(\frac{\xi}{2}\right) + \frac{2\xi+\xi^3}{\sqrt{4\pi}}\, e^{-\xi^2/4}$$

continuum≠discrete

The Scaling Function



Diversity: Degree Fluctuations

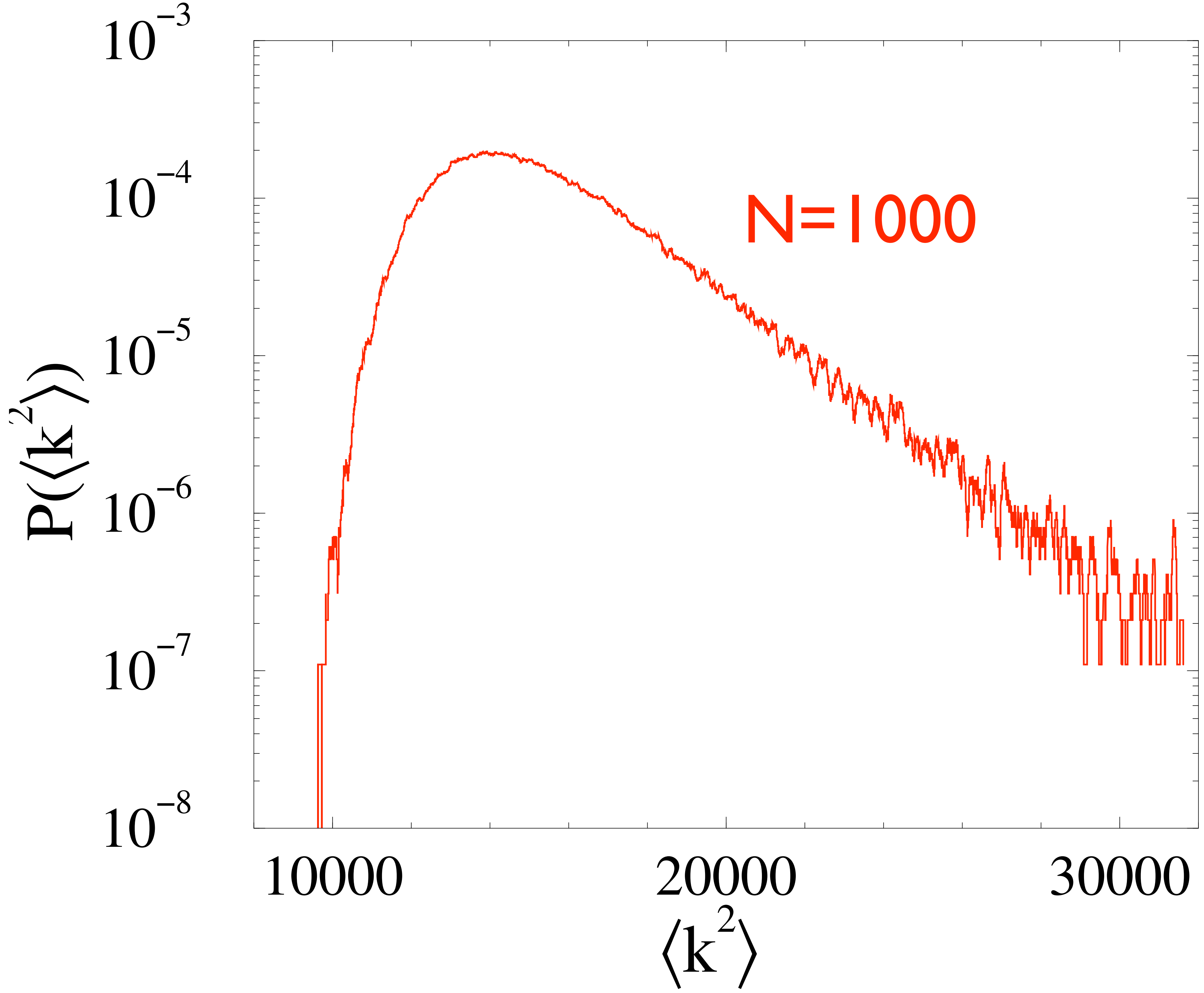
Finite-N corrections: $N_k(N) = Nn_k + \frac{a_k}{N^{1/2}} + \ldots$

Distribution for *fixed* degree (e.g., k=1):

$$N_1(N+1) = \begin{cases} N_1(N) & \text{prob. } \frac{N_1}{2N} \\ N_1(N)+1 & \text{prob. } 1 - \frac{N_1}{2N} \end{cases} \quad \rightarrow \langle N_1(N) \rangle, \quad \langle N_1^2(N) \rangle$$

$$\longrightarrow \sigma_k^2 \equiv \langle N_k^2(N) \rangle - \langle N_k(N) \rangle^2 = \mu_k N$$

Diversity:
Distribution of Mean-Square Degree



Outlook

Two basic questions:

Why does *linear* preferential attachment arise so generically?

What is the “imprint” of the ensemble of all networks with a given parameter set?