

enumeration (not necessarily as the same segment). Thus, it is wasteful to generate such walks repeatedly; our technique has been to generate walks of moderate length beforehand, and to keep their details readily available in a major area of computer memory.

This in itself would already make the program very fast in execution. A further substantial increase in speed (of a factor of 2 or 3) is obtained by noticing that it is now never necessary to lay the ordinary sites of the *final* segment on the lattice (and, by implication, to remove them later); such sites are already mutually self-avoiding as a result of their generation at the preliminary stage. All that is needed for the final segment is to check that the ordinary sites *can* be laid on the lattice without interfering with the earlier segments of the configuration. Provided that the program information is properly organised, such a test is very simple to carry out. In our version of the program the test is coded as a loop of only seventeen computer instructions for the IBM360. This program fragment has been designed with great care, as it is estimated that in some enumerations (in particular on the f.c.c. lattice) it may be executed for 90% or 95% of the total running time.

A condensed flowchart for the process is shown in Fig. 8, the upper part of which shows how the topology vertices are staked out, while the lower part describes the procedure for filling in the bridges. Each part has the basic structure of Fig. 3, though it should be noted that the parameters j_1 and j_2 here label segments and not points. Where the words *select* and *find* appear in heavy type, they refer to the body of information about self-avoiding walks which is assumed to be already available in the store.

Programs embodying these principles have been written for the National Physical Laboratory ACE (now obsolete), the English Electric KDF9 and the IBM360 computers. In favourable cases, topology embeddings have been counted at speeds of about 10^6 successes per minute (on the IBM360).

References

There is much published work in statistical physics which relies heavily on configuration-counting by computer. Not many publications give details of computing procedures, however, and it seems inappropriate to list them here; in any case, they are almost certainly to be found in the bibliographies of the other articles in this volume. An exception is W. F. Lunnon, (1971), "Counting Polyominoes": article in "Computers in Number Theory" (A. O. L. Atkin and B. J. Birch, eds.). Academic Press, London and New York.

3. Linked Cluster Expansion

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I. Introduction	114
A. Motivation	114
B. Derivation of series expansion	116
C. The linked-cluster expansion: historical	117
D. Format and aim of this article	119
II. The Linked-cluster Expansion with Commuting Variables: The Ising Model	120
A. Notation and preliminaries	120
B. Unrenormalised linked-cluster expansion	122
Graphical definitions I:	124
Rule 1: free energy W , unrenormalised form	125
Rule 2: renormalised semi-invariants, unrenormalised form	129
Rule 3: pair correlations, unrenormalised form	130
C. Proof of the unrenormalised linked-cluster expansion	131
D. Relation of the linked-cluster and weak-embedding expansions for the free energy	133
E. Vertex renormalisations: the correlations	135
Graphical definitions II	137
Rule 4: self-field, unrenormalised form	138
Rule 5: pair correlations, vertex renormalised form	140
Rule 6: self-field, vertex renormalised form	140
F. Vertex renormalisation: the free energy	141
Rule 7: the functional ϕ , vertex renormalised form	142
G. The classical-fluid linked-cluster expansion as a special case	144
Rule 8A: classical-fluid free energy W , unrenormalised v -form	146
Rule 8B: classical-fluid free energy W , unrenormalised f -form	147
Rule 9: classical-fluid self-field, vertex renormalised f -form	147
Rule 10: classical-fluid functional ϕ , vertex renormalised f -form	148
H. ϕ -derivable approximations	148

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1969, 1970; Polyakov, 1969, 1970) may point in the right direction. Series expansions, although they provide little physical understanding of the critical processes, have made possible precise and reliable estimates of a variety of critical properties for a wide class of model systems. Critical exponents derived from series were important data in the formation of the scaling hypothesis (Essam and Fisher, 1963; Fisher, 1965; Kadanoff, 1966; Kadanoff *et al.*, 1967; Widom, 1965a, b; Domb and Hunter, 1965). Similarly, series-derived model information is often compared quantitatively with measurements in the analysis of experimental data. This chapter will attempt neither a review of important results derived by series methods nor a detailed assessment of the contribution of these results to the understanding of critical phenomena. The interested reader is referred to several useful review articles (Domb, 1960; Fisher, 1963, 1965, 1967; Helfand, 1963). Our present focus is methodology rather than results.

Domb and Sykes (Domb, 1949, 1960; Domb and Sykes, 1956, 1957, 1961) were the first to develop series methods for the study of critical phenomena. The volume of subsequent work, following this initial exploration, has been considerable. Many refinements of techniques and extensions to other model systems are discussed elsewhere in this volume; however, the main lines of the original work remain clear. There are three stages in the argument:

- (i) a parameter x is selected in terms of which the desired thermodynamic or correlation function of the chosen model system may perturbatively be expanded;
- (ii) a finite number of coefficients in the perturbation series expansion are computed;
- (iii) extrapolative analysis of the computed coefficients *in the light of the expected form of the critical behaviour* locates the critical point and gives estimates of singular behaviour.

The parameter x selected at stage (i) is often a typical interaction energy (such as the exchange energy J) divided by the thermal energy $kT = \beta^{-1}$, $x = \beta J$. For certain special cases such as the Ising model the low-temperature parameter $x = \exp(-\beta J)$ may also be used. The quantity Q whose critical behaviour as a function of x is to be studied must have an expansion about $x = 0$,

$$Q(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with a finite radius of convergence. At stage (ii) a finite number of the coefficients a_n are determined, $a_0, a_1, \dots, a_{n_{\max}}$. Typically the calculation is in principle straightforward; however, the increased labour necessary for calculating each succeeding coefficient is large. A rule of thumb is that the

I. Analysis of the two-point function	152
Graphical definitions III	153
Rule 11: l , n -valent correlation factors vertex, renormalised form	154
J. Renormalisation of the interaction	157
Rule 12: three-point correlations, vertex and bond renormalised form	158
Rule 13: elementary correlation factors, vertex and bond renormalised form	159
Rule 14: the functional $Q[M, C]$, vertex and bond renormalised form	162
K. Further renormalisation: comments	162
III. Further Applications and Extensions of the Linked-cluster Expansion	162
A. Orientation	162
B. Inclusion of additional local fields	163
C. Several commuting variables: the classical Heisenberg model	165
Rule 15: Classical Heisenberg model free energy W , unrenormalised form	167
Rule 16: Classical Heisenberg model pair correlations, unrenormalised form	168
Rule 17: Classical Heisenberg model self-field, vertex renormalised form	168
D. Non-commuting variables: the quantum Heisenberg model	169
Rule 18: Heisenberg model free energy W , unrenormalised form	173
Rule 19: Heisenberg model pair correlations, unrenormalised form	174
E. Fermion variables	175
Appendix: The Relation of the Free Multiplicities to the Weak Lattice Constants	176
References	178

I. Introduction

A. Motivation

The study by the method of perturbative series expansions of the thermodynamic properties of models described by lattice Hamiltonians has historically been of great value in the understanding of critical phenomena. Onsager's brilliant solution of the two-dimensional nearest-neighbour zero-field spin $\frac{1}{2}$ Ising model (Onsager, 1944) proved beyond question that mean-field, Landau-like descriptions of the critical region were qualitatively, quantitatively, and fundamentally false. On the other hand, Onsager's methods and those used by subsequent workers in obtaining rigorous solutions for particular one- and two-dimensional models were quite specialised and have done little to suggest viable techniques for first-principle calculations—even approximate—of the critical properties of more complicated systems, particularly in three dimensions. Ideally a general technique is required, embodying a physically correct understanding of the large fluctuations dominating the singular, critical behaviour and from which all relevant quantities could be calculated qualitatively accurately and to any desired degree of precision, given only the requisite labour. Such a general technique is not currently available, though recent Russian work (Gribov and Migdal, 1969; Migdal,

computation of a_{n+1} involves at least as much labour as the cumulative calculation of $a_0, \dots, a_n$. Thus, while there is in principle no limit to the number of calculable coefficients, there is in practice a rather sharp upper bound $n_{\max}$ (which depends on the details of the model being considered) determined by such practical considerations as time and patience and, at the next level, electronic computer capacity and funding. Finally at stage (iii) critical properties are extracted from the available coefficients by methods described in this volume (Gaunt and Guttman, Chapter 4) and elsewhere (Baker, 1965; Domb and Sykes, 1961; Fisher, 1967).

The limiting link in the above schema is stage (ii), determination of the expansion coefficients. Typically the first few coefficients are trivial and no special methodology is necessary. However, in higher orders the bookkeeping is extremely involved; and, scrupulous accuracy is necessary in determining the coefficients, since (in stage (iii)) extrapolation makes apparent critical-point behaviour exceedingly sensitive to tiny fractional changes in the last few available coefficients. For these reasons, it becomes of paramount importance to have a well-defined, systematic procedure for computing coefficients, which incorporates as many short cuts as possible and minimises the opportunity for careless error. It is to this systematisation which we now turn.

B. Derivation of series expansions

All systematic methods for the determination of series coefficients are at some level graphical or diagrammatic.[†] With each coefficient is associated a set of graphs of some given topological type. To each graph corresponds a numerical contribution according to a well-defined rule. To calculate the required coefficient, one simply sums all contributions. There are, thus, three steps in the calculation:

- (i) enumerate the contributing graphs;
- (ii) calculate the contribution of each;
- (iii) add.

Accordingly, graphical methods differ in (a) which diagrams contribute and (b) what rules determine the contributions.

The rule for calculating the contribution (ii) of each graph generally consists of two parts:

- (a) a factor, *the weight of the graph*, which depends only on its topology, and

[†] We shall use the terminology "graph" rather than "linear graph" or "diagram." Although an extensive mathematical literature on graph theory exists (Berge, 1958; Harary, 1969; Ore, 1962), physicists have been far from universal in adopting its terminology. We shall introduce terminology in detail as we proceed, conforming in general to the recent compilation of Essam and Fisher (1970). (See also Domb, this volume, Chapter 1).

- (b) the configurational multiplicity of the graph, determined by the number of different ways it can be *embedded* in the underlying lattice subject to certain constraints.

Different graphical expansions differ with respect to both weights and embedding constraints. The three main embedding systems are *strong*, *weak*, and *free* (Domb, 1960; Sykes *et al.*, 1966; Jasnow, 1969; Jasnow and Wortis, 1968) in order of decreasingly stringent constraints. The strong and weak systems are discussed elsewhere in this volume (Chapter 1), and the relation between them has been investigated by Sykes *et al.* (1966). The free embedding system and its relation to the weak system is considered in the appendix to this article.

In high-order calculations the number of graphs which potentially contribute in (i) above is enormous. Many of these may have no possible embeddings once the constraints (b) have been applied. Broadly speaking, the more restrictive the constraints, the fewer the possible embeddings. This very significant simplification favours restrictive expansions.

An alternative and in many ways complementary method of cutting down the number of contributing graphs is to perform a selective resummation or *renormalisation* in which classes of graphs with related topologies are lumped together. The calculation of the weights (a) of the remaining graphs is then more complicated, and the net effect is a trade-off of algebraic complexity for topological complexity. The special simplicity of the configurational factors (b) for the free or unrestricted embeddings makes the renormalisation method most appropriate. As a rule the restrictive embeddings are best in low dimensionality and for rather open lattice structures. For close-packed lattices and in higher dimensionality the renormalisation method seems preferable. In any given study, there may be additional considerations favouring one method or another. For example, for the high-temperature Ising model the weak-embedding method (Section II.D) becomes increasingly complex for higher spin, while for the free-embedding linked-cluster expansion the spin enters merely as a parameter.

This article will deal exclusively with the free-embedding expansions and their renormalisation. Connection with the weak-embedding expansion is discussed in Section II.D for the $s = \frac{1}{2}$ Ising model.

C. The linked-cluster expansion: historical

The prototypical free-embedding expansion is the linked-cluster expansion for the classical fluid, dating from the work of Ursell (1927), Yvon (1935), and Mayer (1937). Good reviews of more recent work are given by Uhlenbeck and Ford (1962) and by Stell (1964) (see also Chapter 1). The original expansion is in terms of the non-interacting single particle density n_0 , with the pair-potential in the combination $[\exp(-\beta v(\mathbf{r}_1 - \mathbf{r}_2)) - 1]$ as expansion

parameter. The simplest (one-point) renormalisation produces the familiar virial expansion (Huang, 1963), in which n_0 is replaced by the fully interacting particle density n . This renormalisation can be interpreted as a Legendre transformation in which the (inhomogeneous) particle density replaces the (non-uniform) one-particle potential as fundamental variables. De Dominicis (1962) showed that a second renormalisation can be done in which two-point correlations replace the pair potentials in a Legendre transformation generated by the entropy. A truncation of this two-point renormalisation yields the hypernetted chain approximation (Van Leuwen *et al.*, 1959; Meeron, 1960).

The linked-cluster expansion for the *quantum* fluid has been studied by many authors (Martin and Schwinger, 1959; Kadanoff and Baym, 1962; Arikosov *et al.*, 1965). Because of the non-commutativity of operators, the extra complexity of time-dependence enters the calculation even for thermodynamic quantities. The expansion is in powers of the pair-potential; however, there are a number of energy variables, so there is no single dimensionless expansion parameter. The one-point renormalization, first done by Luttinger and Ward (1960) for the thermodynamic properties, replaces the non-interacting one-particle propagator by the fully interacting one-particle propagator. The two-point renormalisation was carried through by De Dominicis and Martin (1964a, b), again in terms of a generalised entropy functional.

The early expansions for magnetic systems, especially the Ising and Heisenberg models which have been important in the study of critical phenomena, were all of the weak (high-temperature) and strong (low-temperature) embedding types. Horwitz and Callen (1961) showed how an earlier expansion due to Brout (1959, 1960, 1961) could be rearranged to eliminate excluded volume restrictions. A direct derivation of the free, linked-cluster expansion for the Ising model in terms of the expansion parameter βJ was first given by Englert (1963). Because of the non-particle-like kinematics, the class of graphs contributing to this expansion contains multiple bonds and is considerably larger than the class which contributes to the classical fluid expansion. The reason for this is explained in Section II.G. Englert (1963) also performed the one-point renormalisation of the Ising linked-cluster expansion, though it was not until the work of Bloch and Langer (1965) that the connection with a Legendre transformation was understood. The parallel development for the classical Heisenberg model was done by Jasnow and Wortis (1968) and used for the first time to generate high-temperature series expansions (Jasnow and Wortis, 1968; Jasnow and Moore, 1968). The two-point renormalisation was originally carried through by Wortis *et al.* (1969) and has proved very useful in obtaining high-temperature series (Ferrer *et al.*, 1969; Moore, 1969, 1970; Moore *et al.*, 1969).

The generalisation of the Englert formalism to the spin s quantum mechanical Heisenberg model was first done by Stinchcombe *et al.* (1963). It was not until the work of Vaks *et al.* (1968a, b), who employed a partial one-particle renormalisation, that the expansion was used systematically in calculations. The full renormalisations are carried out in Section III.D, where strong parallels with the quantum fluid are pointed out. However, in the calculation of those high-temperature series of relevance to the study of critical phenomena, no form of the linked-cluster expansion has yet been made competitive with the weak-embedding method of Baker *et al.* (1967) for the $s = \frac{1}{2}$ Heisenberg model (see also this Volume, Chapter 5).

In studying the properties of an assembly of electrons on a lattice, interacting via a zero-range interaction, Hubbard (1963, 1964a, b, 1965, 1966) has developed a linked-cluster expansion in powers of the kinetic energy, structurally identical to the unrenormalised expansion for the quantum Heisenberg model, only with the added complication of Fermi statistics.

D. Format and aim of this article

The intent of this article is frankly pedagogical. We hope to make the linked-cluster expansion for systems like the Ising and Heisenberg models available to the reader as a *working tool*. To this end we have included a considerable number of simple, worked examples. While these may strike the initiate as superfluous, it is hoped that the non-expert may find them useful. The central results of the theory are the rules (in the sense of I.B (i)-(iii)) for writing down the original graphical expansions and their renormalisations. From a practical point of view, the important thing is to be able to apply the rules in actual calculation. Characteristically, the detailed derivation of the rules, while logically necessary, is often not very illuminating. Such derivations tend to involve cumbersome combinatorial arguments with little obvious physical meaning. Without pretending to rigor, we have attempted to sketch all important steps and to point out—often by a simple example—what the essential topological content of the general proof must be. References are given to more detailed proofs, when they are available in the literature.

The usefulness of the linked-cluster expansion is by no means restricted to the derivation of series expansions for critical properties. Many approximate closed-form solutions for thermodynamic and correlation functions can be derived by graphical methods by selective truncation of graphical sums. Despite much effort, no such approach has yet yielded correct properties in the critical region; however, qualitatively correct phase diagrams and quantitative non-critical results can be obtained for a variety of systems. While our primary orientation is towards the derivation of series expansions for problems in critical phenomena, we have attempted to develop the theory in

such a way that it applies immediately to a variety of non-critical calculations as well.

The methods illustrated are sufficiently general to apply to any system which is spatially local in the absence of the perturbing interaction. For the sake of clarity it seemed wise, however, to couch the discussion in terms of specific examples. Directions of possible generalisation are remarked.

Section II develops the linked-cluster expansion and its various renormalisations for the spin s Ising model. This development, which illustrates all the topological content of the renormalisation method in the simplest possible context, is carried through in full detail. Section II.D explores the relation between the linked-cluster and weak-embedding expansions for the $s = \frac{1}{2}$ Ising model. Section II.G shows how the magnetic expansion reduces for the classical fluid to the Ursell-Yvon-Mayer expansion.

In Section III linked-cluster expansions for a variety of other more complicated systems are sketched. The topological content of these expansions and their renormalisations are identical to those of the Ising model. Omitting redundant derivations, we have attempted to bring out the parallels with the Ising development. Once these are understood, all results may be written down by inspection.

II. The Linked-Cluster Expansion with Commuting Variables: The Ising Model

A. Notation and preliminaries

The Hamiltonian for the spin s Ising model is usually written,

$$\mathcal{H} = -\frac{1}{s^2} \sum_{\langle ij \rangle} J_{ij} s_i s_j - \frac{m}{s} \sum_i H_i s_{zi}, \quad (1)$$

where $\sum_{\langle ij \rangle}$ denotes the sum over distinct pairs. Each spin s_i ($i = \mathbf{R}_1, \dots, \mathbf{R}_N$) is situated at one of the N sites of a regular lattice and interacts via its z -component magnetic moment ms_{zi} ($s_z = -s, -s+1, \dots, s$) with an inhomogeneous external magnetic field $(1/s)H_i$. Each spin pair is coupled by the spin-dependent energy $(-J_{ij}/s^2)s_i s_j$, where the exchange coupling J_{ij} , which may in principle be different for each pair, is normally a function only of the difference variable $(\mathbf{R}_i - \mathbf{R}_j)$. Note that $J_{ii} = 0$. The factors of $1/s$ are chosen so as to normalise the maximum interaction between parallel spins. The classical limit, in which the spins become classical unit vectors, is achieved by taking $s \rightarrow \infty$.

It will be convenient to introduce as variables the operators $\mu = s_z/s$, which range in the interval $[-1, 1]$. We then rewrite (1) in terms of dimensionless variables,

$$-\beta\mathcal{H} = \frac{1}{2} \sum_{\langle ij \rangle} v(12)\mu(1)\mu(2) + \sum_i h(1)\mu(1), \quad (2)$$

where $\beta \equiv 1/kT$ and the numerical arguments, which will henceforth stand for lattice sites, run over the points $\mathbf{R}_1, \dots, \mathbf{R}_N$. The dimensionless exchange coupling is $v(ij) \equiv \beta J_{ij}$; the reduced magnetic field is $h(i) \equiv \beta m H_i$. The partition function and free energy of the Ising model are given by

$$\text{Tr} \exp(-\beta\mathcal{H}) = Z[h, v] = \exp W[h, v]$$

and

$$F = -kT W[h, v], \quad (3)$$

where the functional dependence of the thermodynamics on the fields and exchange couplings has been emphasised by the use of square brackets.

The advantage of maintaining the fields and couplings variable *independently* for each site and pair is that all correlation functions may be found by differentiation directly from the dimensionless free energy W . For example,

$$\mathcal{M}_1(1) \equiv \langle \mu(1) \rangle \equiv \frac{\langle s_{z1} \rangle}{s} = \frac{1}{Z} \frac{\delta Z}{\delta h(1)} = \frac{\delta W}{\delta h(1)} = \frac{dW}{d(\beta m H_1)}. \quad (4)$$

We use the standard notation $\langle X \rangle = (1/Z) \text{Tr} X \exp(-\beta\mathcal{H})$. By using the functional notation for the derivative $\delta/\delta h$, we stress that it is the *local field at the site 1* which is being varied. Similarly, the two-point (cumulant) correlations are available as second derivatives:

$$\mathcal{M}_2(12) \equiv \langle \mu(1)\mu(2) \rangle - \langle \mu(1) \rangle \langle \mu(2) \rangle = \frac{\delta^2 W}{\delta h(1)\delta h(2)}. \quad (5)$$

The n -point (cumulant) correlations are defined by

$$\mathcal{M}_n(1 \ 2 \ \dots \ n) \equiv \frac{\delta^n W}{\delta h(1) \dots \delta h(n)}, \quad n = 1, 2, \dots \quad (6)$$

Derivatives of W with respect to the pairwise exchange couplings also generate correlations:

$$\frac{\delta W}{\delta v(12)} = \langle \mu(1)\mu(2) \rangle, \quad (7)$$

which can be rewritten using (4) and (5) in the general form,

$$\frac{\delta W}{\delta v(12)} = \frac{\delta^2 W}{\delta h(1)\delta h(2)} + \frac{\delta W}{\delta h(1)} \frac{\delta W}{\delta h(2)}. \quad (8)$$

Equation (8) constitutes a non-linear functional differential equation for the free energy. The appropriate initial condition is

$$W[h, v=0] \equiv W_0[h] = \ln \text{Tr} \exp \left(\sum_i h(1) \mu(1) \right) = \sum_i M_0^0(1), \quad (9)$$

with

$$M_0^0[1; h(1)] \equiv \ln \left[\sum_{\mu=-1}^1 e^{h(1)\mu} \right] = \ln \left[\frac{\sinh \left(\frac{h(1)}{2s} (2s+1) \right)}{\sinh \left(\frac{h(1)}{2s} \right)} \right], \quad (10)$$

which is even in h . While the system (8)–(10) cannot in general be solved, it can be used to develop W in powers of v . This development, interpreted graphically, is just the linked-cluster expansion for the Ising model free energy W . Once W is known, eqn (6) yields the corresponding linked-cluster expansions for all correlation functions. In actual calculation—i.e. once the graphical rules are known—it is often most convenient (Stanley and Kaplan, 1966; Stanley, 1967) to find the correlations directly and evaluate physical quantities from them. For example, the susceptibility per site and the total energy are respectively (in dimensionless variables),

$$\chi = \sum_i \mathcal{M}_2(12),$$

$$-E = \frac{1}{2} \sum_{i,j} v(12) [\mathcal{M}_2(12) + \mathcal{M}_1(1) \mathcal{M}_1(2)] + \sum_i h(1) \mathcal{M}_1(1).$$

B. Unrenormalised linked-cluster expansion

The linked-cluster expansion for $W[h, v]$ is nothing more than the many-variable Taylor development,

$$W[h, v] = \left[\exp \left(\sum_{\text{pairs}} v(12) \frac{\delta}{\delta \tilde{v}(12)} \right) \right] W[h, \tilde{v}] \Big|_{\tilde{v}=0}. \quad (11)$$

The first few terms read,

$$W[h, v] = W_0[h] + \sum_{\langle 12 \rangle} v(12) \frac{\delta}{\delta \tilde{v}(12)} W[h, \tilde{v}] \Big|_{\tilde{v}=0}$$

$$+ \frac{1}{2} \sum_{\langle 12 \rangle \langle 34 \rangle} v(12) v(34) \frac{\delta}{\delta \tilde{v}(12)} \frac{\delta}{\delta \tilde{v}(34)} W[h, \tilde{v}] \Big|_{\tilde{v}=0} + \dots \quad (12)$$

In each order the coefficient of v^n has the form of $\frac{\delta^n W}{\delta v^n} \Big|_{v=0}$, which by successive use of (8) can always be expressed in terms of products of the correlation functions $\mathcal{M}$ evaluated in the non-interacting limit $v=0$. At $v=0$, all the lattice sites are entirely uncorrelated with one another, so the correlations are spatially local,

$$\mathcal{M}_n(1 \dots n) \Big|_{v=0} = \delta(1 \dots n) M_n^0(1), \quad (13)$$

where $\delta(1 \dots n) = 1$ when all lattice points, 1, ..., n , are identical and vanishes otherwise. The quantities $M_n^0(1)$ are called *bare* or *unrenormalised semi-invariants* (Englert, 1963). Note that $M_n^0(1)$ depends only on the field $h(1)$ at the site 1. It follows from (9) and (6) that

$$M_n^0(h) = \frac{d^n}{dh^n} M_0^0(h), \quad (14)$$

or

$$W_0[h] = \sum_i \sum_{n=0}^{\infty} \frac{(h(1))^n}{n!} M_n^0(h) = 0. \quad (15)$$

This locality is directly responsible for the *linked*-cluster property, which in turn underlies the extensivity of the thermodynamic functions as $N \rightarrow \infty$ for appropriately regular and short-ranged potentials (Ruelle, 1969).

The above will become clearer as we work out the first few contributions in detail. Using (8), we evaluate,

$$\frac{\delta W}{\delta v(12)} \Big|_{v=0} = [\mathcal{M}_2(12) + \mathcal{M}_1(1) \mathcal{M}_1(2)]_{v=0} = M_1^0(1) M_1^0(2), \quad (16)$$

where in the final equality we have used (13) and explicitly noted that $1 \neq 2$. Thus, the first order term in W is just

$$\sum_{\langle 12 \rangle} M_1^0(1) v(12) M_1^0(2) = \frac{1}{2} \sum_{i,j} M_1^0(1) v(12) M_1^0(2). \quad (17)$$

Evaluation of the third term in (12) necessitates using eqn (8) twice:

$$\frac{\delta^2 W}{\delta v(12) \delta v(34)} = \frac{\delta}{\delta v(34)} \frac{\delta W}{\delta v(12)} = \frac{\delta^2}{\delta h(1) \delta h(2)} \frac{\delta W}{\delta v(34)}$$

$$+ \left[\frac{\delta}{\delta h(1)} \frac{\delta W}{\delta v(34)} \right] \frac{\delta W}{\delta h(2)} + \frac{\delta W}{\delta h(1)} \left[\frac{\delta}{\delta h(2)} \frac{\delta W}{\delta v(34)} \right],$$

$$\frac{\delta^2 W}{\delta v(12)\delta v(34)} \Big|_{v=0} = (\delta(13)\delta(24) + \delta(14)\delta(23))M_2^0(1)M_2^0(2)$$

$$\begin{aligned} &+ M_1^0(1)M_1^0(3)\delta(24)M_2^0(2) \\ &+ M_1^0(1)M_1^0(4)\delta(23)M_2^0(2) \\ &+ M_1^0(2)M_1^0(3)\delta(14)M_2^0(1) \\ &+ M_1^0(2)M_1^0(4)\delta(13)M_2^0(1). \end{aligned} \quad (18)$$

Thus,

$$\begin{aligned} W[h, v] &= \sum_i M_i^0(1) + \frac{1}{2} \sum_{i,j} M_1^0(1)v(12)M_1^0(2) \\ &+ \frac{1}{4} \sum_{i,j,k} M_2^0(1)v^2(12)M_2^0(2) \\ &+ \frac{1}{2} \sum_{1,2,3} M_1^0(1)v(12)M_2^0(2)v(23)M_1^0(3) \\ &+ \text{order } v^3 \text{ and higher.} \end{aligned} \quad (19)$$

It should be apparent at this point that finding the coefficient of v^n is a trivial but tedious task.

Luckily there is a general method for writing down immediately all terms in the expansion (11) (or (19)) for $W[h, v]$. The rule associates the terms of the expansion in a one-to-one manner with a set of linear graphs. After stating the rule and discussing its application, we shall return in Section II.C to its derivation. Before setting down the rule, we must introduce some terminology from graph theory (Essam and Fisher, 1970, see also Domb this Volume, Chapter 1):

Graphical definitions I:

A graph is a set of vertices (points) and edges (lines). Each edge is incident at two distinct vertices.† An n -rooted graph, $n = 0, 1, 2, \dots$, has n fixed, distinguishable, external or rooted vertices. All remaining internal vertices are free.

The valence of a vertex is the number of edges incident at that vertex. A vertex of valence n is called n -valent.

A connected graph is a graph with the property that any pair of its vertices is joined by a continuous sequence of edges. A graph which is not connected is disconnected.

Two n -rooted graphs are isomorphic if, when their external vertices are put in one-to-one correspondence, there exists at least one way in which their internal vertices and edges may be placed in one-to-one correspondence in such a way that all connectivity is maintained. The number of distinct ways in which a

† "Loops," in which a single edge is incident with both ends at the same vertex, are prohibited.

graph can be made isomorphic to itself is called the symmetry factor† of the graph. A set of graphs are topologically distinct if no two of them are isomorphic.



FIG. 1. The fundamental graphical elements: (a) an edge, (b) an external vertex, (c) an internal vertex, (d) a graph in which the external vertex carries no vertex factor.

When picturing graphs (see Fig. 1), we draw the edges as lines and the external and internal vertices as open and filled circles, respectively. The only exception to this comes in Section II.E and after, where, in the process of



FIG. 2. Symmetry factors for the unrooted graphs (a), (b), and (c) are 12, 48, and 24 respectively. The general n -gon has a symmetry factor $2n$. The n -fold multiedge has a symmetry factor $2n!$. The complete n -vertex graph (all pairs connected by a single edge) has a symmetry factor $n!$.

renormalisation, we shall need a vertex with which no algebraic factor is associated. For such a vertex the circle will be omitted. Some unrooted graphs and their symmetry factors are shown in Fig. 2.

In order to calculate $W[h, v]$ one takes the set of all topologically distinct 0-rooted connected graphs. One associates with each graph a contribution according to Rule 1 below, and one sums all such contributions:

Rule 1: Free Energy W , Unrenormalised Form

- (a) Assign a dummy label to each internal vertex.
- (b) For each edge joining vertices i and j write a factor $v(ij)$.
- (c) For each l -valent internal vertex i write a factor $M_l^0(i)$ (eqn (10), (14)).
- (d) Sum each internal vertex label freely over the entire lattice.
- (e) Divide by the symmetry factor of the unrooted graph.

† This is termed "symmetry number" in Chapter 1. The graphs in the present chapter may contain multiple bonds between vertices which are treated as non-identical.

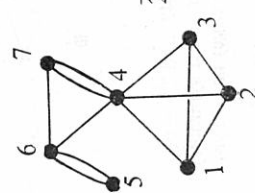
In this spirit we write,

$$W[h, v] = \left\{ \begin{array}{l} \text{sum of all topologically} \\ \text{distinct unrooted connected} \\ \text{graphs according to Rule 1} \end{array} \right\} \quad (20)$$

$$= \begin{array}{c} \bullet \quad + \frac{1}{2} \text{---} \bullet + \frac{1}{4} \text{---} \bullet \text{---} \bullet + \frac{1}{2} \text{---} \bullet \text{---} \bullet \text{---} \bullet + \frac{1}{12} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + \frac{1}{6} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ + \frac{1}{2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + \frac{1}{6} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + \frac{1}{2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array}$$

+ order v^4 and higher.

The symmetry factor for each graph is given in the form $1/g$. The reader will verify that the first four terms of (20) correspond in sequence to the first four terms of (19), according to Rule 1. The remaining terms in (20) give all contributions to W which are third order in the interaction, i.e., which come from graphs with exactly three edges. Figure 3 gives the contribution to W of a more complicated graph.



$$g = 6(2!)^2 = 24$$

$$\frac{1}{24} \sum_{1, \dots, 7} v(12)v(23)v(13)v(14)v(14)v(24)v(34)v(34)v(46)v(46)v(67)v(67)v^2(56) \times M_3^0(1)M_3^0(2)M_3^0(3)M_3^0(4)M_2^0(5)M_4^0(6)M_3^0(7) \quad (56)$$

Fig. 3. A twelfth order contribution to the Ising free energy according to Rule 1.

Equation (20) gives $W[h, v]$ for arbitrary inhomogeneous fields and interactions. Although this freedom will be useful in what follows, it is worth pointing out that, if the free energy is all we wish to calculate, considerable simplification is gained by immediately specialising to the translationally invariant case,

$$h(1) = h, \quad v(12) = V(\mathbf{R}_1 - \mathbf{R}_2). \quad (21)$$

The bare semi-invariants $M_n^0(h)$ are independent of position, and the formal extensivity of W is realised,

$$\begin{aligned} \frac{W}{N} &= M_0^0(h) + \frac{1}{2}(M_1^0(h))^2 \sum_{\mathbf{R}} V(\mathbf{R}) \\ &+ \frac{1}{4}(M_2^0(h))^2 \sum_{\mathbf{R}} V^2(\mathbf{R}) + \frac{1}{2}(M_1^0(h))^2 M_2^0(h) \left[\sum_{\mathbf{R}} V(\mathbf{R}) \right]^2 \\ &+ \text{order } v^3 \text{ and higher,} \end{aligned} \quad (22)$$

where $\mathbf{R}$ runs over the entire lattice and boundary effects have been neglected (periodic boundary conditions). It is easy to see from (10) and (14) that $M_n^0(h)$ has parity $(-1)^n$, so that, when $h = 0$, all odd semi-invariants vanish and, of the explicit terms in (22), only the first and third survive.

The most studied example of (22) is when, in addition to (21), the potential acts between nearest-neighbour sites only,

$$v(12) = \begin{cases} v, & \text{when 1 and 2 are nearest neighbours,} \\ 0, & \text{otherwise.} \end{cases} \quad (23)$$

In this case (22) simplifies to

$$\begin{aligned} \frac{W}{N} &= M_0^0(h) + \frac{q}{2}(M_1^0(h))^2 v + \left[\frac{q}{4}(M_2^0(h))^2 + \frac{q^2}{2}(M_1^0(h))^2 M_2^0(h) \right] v^2 \\ &+ \text{order } v^3 \text{ and higher,} \end{aligned} \quad (24)$$

where q , the *coordination number* of the lattice, is just the number of nearest neighbours of any site. Let us in this case consider a given n -edged graph G_n and examine its contribution $W(G_n)$ to the full free energy. Because of the nearest-neighbour restriction, all terms in the sum required by Rule 1(d) are identical. The number of such terms we write as $Nm(G_n)$, where $m(G_n)$ is the *free multiplicity of the graph* G_n . $m(G_n)$ measures the number of distinct ways per site of the underlying lattice† in which the *vertex-labelled graph* G_n can be embedded in the lattice with each vertex assigned to a site of the lattice and each edge lying along a nearest-neighbour bond. In the appendix to this article the free multiplicity is discussed, some of its simple properties are given, and it is related to the weak lattice constants. Jasnow (1969) has tabulated the free multiplicities of 2-rooted graphs for a variety of lattices. It is important to notice that, because the sum in Rule 1(d) is unrestricted, it is *quite permissible for graphically distinct vertices to occupy the same lattice site*. This is in contrast to the weak and strong lattice constant systems used in other types of expansions and discussed for example by Domb (1960) and by Sykes *et al.* (1966). (See Domb this Volume, Chapter 1.) The contribution $W(G_n)$ now is

$$W(G_n) = \frac{N}{g(G_n)} m(G_n) v^n \left(\prod_{\mathbf{G}_n} M^0 \right), \quad (25)$$

where, by the notation $\prod_{\mathbf{G}_n} M^0$, we intend to evoke the product of semi-invariants required for the graph G_n by Rule 1(c). To find the coefficient of v^n in the

† For a finite lattice with free surfaces $m(G_n)$ is not strictly intensive. This remark is not entirely pedantic, since it shows where in the expansion the boundary contribution to W may be found. (See Watson, Vol. 1, Chapter 4.) We shall in the text continue to assume periodic boundary conditions, thus guaranteeing intensity of the multiplicity.

expansion for W we have only to sum up the contributions of all n -edged graphs. If one is given a list of graphs G_n along with corresponding symmetry factors and multiplicities, the labour, which is entirely clerical, might be organized as shown in Fig. 4.

G	$g(G)$	$m(G)$	$W(G)/N$
	12	q	$\frac{1}{12}qv^3(M_3^0)^2$
	6	q^3	$\frac{1}{6}q^3v^3(M_1^0)^3M_3^0$
	2	q^2	$\frac{1}{2}q^2v^3M_1^0M_2^0M_3^0$
	6	$6p(\Delta)$	$p(\Delta)v^3(M_2^0)^3$
	2	q^3	$\frac{1}{2}q^3v^3(M_1^0)^2(M_2^0)^2$

FIG. 4. Bookkeeping for the evaluation of the v^3 term in the nearest-neighbour Ising model free energy. The total v^3 contribution is the sum of the five terms in the right hand column. q is the coordination number of the lattice and $p(\Delta)$ is its weak lattice constant for the triangle graph. For loose-packed lattices $p(\Delta) = 0$. $p(\Delta) = 2, 8$ on the triangle and f.c.c. lattices, respectively.

So far we have discussed only the free energy expansion. Equations (4)–(6) allow corresponding linked-cluster expansions to be written down for all Ising model cumulant correlations $\mathcal{M}_n$ simply by term-by-term differentiation of the free energy. Note that (i) the fields appear in W only through the semi-invariants M_n^0 and (ii) differentiation of an $M_n^0(h)$ has by (14) the very simple effect,

$$\left(\frac{d}{dh}\right) M_n^0(h) = M_{n+1}^0(h). \quad (26)$$

Consider, for example, differentiation of the first few terms (19) of W :

$$\begin{aligned} \frac{\langle s_{z1} \rangle}{s} &= \langle \mu(1) \rangle \equiv \mathcal{M}_1(1) \equiv \frac{\delta W}{\delta h(1)} \equiv M_1(1) \\ &= M_1^0(1) + \sum_2 M_2^0(1)v(12)M_1^0(2) \\ &\quad + \frac{1}{2} \sum_3 M_3^0(1)v^2(12)M_2^0(2) \end{aligned}$$

$$\begin{aligned} &+ \sum_{2,3} M_2^0(1)v(12)M_2^0(2)v(23)M_1^0(3) \\ &+ \frac{1}{2} \sum_{2,3} M_3^0(1)v(12)M_1^0(2)v(13)M_1^0(3) \\ &+ \text{order } v^3 \text{ and higher.} \end{aligned} \quad (27)$$

The effect of differentiation is to fix by turn each graphical vertex and to increase by one the order of its bare semi-invariant. When two vertices of the same graph are topologically equivalent, their contributions are, of course, identical.

The result of the above procedure is easily expressed in terms of a graphical rule; however, while we are at it, it will be useful to define a somewhat more general quantity, the full significance of which will only appear in Section II.E. In this spirit we define $M_n(1)$ the n th order renormalised semi-invariant,

$$M_n(1) = \left\{ \begin{array}{l} \text{sum of all topologically} \\ \text{distinct 1-rooted connected} \\ \text{graphs according to Rule 2.} \end{array} \right\} \quad (28a)$$

Rule 2: Renormalised Semi-Invariants, Unrenormalised Form

- Assign the label 1 to the external (root) vertex and a dummy label to each internal vertex.
- For each edge joining vertices i and j write a factor $v(ij)$.
- For each l -valent internal vertex i write a factor $M_l^0(i)$. For each l -valent external vertex write a factor $M_{n+1}^0(1)$.
- Sum each internal vertex label freely over the entire lattice.
- Divide by the symmetry factor of the 1-rooted graph.

The graphical representation (cf. (20)) starts,

$$\begin{aligned} M_n(1) &= \text{graphical representation} \\ &= \text{order } v^4 \text{ and higher.} \end{aligned} \quad (28b)$$

The open circle represents the external vertex 1. Notice that the symmetry factors for the rooted graphs differ from those of their unrooted analogs. When $n = 1$, Rule 2 gives the magnetisation (27) of the Ising model.

An additional derivative of (27) gives the pair correlations (5). Note that there are two types of terms, those in which the $\delta/\delta h(2)$ derivative hits the

δ -functions and bare semi-invariants (13). Because $v(ij) = 0$ when $i = j$, the locality property eliminates from (32) any term in $\langle \mu(1) \dots \mu(2n) \rangle_0$ in which both ends of the same bond $v(ij)$ have been assigned to the same $\mathcal{M}^0$, i.e., $v(12)\mathcal{M}^0(12\dots) = 0$. The remaining bracketed quantities can be expressed graphically in the sense of Rule 1 (a)-(d) if we include both *connected and disconnected graphs*. The $n = 1$ and $n = 2$ brackets are, for example, respectively,

$$\left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \text{ and } \left[\begin{array}{c} \bullet \bullet \\ | \quad | \\ \bullet \bullet \end{array} \right] + 4 \left[\begin{array}{c} \bullet \bullet \\ | \quad | \\ \bullet \bullet \end{array} \right] + 2 \left[\begin{array}{c} \bullet \bullet \\ | \quad | \\ \bullet \bullet \end{array} \right], \quad (34)$$

where the numerical factors come from different permutations of the dummy labels. Disconnected contributions are products of *connected components*. Let us label all topologically distinct connected components by some index $\alpha = 1, 2, \dots$ and the corresponding contributions according to Rule 1 (a)-(d) by w_α . Figure 5 shows how the first few lines of such a listing might look.

α	G_α	w_α
1		$\sum_{1,2} M_1^0(1)v(12)M_1^0(2)$
2		$\sum_{1,2,3} M_1^0(1)v(12)M_2^0(2)v(23)M_1^0(3)$
3		$\sum_{1,2} M_2^0(1)(v(12))^2 M_2^0(2)$

FIG. 5. A partial listing of the contributions of the various connected components in the bracket of eqn (32).

Any term in the bracket is simply a product of one or more w_α , $\prod_{\alpha=1}^{\infty} w_\alpha^{n_\alpha}$, where $n_\alpha = 0, 1, \dots$ is the number of times each G_α appears. For example, the terms in (34) are just $[w_1]$ and $[w_1^2 + 4w_2 + 2w_3]$. Note that each term may appear a number of times. This is because the arguments 1, 2, ..., $2n$ are summed, so contributions in which these arguments have been *permuted in such a way as to leave the topology unchanged* are identical. The number of such permutations for a graph specified by a given set of $\{n_\alpha\}$ is just

$$2^n n! \prod_{\alpha=1}^{\infty} \frac{1}{n_\alpha!} g_\alpha^{n_\alpha},$$

where g_α is the symmetry factor referred to in Rule 1(e). The total contribution to the bracket in (32) of each topology $\{n_\alpha\}$ is just

$$2^n n! \prod_{\alpha} \frac{1}{n_\alpha!} \left(\frac{w_\alpha}{g_\alpha} \right)^{n_\alpha}. \quad (35)$$

Substitution of (35) into (32) gives

$$W[h, v] = W_0[h] + \sum_{\alpha=1}^{\infty} \frac{w_\alpha}{g_\alpha}, \quad (36)$$

which is identical to Rule 1.

The graphical rules for correlation functions may be derived by arguments essentially similar to the above (Englert, 1963). In the evaluation of $\langle X \rangle = \text{Tr } X \exp(-\beta \mathcal{H}) / Z[h, v]$, the denominator divides out all unrooted graphs. The whole procedure is entirely analogous to our treatment of W , and we prefer the method of term-by-term differentiation described in II.B.

D. Relation of the linked-cluster and weak-embedding expansions for the free energy

The relation between the linked-cluster and weak-embedding expansions is simple in principle, although, as will be seen in what follows, the enumeration of those terms in one which correspond to a given term in the other can be quite involved. The relation is most easily studied by means of an example. We choose for this purpose the $s = \frac{1}{2}$ Ising model free energy W .

The standard weak-embedding expansion (Newell and Montroll, 1953; Domb, 1960) uses the operator identity $\exp(\alpha A) = \cosh \alpha + A \sinh \alpha$, valid for any operator A such that $A^2 = 1$, to rewrite the $s = \frac{1}{2}$ Ising partition function (3). Thus,

$$W[h, v] = \sum_i \ln(2 \cosh h(1)) + \frac{1}{2} \sum_{i,j} \ln \cosh v(12) + \ln 2^{-N} \text{Tr} \left[\prod_i (1 + \mu(1) \tanh h(1)) \prod_{\langle ij \rangle} (1 + \mu(i) \mu(j) \tanh v(12)) \right], \quad (37)$$

where $\prod_{\langle ij \rangle}$ denotes the product over all pairs of sites. Development of the trace in powers of $\tanh v$ yields the weak-embedding expansion. We maintain as *individually* variable the local fields and bond strengths in order to facilitate detailed comparison with the linked-cluster form of the development. Terms which survive the trace must have an even number of factors $\mu(i)$ at each site i and, thus, can be designated uniquely by listing the contributing bonds $\tanh v(ij)$. The topology of the contributing bonds constitutes a graph. Both connected and disconnected graphs appear. When the logarithm is taken, all those parts of the disconnected contributions which go as N^n , $n > 1$,

cancel against iterations of lower orders, thus guaranteeing extensivity in the thermodynamic limit.

The first few terms in the resulting expression are shown in equation (38):

$$\begin{aligned}
 W[h, v] - W_0[h] - \frac{1}{2} \sum_{1,2} \ln \cosh v(12) \\
 = \frac{1}{2} \sum_{1,2} \tanh h(1) \tanh v(12) \tanh h(2) \\
 + \frac{1}{2} \sum_{1 \neq 3} \tanh h(1) \tanh v(12) \tanh v(23) \tanh h(3) \\
 - \left[\frac{1}{4} \sum_{1,2} \tanh^2 h(1) \tanh^2 v(12) \tanh^2 h(2) \right. \\
 \left. + \frac{1}{2} \sum_{1 \neq 3} \tanh h(1) \tanh v(12) \tanh^2 h(2) \tanh v(23) \tanh h(3) \right] + \dots \\
 = \text{---} + \text{---} + \text{---} + \dots, \quad (38)
 \end{aligned}$$

where we have indicated the graph corresponding to each term in the weak-embedding sense. Each edge i, j carries a factor $\tanh v(ij)$. Each vertex i with odd valence carries a factor $\tanh h(i)$. The disconnected graphs come from the expansion of the logarithm.

On the other hand, the bare semi-invariants (14) for the linked-cluster expansion are for $s = 1/2$,

$$\begin{aligned}
 M_n^0(h) &= \frac{d^n}{dh^n} \ln (2 \cosh h), \\
 M_1^0 &= \tanh h, \quad M_2^0 = 1 - \tanh^2 h, \dots, \quad (39)
 \end{aligned}$$

Substitution into (19) gives,

$$\begin{aligned}
 W[h, v] - W_0[h] &= \frac{1}{2} \sum_{1,2} \tanh h(1) v(12) \tanh h(2) \\
 &+ \frac{1}{4} \sum_{1,2} (1 - \tanh^2 h(1)) v^2(12) (1 - \tanh^2 h(2)) \\
 &+ \frac{1}{2} \sum_{1,2,3} \tanh h(1) v(12) (1 - \tanh^2 h(2)) v(23) \tanh h(3) + \dots \\
 &= \frac{1}{2} \text{---} + \frac{1}{4} \text{---} + \frac{1}{2} \text{---} + \dots \quad (40)
 \end{aligned}$$

To compare (38) and (40), it is necessary (i) to expand $\ln \cosh v$ and $\tanh v$ in powers of v and (ii) to dissect out of the free linked-cluster sums those contributions from each set of distinct bonds. The connections in Fig. 6 show which set of graphs in one expansion contribute to a given term of the other.

Note that each weak graph will, in a complete tabulation, contribute to an infinite number of linked-cluster graphs (since $\tanh v$ has all odd powers of v), while each linked-cluster graph contributes only to a finite number of weak graphs (with the same number of lines or fewer).

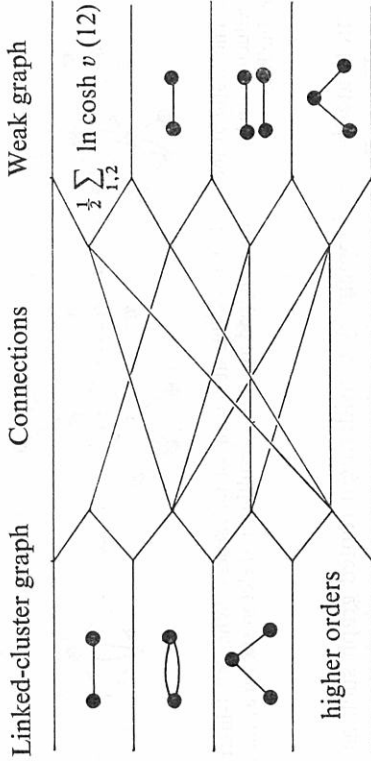


FIG. 6. Graphical correspondences between the linked-cluster and weak-embedding expansions for the $s = \frac{1}{2}$ Ising free energy. Connections show which graphs in one expansion contribute to a given term in the other.

To clarify this relation it is useful to sort the terms of both expansions according to which connected set of bonds $\{v(ij)\}$ is used. This sorting according to a *physical cluster* of bonds has been discussed by Rushbrooke and Morgan (Rushbrooke and Morgan, 1961; Morgan and Rushbrooke, 1961; Rushbrooke, 1964), Elliott and Heap (Elliott and Heap, 1962; Heap, 1963), Jasnow and Wortis (1967), and others. Such a physical cluster receives on the one hand contributions from all *linked-cluster* graphs which can be embedded in it using every physical bond *at least once* (note that the free sums allow multiple usage of each site and bond—quite complicated graphs can be “collapsed” to fit into simple physical clusters) and on the other hand contributions from all weak graphs which can be embedded in it using every bond *once and only once*. A further sorting of these correspondences by powers of $v(ij)$ for each bond extends Fig. 6 as far as patience permits.

The complexity of the interrelation between the two types of expansion rests—in a language restricted to the nearest-neighbour case (23)—on the connection between the free multiplicities $m(G)$ defined after (24) and the weak lattice constants. This connection, which is touched upon in the appendix, has not yet been seriously explored.

E. Vertex renormalisation: the correlations

To make the linked-cluster expansion practical from a calculational point of view, one must find a way of reducing the number of contributing graphs.

This is done by selective resummation of classes of graphs. The effect is to introduce a degree of implicitness into the procedure and thus to exchange topological complexity for algebraic complexity.

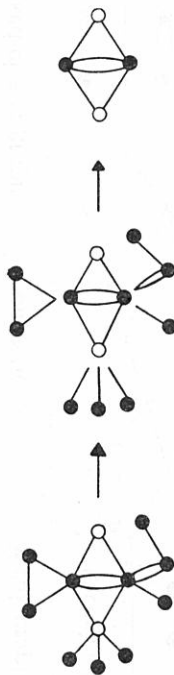


FIG. 7. The graph at the left is 1-reducible at several of its vertices. When all 1-insertions at the reducible vertices have been clipped away, the resulting 1-skeleton shown at the right is 1-irreducible.

We motivate the development by considering a rooted graph such as that shown in Fig. 7. Every such graph can be regarded as composed of a basic *skeleton graph* decorated by additions or *insertions*. There are an infinite number of insertions, which can be made singly or in combinations at each vertex of such a skeleton. The contributions to each graph corresponding to the insertions at a given vertex enter multiplicatively and are independent of the rest of the graph, *since the sum in part (d) of the Rules is always free*. The summed contributions of a fixed bare skeletal vertex with all its possible decorations defines a factor we shall call the *renormalised semi-invariant* M_n , where n gives the number of incident *skeletal* lines. See Fig. 8. Calculation of a skeleton with renormalised semi-invariants at the vertices sums the infinite number of bare-perturbation-theory graphs having that skeleton.

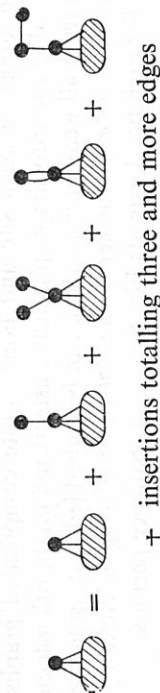


FIG. 8. Vertex renormalisation sums graphs differing only by insertions at a given vertex. The black circle in the graph at the left represents the renormalised vertex factor M_3 . The cross-hatched oval, which stands for the remainder of the graph contains one or more rooted vertices. The vertex undergoing renormalisation may itself be a root.

Before continuing, it will be useful to supplement the graphical terminology introduced after eqn (19). In the discussion below all graphs will be assumed *connected* unless explicitly stated to the contrary.

Disassembly of a Vertex†

A vertex V of a graph G is disassembled by the following procedure:

- (i) Clip away all edges from V (thereby dividing G into a number of disconnected parts).
- (ii) Reattach to V all clipped edges belonging to disconnected parts containing external vertices.
- (iii) Join to one another the clipped edge ends belonging to each additional connected part (if any).

This process is pictured for several graphs in Fig. 9.

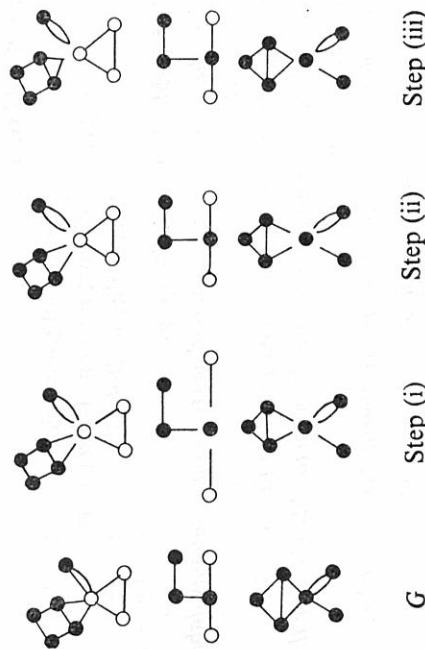


FIG. 9. Disassembly of various graphical vertices.

Graphical Definitions II

The unrooted parts formed in the process of disassembly are called 1-insertions. The clipped vertex of a 1-insertion will be regarded as the external vertex of the 1-insertion and will be drawn without a circle.‡

An articulation point of a rooted graph is a vertex the disassembly of which produces at least one 1-insertion. An articulation point of an unrooted graph is a vertex the disassembly of which produces at least two 1-insertions. A graph which contains no articulation points is 1-irreducible. A graph which contains one or more articulation points is 1-reducible.

The 1-skeleton of a rooted graph is the graph produced by disassembling in turn all vertices of the graph and discarding all 1-insertions thereby produced.

† This operation differs, for example, from deletion of a vertex, which removes all incident edges (Essam and Fisher, 1970), in that no graphical elements are gained or lost. From the algebraic point of view, we are simply regrouping the factors in each perturbation theoretic term. This is the origin of the slight differences which will appear between our terminology and that of Essam and Fisher.

‡ This notation is intended to suggest that such a vertex carries no semi-invariant factor.

Any rooted graph has a *unique 1-irreducible 1-skeleton*, as illustrated in Fig. 7, to which it can be reduced by clipping away all 1-insertions. (Note that the open circle is the 1-skeleton of every 1-rooted graph.) Conversely any rooted graph can be constructed from a 1-irreducible 1-skeleton in a unique way by appending appropriate combinations of 1-insertions at each skeletal vertex.[†]

Because of the free sums in parts (d) of Rules 1–3, each 1-insertion enters as a factor in the overall graphical contribution. To keep track of the appropriate subscript on the bare semi-invariant, it is necessary to classify 1-insertions according to the *valence of the external vertex*. We therefore define the *self-field* $G_n(1)$ as,

$$G_n(1) = \left\{ \begin{array}{l} \text{sum of all topologically} \\ \text{distinct } n\text{-valent 1-insertions} \\ \text{according to Rule 4} \end{array} \right\} \quad (41a)$$

Rule 4: Self-Field, Unrenormalised Form

- Assign the label 1 to the external vertex and a dummy label to each internal vertex.
- For each edge joining vertices i and j write a factor $v(ij)$.
- For each 1-valent internal vertex i write a factor $M_i^0(i)$. For the external vertex write a factor 1.
- Sum each internal vertex label freely over the entire lattice.
- Divide by the symmetry factor of the 1-insertion.

In this form G_n depends on v and h (via the M_n^0 's). The graphical representation of the first few G_n 's looks like:

$$\begin{aligned} G_1(1) &= \text{graph 1} + \frac{1}{2} \text{graph 2} + \frac{1}{2} \text{graph 3} + \text{order } v^4 \text{ and higher.} \\ G_2(2) &= \frac{1}{2} \text{graph 4} + \frac{1}{2} \text{graph 5} + \frac{1}{2} \text{graph 6} + \text{order } v^4 \text{ and higher} \\ G_3(1) &= \frac{1}{6} \text{graph 7} + \text{order } v^4 \text{ and higher.} \end{aligned} \quad (41b)$$

The factors G_n play the physical role of a generalized 1-body effective field. They have a parallel in the classical-fluid expansion (see II.F), and are closely analogous to the quantum-mechanical self-energy operator Σ (Kadanoff and Baym, 1962).

We are now in a position to calculate the net contribution of all possible

[†] Note that an *unrooted* graph does not possess a unique construction from an underlying 1-skeleton. It can, for example, be constructed by decorating *any* isolated vertex. Because of this ambiguity the free energy must be handled separately from the correlations (see Section II.F).

decorations of a single given skeletal vertex, as pictured for example in Fig. 8. The presence of k identical 1-insertions at the same vertex makes a multiplicative contribution of $k!$ to the overall symmetry factor for the decoration of an n -valent skeletal vertex:

$$\begin{aligned} M_n(1) &= M_n^0(1) + \sum_{l=1}^{\infty} G_l(1) M_{n+l}^0(1) + \frac{1}{2} \sum_{l,m=1}^{\infty} G_l(1) G_m(1) M_{n+l+m}^0(1) + \dots \\ &= \left[\exp \sum_{l=1}^{\infty} G_l(1) \frac{\delta^l}{\delta \tilde{h}} \right] M_n^0[1; \tilde{h}] \Big|_{\tilde{h}=h(1)}, \end{aligned} \quad (42)$$

with $n = 1, 2, 3, \dots$, which is a re-expression of (28) and includes the magnetisation $\mathcal{M}_1(1) = M_1(1)$ as a special case.[†]

Summation of all rooted graphs having the same 1-skeleton is now accomplished by the replacement $M_n^0 \rightarrow M_n$ at the skeletal vertices. The only difficulty in passing from (42) to the above statement is the question of symmetry factors. Suppose a given rooted graph G is composed of a 1-skeleton, G_s , decorated by a series of 1-insertions $G_{i1}, G_{i2}, \dots$. The symmetry factor $g(G)$ is composed of several factors,

$$\frac{1}{g(G)} = \frac{1}{g(G_s)} \frac{1}{\Pi g(G_i)} \frac{F_{EV}}{F_{SV}}. \quad (43)$$

F_{SV} (SV = single vertex) stands for the combinatorial factors arising from the presence of identical 1-insertions at a single vertex. F_{EV} (EV = equivalent vertex) stands for the combinatorial factors by which the skeletal symmetry factor $g(G_s)$ is reduced due to distinct decoration of previously equivalent skeletal vertices. For example, for the graph of Fig. 7, $g(G_s) = 4$, $\Pi g(G_i) = 2^2 = 4$, $F_{SV} = 3! = 6$, $F_{EV} = 2$, and $g(G) = 48$. In the renormalised form only the factor $1/g(G_s)$ is explicit. The $g(G_i)$ are absorbed in the self-fields; the F_{SV} are incorporated in the exponential form of (42); and, the F_{EV} are supplied by equivalent cross-terms in the expansion of the product of renormalised semi-invariants.

The procedure for finding the vertex renormalised form of a bare expansion is to *group all contributing graphs with the same 1-skeleton and to sum them by putting $M_n^0 \rightarrow M_n$* . For example, the two-point part of the pair correlations given by (29), (30), and Rule 3 may be re-expressed as,

$$\mathcal{M}_2(12) = \delta(12) M_2(1) + \left\{ \begin{array}{l} \text{sum of all topologically distinct} \\ \text{2-rooted 1-irreducible connected} \\ \text{graphs according to Rule 5.} \end{array} \right\}. \quad (44)$$

[†] Equation (42) is the classical analog of the quantum many-body Dyson equation (Kadanoff and Baym, 1962). M_n^0 corresponds to the bare propagator G_0 , and M_n corresponds to the renormalised propagator G . The fact that M_n^0 and M_n are spatially local reflects the locality of the Hamiltonian (1) in the absence of interaction.

Rule 5: Pair Correlations, Vertex Renormalised Form

- Assign labels 1 and 2 to the external vertices and dummy labels to the internal vertices.
- For each edge joining vertices i and j write a factor $v(ij)$.
- For each l -valent internal vertex i write a factor $M_l(i)$. For each l -valent external vertex $j = 1, 2$ write a factor $M_{l+1}(j)$.
- Sum each internal vertex label freely over the entire lattice.
- Divide by the symmetry factor of the 2-rooted graph.

Graphically,

$$\mathcal{M}_2(12) - \delta(12)M_2(1) = \text{diagram 1} + \frac{1}{2} \text{diagram 2} + \frac{1}{2} \text{diagram 3} + \frac{1}{2} \text{diagram 4} + \text{order } v^4 \text{ and higher.} \quad (45)$$

Equation (45) sums an infinite number of terms in the unrenormalised expansion and is exact through order v^3 with fewer graphs than the corresponding bare expression (30). The simplification is even more impressive in higher orders.

In contrast to the bare expansion, for which the vertex factors M_n^0 are known directly (see (10) and (14)), the renormalised expansion is only useful after the semi-invariants M_n have been calculated (e.g. from (41) and (42)). This is facilitated by using a renormalised expression for the self-fields G_n :

$$G_n(1) = \left\{ \begin{array}{l} \text{sum of all topologically distinct 1-irreducible} \\ \text{1-insertions which are } n\text{-valent at the external} \\ \text{vertex according to Rule 6.} \end{array} \right. \quad (46)$$

Rule 6: Self-Field, Vertex Renormalised Form

Same as Rule 4 except:

- For each l -valent internal vertex i write a factor $M_l(i)$. For the external vertex write a factor 1.

$$\begin{aligned} G_1(1) &= \text{diagram 1} \quad (\text{exact!}), \\ G_2(1) &= \frac{1}{2} \text{diagram 2} + \frac{1}{2} \text{diagram 3} + \frac{1}{2} \text{diagram 4} + \text{order } v^5 \text{ and higher,} \\ G_3(1) &= \frac{1}{6} \text{diagram 5} + \frac{1}{2} \text{diagram 6} + \text{order } v^5 \text{ and higher.} \end{aligned} \quad (47)$$

In this form the self-field depends functionally on v and the set M_n , $n = 1, 2, \dots$. Thus, $G(1) = G[1; M, v]$, and (42) becomes a non-linear equation, schematically,

$$M = \exp(G[M, v])M^0 \quad \text{or, more symmetrically, } M^0 = \exp(-G[M, v])M, \quad (48)$$

which can in principle be inverted to give $M[M^0, v]$. Once the M 's are known as functions of the variables h and v , then eqns (44) and (45) and their higher-order analogs evaluate the cumulant correlations.

In the derivation of high-temperature series expansions (Jasnow, 1969; Jasnow and Wortis, 1968) eqns (47) and (48) are solved iteratively in powers of v . To zeroth order $M_n = M_n^0$, which are given by (10) and (14). Note that $G_n \sim v^n$. Once the M_n 's are known to order v^l , (47) gives the G 's to order $l+1$. These new G 's in (48) give the M_n 's to order $l+1$, and so on. Stanley (1967) lists all $h = 0$ M_2 graphs necessary through order v^8 (close-packed lattices) and v^9 (loose-packed lattices).

To go beyond iteration, it is necessary to use other methods (such as those discussed in Section II.H) to find $G[M, v]$. Inverting (48) may be facilitated by forming the Fourier transforms of the bare semi-invariants,

$$M_n^0(h) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikh} (ik)^n M_0^0(k), \quad (49)$$

where some care must be taken for $n = 0, 1$. Then, from (42),

$$M_n(h) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\exp \sum_{l=1}^{\infty} (ik)^l G_l[M, v] \right) e^{ikh} (ik)^n M_0^0(k). \quad (50)$$

In the translationally invariant situation (21), (50) is a set of coupled algebraic equations for the numbers M_n , $n = 1, 2, \dots$.

F. Vertex renormalisation: the free energy

Free energy graphs are unrooted and hence have no unique 1-skeleton. The direct regrouping described in Section II.E is therefore inapplicable. This obstacle can be circumvented by two routes. De Dominicis (1962), Englert (1963), and Bloch and Langer (1965) use a topological method. Following an earlier publication (Wortis, *et al.*, 1969), we prefer simply to integrate eqn (4),

$$M_1(1) = \frac{\delta W[h, v]}{\delta h(1)}, \quad (51)$$

subject to the boundary condition that, as $h(1) \rightarrow \infty$, the only contributing state is $\mu(1) = 1$ for all sites 1. Thus, $M_1^0(1) = 1$, $M_n^0(1) = 0$, $n > 1$, and only the first two terms in (20) are non-vanishing, with analogous simpli-

fications for other quantities: as $h(1) \rightarrow \infty$ for all sites 1,

$$W = \sum_1 h(1) + \frac{1}{2} \sum_{1,2} v(12),$$

$$M_1(1) = 1, \quad G_1(1) = \sum_1 v(12),$$

$$M_n(1) = G_n(1) = 0 \text{ for } n > 1. \quad (52)$$

Equation (42) with $n = 1$ provides a vertex renormalised expression for $M_1(1)$. Our task is to construct a functional W which satisfies (51) subject to (52).

Naively, it might seem that a good candidate for W would be $\Sigma_1 M_0(1)$, with $M_0(1)$ defined by (42) with $n = 0$. On closer inspection we note that G_n depends on h through the renormalised semi-invariants $G[M(h), v]$. Thus, direct differentiation of (42) gives

$$\frac{\delta}{\delta h(1)} \sum_2 M_0(2) = M_1(1) + \sum_2 \sum_{n=1}^{\infty} \frac{\delta G_n(2)}{\delta h(1)} M_n(2). \quad (53)$$

Another appealing candidate is the symmetrical quantity,

$$\Phi[M, v] = \left\{ \begin{array}{l} \text{sum of all topologically distinct 1-irreducible} \\ \text{0-rooted graphs}^\dagger \text{ according to Rule 7.} \end{array} \right\}. \quad (54)$$

Rule 7: The Functional Φ , Vertex Renormalised Form

Same as Rule 1 except:

(c) For each 1-valent internal vertex i write a factor $M_i(i)$.

$$\Phi[M, v] = \frac{1}{2} \left(\text{diagram 1} + \frac{1}{4} \text{diagram 2} + \frac{1}{12} \text{diagram 3} + \frac{1}{48} \text{diagram 4} + \frac{1}{8} \text{diagram 5} \right) + \text{order } v^5 \text{ and higher.} \quad (55)$$

Φ , which is like a thermodynamic potential, has as its natural variables v and the renormalised semi-invariants. It is easy to verify that

$$\frac{\delta \Phi[M, v]}{\delta M_n(1)} = G_n(1). \quad (56)$$

For example, consider the first term of (55):

$$\frac{1}{2} \text{diagram 1} = \frac{1}{2} \sum_{2,3} M_1(2) v(23) M_1(3). \quad (57)$$

Differentiation of (57) with respect to $M_1(1)$ yields $G_1(1) = \sum_2 v(12) M_1(2)$. (Note that (57) is the only 1-irreducible graph containing a 1-valent vertex.) In general, term-by-term differentiation of (55) leads directly to (47). The

[†] The single vertex is specifically omitted from this sum.

only point requiring comment is the symmetry factors. It suffices to note that, in the differentiation (56), vertices in (55) equivalent under symmetry operations of the graph contribute additively. The factor thus produced serves to lower the symmetry factor of the unrooted graph to that of the 1-rooted graph as required by Rule 6(e).

What is the field derivative of Φ ? Use of (56) and the chain rule gives,

$$\frac{\delta \Phi}{\delta h(1)} = \sum_2 \sum_{n=1}^{\infty} \frac{\delta \Phi}{\delta M_n(2)} \frac{\delta M_n(2)}{\delta h(1)} = \sum_2 \sum_{n=1}^{\infty} G_n(2) \frac{\delta M_n(2)}{\delta h(1)}. \quad (58)$$

The sum of (53) and (58) yields,

$$\frac{\delta}{\delta h(1)} \left[\sum_2 M_0(2) + \Phi \right] = M_1(1) + \frac{\delta}{\delta h(1)} \left[\sum_2 \sum_{n=1}^{\infty} G_n(2) M_n(2) \right]. \quad (59)$$

Thus,

$$W = \sum_2 M_0(2) + \Phi - \sum_2 \sum_{n=1}^{\infty} G_n(2) M_n(2), \quad (60)$$

which can easily be shown to satisfy (52). When once the renormalised semi-invariants have been computed (as discussed in II.E), then (60) evaluates the free energy.

If properly interpreted, (60) has an interesting stationarity property. Consider the functional

$$\begin{aligned} \mathcal{W}_1[M, v; h] \equiv \sum_2 \left[\left(\exp \sum_{n=1}^{\infty} G_n[2; M, v] \frac{d^n}{d\tilde{h}^n} \right) M_0^0(\tilde{h}) \right]_{\tilde{h}=h(2)} \\ + \Phi[M, v] - \sum_2 \sum_{n=1}^{\infty} G_n[2; M, v] M_n(2), \end{aligned} \quad (61)$$

which is equal to W provided the M_n , $n \geq 1$, are related to the fields h via (42). We choose to regard $\mathcal{W}_1$ as depending separately on the M_n 's, $n \geq 1$, (via $\Phi[M, v]$ and $G[M, v]$) and the fields h which appear explicitly in the M_n^0 's of first term on the right. Direct differentiation of (61) using (56) gives,

$$\begin{aligned} \frac{\delta \mathcal{W}_1}{\delta M_1(1)} \Big|_{h \text{ fixed}} &= \sum_2 \sum_{n=1}^{\infty} \frac{\delta^2 \Phi}{\delta M_1(1) \delta M_n(2)} \\ &\times \left[\left(\exp \sum_{k=1}^{\infty} G_k(2) \frac{d^k}{d\tilde{h}^k} \right) M_n^0(\tilde{h}) \right]_{\tilde{h}=h(2)} - M_n(2), \end{aligned} \quad (62)$$

so that the physical free energy W is a stationary point of $\mathcal{W}_1$. An additional derivative of (62) shows that the stationary point is not necessarily an extremum.

There is another related variational principle (Bloch and Langer, 1965) which can be proved to be a maximum principle. The free energy $W[h, v]$ is

ultimately a functional of the inhomogeneous applied magnetic field h . We are free, equivalently, to regard W as functional of the inhomogeneous magnetisation M_1 via the dependence $W[h[M_1, v], v]$ provided that the magnetisation $M_1 = M_1[h, v]$ is invertible. This will be so, provided that the Jacobian connecting M_1 and h is non-vanishing, which in turn is proved by the observation that the real symmetrical fluctuation matrix,

$$\frac{\delta M_1(1)}{\delta h(2)} \equiv \frac{\delta^2 W}{\delta h(1) \delta h(2)} = \langle (\mu(1) - \langle \mu(1) \rangle) (\mu(2) - \langle \mu(2) \rangle) \rangle, \quad (63)$$

is positive definite. The generator of the Legendre transformation to the variables $[M_1, v]$ is,

$$\Psi[M_1, v] \equiv W[h[M_1, v], v] - \sum_2 h[2; M_1, v] M_1(2),$$

with

$$\frac{\delta \Psi}{\delta M_1(1)} = \sum_2 \frac{\delta W}{\delta h(2)} \frac{\delta h(2)}{\delta M_1(1)} - \sum_2 \frac{\delta h(2)}{\delta M_1(1)} M_1(2) - h_1(1) = -h[1; M_1, v]. \quad (64)$$

Notice that

$$-\frac{\delta^2 \Psi}{\delta M_1(1) \delta M_1(2)} = \frac{\delta h[1; M_1, v]}{\delta M_1(2)}$$

is the inverse of (63) and, therefore, positive definite. The functional,

$$\mathcal{W}_2[M_1, v; h] \equiv \Psi[M_1, v] + \sum_1 h(1) M_1(1), \quad (65)$$

(as in (61) the field h is to be regarded as a parameter) has the property that it is a maximum and equal to the physical free energy W when M_1 is put equal to the physical magnetisation distribution corresponding to the field h .

The functional $\mathcal{W}_2$ is asymmetrical in the renormalised semi-invariants because of the special relationship (4) between h and M_1 . By defining a set of fictitious fields (Bloch and Langer, 1965; Wortis *et al.*, 1969) related to the higher-order semi-invariants as h is to M_1 , it is possible to define an entirely symmetrical generalisation of $\mathcal{W}_2$, which is stationary under variations of all M_n 's, $n \geq 1$, but, unfortunately, not maximal.

G. The classical-fluid linked-cluster expansion as a special case

It is not our purpose in this section to review the extensive literature on the classical fluid (De Dominicis, 1962; Stell, 1964). Rather, we wish to sketch the classical-fluid linked-cluster development only sufficiently so that the reader familiar with the classical situation will appreciate the very close parallel with the present development. Skipping this section will not destroy the continuity of the text.

It is convenient to introduce a single-particle potential field $u(\mathbf{r})$, which plays a role analogous to that of the magnetic field h . The Hamiltonian for N particles can then be written,

$$-\beta \mathcal{H}_N = \sum_{i=1}^N \left(-\frac{\mathbf{p}_i^2}{2m} + u(\mathbf{r}_i) \right) + \sum_{\langle ij \rangle} v(\mathbf{r}_i - \mathbf{r}_j), \quad (66)$$

where all energies are in units of kT and we have chosen u and v positive when they tend to bind. All momentum integrals are trivial and the grand partition function is written (Huang, 1963),

$$\Xi[u, v] = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int (d\mathbf{r}_1) \dots (d\mathbf{r}_N) \exp \left(\sum_{i=1}^N u(\mathbf{r}_i) + \sum_{\langle ij \rangle} v(\mathbf{r}_i - \mathbf{r}_j) \right), \quad (67)$$

where z is the fugacity $e^{\beta\mu}/\lambda^d$ (μ = chemical potential, λ = thermal wavelength, d = dimensionality). In analogy to (3) we write

$$W[u, v] = \ln \Xi. \quad (68)$$

The connection with thermodynamics is $\beta PV = \ln \Xi$, where P is the pressure and V the volume. The usual n -particle distribution functions are,[†]

$$\begin{aligned} \rho_n(\mathbf{r}_1, \dots, \mathbf{r}_n) &\equiv \rho_n(1 \dots n) \\ &= \frac{1}{\Xi} \sum_{l=n}^{\infty} \frac{z^l}{(l-n)!} \int (d\mathbf{r}_{n+1}) \dots (d\mathbf{r}_l) \exp \left[\sum_{i=1}^l u(\mathbf{r}_i) + \sum_{\langle ij \rangle} v(\mathbf{r}_i - \mathbf{r}_j) \right] \end{aligned} \quad (69)$$

which may be generated by differentiation,

$$\frac{1}{\Xi} \frac{\delta \Xi}{\delta u(1)} = \rho_1(1), \quad \frac{1}{\Xi} \frac{\delta^2 \Xi}{\delta u(1) \delta u(2)} = \rho_2(12) + \delta(12) \rho_1(1), \text{ etc.}$$

Cluster functions $\bar{U}$ may be derived by direct differentiation of W ,

$$\bar{U}_n(1 \dots n) \equiv \frac{\delta^n W}{\delta u(1) \dots \delta u(n)}, \quad (70)$$

which are the analogs of our $\mathcal{M}_n(1 \dots n)$ of (6) and differ from the usual Ursell functions U_n (Stell, 1964) by the presence of δ -functions, e.g.,

$$\begin{aligned} \bar{U}_2(12) &\equiv \frac{\delta^2 W}{\delta u(1) \delta u(2)} = \rho_2(12) - \rho_1(1) \rho_1(2) + \delta(12) \rho_1(1) \\ &= U_2(12) + \delta(12) \rho_1(1). \end{aligned}$$

[†] Functional differentiation with respect to a continuous function at a point requires a little discussion (Morita and Hiroike, 1961a, b; Volterra, 1959; Berezin, 1966). Since this discussion is entirely irrelevant to our main purposes, we simply remark that the concerned reader may henceforth imagine himself to be dealing with a gas confined to a lattice.

The non-interacting free energy is (cf. (9)):

$$W_0[u] \equiv W[u, v=0] = \int (d\mathbf{r}) M_0^0(\mathbf{r}) = z \int (d\mathbf{r}) e^{u(\mathbf{r})}, \quad (71)$$

from which it follows that (cf. 14),

$$M_n^0(\mathbf{r}) = z e^{u(\mathbf{r})} \text{ for all } n = 0, 1, 2, \dots \quad (72)$$

By choosing $n = 1$ and comparing with (69), $z e^{u(\mathbf{r})}$ may be identified with $\rho_1^0(\mathbf{r})$, the non-interacting particle density. The unrenormalised rule for the free energy may now be derived just as in Sections II.B or II.C. Everything is exactly the same except that the unrenormalised semi-invariants are given by (72) in place of (10) and (14). To compute the difference $(W[u, v] - W_0[u])$, one sums all topologically distinct unrooted connected graphs according to (cf. Rule 1):

Rule 8A: Classical-Fluid Free Energy W , Unrenormalised v -form

- (a) Assign a dummy label to each internal vertex.
- (b) For each edge joining vertices i and j write a factor $v(ij)$.
- (c) For each vertex i write a factor $z e^{u(i)} = \rho_1^0(i)$.
- (d) Integrate each internal vertex label freely over the volume of the container.
- (e) Divide by the symmetry factor of the unrooted graph.

Equation (20) gives the first few terms of the sum.

The important difference between Rule 8A and Rule 1 is in part (c), where Rule 8A requires a vertex factor $\rho_1^0(i)$ independent of valence. This valence-independence, which follows directly from the exponential form of M_0^0 (eqn (72)), is the only important respect in which the magnetic expansion differs from the classical-fluid expansion. This difference produces a number of simplifications.

The reader will have noticed that the expansion given by Rule 8A is in terms directly of the pair potential rather than of the Mayer f -function, $f(12) \equiv (e^{v(12)} - 1)$. To obtain the traditional form we sum over multiedges:

$$f(12) = | + \frac{1}{2} \bigcirc + \frac{1}{6} \bigcirc + \frac{1}{6} \bigcirc + \frac{1}{n!} \left[\bigcirc \right]_n + \dots \quad (73)$$

Note that each edge of multiplicity n contributes to the symmetry factor g a factor $n!$. By omitting the vertex circles in (73), we wish to emphasise that f contains none of the vertex factors 8A(c). We now group together in a class all graphs contributing to 8A which differ only in the degree of multiplicity of corresponding edges. The replacement $v(12) \rightarrow f(12)$ sums each such class, producing the familiar result that to calculate $(W[u, v] - W_0[u])$ one may

sum all topologically distinct unrooted connected graphs with no multiedges according to:

Rule 8B: Classical-Fluid Free Energy W , Unrenormalised f -form
Same as Rule 8A except:

- (b) For each joining vertices i and j write a factor $f(ij) = (e^{v(ij)} - 1)$.

The first few terms in this form of the expansion are (cf. (20)),

$$\begin{aligned} W[u, f] - W_0[u] &= \frac{1}{2} \bigcirc + \frac{1}{2} \bigcirc + \frac{1}{6} \bigcirc + \frac{1}{6} \bigcirc + \frac{1}{6} \bigcirc + \frac{1}{6} \bigcirc \\ &\quad + \text{order } f^4 \text{ and higher.} \\ &= \frac{1}{2} \int (d\mathbf{r}_1)(d\mathbf{r}_2) \rho_1^0(1) f(12) \rho_1^0(2) + \dots \end{aligned} \quad (74)$$

Note that the summation over graphs differing only with respect to edge multiplicity cannot be done in the magnetic case, since each level of multiplicity requires a different vertex factor M_n^0 due to the changed valence.

Vertex renormalization goes through just as before. Because the vertex factors M_n^0 are valence-independent, there is no need to classify the self-fields G_n according to their external valence. The renormalised vertex is just the local density $\rho_1(\mathbf{r})$ (cf. the identification $M_1(1) = \rho_1(1)$, the local magnetisation, after (42)). The self-field $G(1)$ is therefore (cf. Rule 6 and eqn (46)),

$$G(1) = \begin{cases} \text{sum of all topologically distinct} \\ \text{1-irreducible 1-insertions with} \\ \text{no multiedges according to Rule 9.} \end{cases} \quad (75)$$

Rule 9: Classical-Fluid Self-field, Vertex Renormalised f -form

- (a) Assign the label 1 to the external vertex and a dummy label to each internal vertex.
- (b) For each edge joining vertices i and j write a factor $f(ij) = (e^{v(ij)} - 1)$.
- (c) For each internal vertex i write a factor $\rho_1(i)$. For the external vertex write a factor 1.
- (d) Integrate each internal vertex label freely over the volume of the container.
- (e) Divide by the symmetry factor of the 1-insertion.

The first few terms are:

$$G(1) = | + \frac{1}{2} \bigcirc + \frac{1}{2} \bigcirc + \text{order } f^5 \text{ and higher.} \quad (76)$$

Of course, corresponding expressions in terms of v rather than f or in terms of the unrenormalised vertex ρ_1^0 (cf. Rule 4) could also be written down.

The analog of (42), expressing the decoration of an unrenormalised vertex is simply,

$$\rho_1(1) = e^{G(1)} \rho_1^0(1). \quad (77)$$

Note the absence of derivatives. The vertex renormalised free energy is given by (cf. (60)),

$$W = \int (d\mathbf{r}_1) \rho_1(1) + \Phi - \int (d\mathbf{r}_1) G(1) \rho_1(1), \quad (78)$$

where, in terms of the Mayer f -function, Φ is given by the sum of all topologically distinct connected 1-irreducible unrooted graphs with *no multiedges* according to Rule 10 (cf. Rule 7):

Rule 10: *Classical-Fluid Functional Φ , Vertex Renormalised f -form.*

Same as Rule 9 except:

(e) *Divide by the symmetry of the unrooted graph.*

Rules 9 and 10 give Φ and G as functions of the particle density ρ_1 . In the translationally invariant situation $u(1) = 0$, $v(12) = v(\mathbf{r}_1 - \mathbf{r}_2)$, (78) may be expanded in powers of the density. Graphs with exactly n vertices contribute to ρ_1^n . Thus,

$$\begin{aligned} \beta P &= \frac{W}{V} = \rho_1 + \frac{\Phi}{V} - \rho_1 G \\ &= \rho_1 - \frac{\rho_1^2}{2} \int (d\mathbf{r}) f(\mathbf{r}) - \frac{\rho_1^3}{3} \int (d\mathbf{r}_1)(d\mathbf{r}_2) f(\mathbf{r}_1) f(\mathbf{r}_2) f(\mathbf{r}_1 - \mathbf{r}_2) \\ &\quad + \text{order } \rho_1^4 \text{ and higher.} \end{aligned} \quad (79)$$

which is just the ordinary virial expansion.

H. Φ -Derivable approximations

Despite renormalisation, the series expansions always involve an infinite number of terms. When there is a small expansion parameter (e.g. v or ρ_1), then it may be that only a finite number of terms contribute appreciably. On the other hand, there are situations where it is convenient to have an analytic solution for the free energy, correlations, etc., which is at least qualitatively valid over a wide range of parameters. Such solutions must sum an infinite subset of perturbation theory graphs. Which subset to choose is dictated by an understanding of the physics. So far it has not been possible to pick out an analytically summable subset of graphs which gives proper critical behaviour; however, these closed-form approximations are of sufficient historical interest and sufficient practical importance in *non-critical* regions that they deserve mention.

We shall not attempt to review the history of approximate solutions to the Ising model (Newell and Montroll, 1953; Brush, 1967; see also Burley Volume 2, Chapter 9). Rather we limit ourselves to the presentation of a certain class of those solutions which are closely related to the development of Sections II.E and II.F and which are complete and internally consistent in a sense which will be made clear below.

The basic philosophy of the Φ -derivable approximations (Baym, 1962; Bloch and Langer, 1965) is that they involve direct approximation for the free energy functional in the presence of arbitrary inhomogeneous fields h . Thus, corresponding approximations to all correlation functions may be obtained from

$$\mathcal{M}_n(1 \dots n)_{\text{approx}} = \frac{\delta^n W_{\text{approx}}}{\delta h(1) \dots \delta h(n)}, \quad (80)$$

with the guarantee that, being related to the same underlying free energy, they will all have singular behaviour at the same points and obey certain self-consistency conditions such as the well-known susceptibility sum rule,

$$N\chi = \frac{d^2 W}{dh^2} = \sum_{i,j} \mathcal{M}_2(12). \quad (81)$$

The procedure is as follows:

Step (a): Choose an approximate $\bar{\Phi}[\bar{M}, v]$ consisting of a subset of the terms of (55) (Rule 7). The approximate free energy $\bar{W}[\bar{M}, v]$ is given by (60):

$$\bar{W}[\bar{M}, v] = \sum_n \bar{M}_0(2) + \bar{\Phi} - \sum_n \sum_{i=1}^{\infty} \bar{G}_n(2) \bar{M}_n(2), \quad (82)$$

where the bar denotes "approximate." The expression (82) is highly implicit and incomplete until a rule is given for computing $\bar{G}$ and $\bar{M}$. We now show how to choose $\bar{G}$ and $\bar{M}$ in such a way as to achieve consistency.

Step (b): Take

$$\bar{G}_n[1; \bar{M}, v] = \bar{G}_n(1) = \frac{\delta \bar{\Phi}}{\delta \bar{M}_n(1)}, \quad (83)$$

which is still a functional of the unknown $\bar{M}$'s.

Step (c): Define $\bar{M}$'s as solutions of (42),

$$\bar{M}_n(1) = \left[\exp \sum_{i=1}^{\infty} G_i(1) \frac{\delta^i}{\delta \bar{h}^i} \right] M_n^0[1; \bar{h}]_{\bar{h}=h(1)}. \quad (84)$$

This finally makes (82) and (83) explicit.

It is now easy to show by direct differentiation of (82) that

$$\frac{\delta \bar{W}}{\delta h(1)} \equiv \bar{M}_1(1) = \bar{M}_1(1). \quad (85)$$

In effect and for the *approximate* $\bar{W}$, this simply reverses the steps which lead from (51) to (60). To get the pair correlations, one need only take another field derivative of (85). It is advantageous to perform the calculation a little more generally. From (84)

$$\frac{\delta \bar{M}_n(1)}{\delta h(2)} = \delta(12) \bar{M}_{n+1}(1) + \sum_{i=1}^{\infty} \bar{M}_{n+i}(1) \frac{\delta \bar{G}_i(1)}{\delta h(2)}. \quad (86)$$

Now, $\bar{G}$ depends on h only through the semi-invariants $\bar{M}$, so, using the chain rule,

$$\frac{\delta \bar{M}_n(1)}{\delta h(2)} = \delta(12) \bar{M}_{n+1}(1) + \sum_{i,k=3} \bar{M}_{n+i}(1) \bar{I}_{i,k}(13) \frac{\delta \bar{M}_k(3)}{\delta h(2)}, \quad (87)$$

where I is the symmetric matrix,

$$\bar{I}_{ik}(13) = \frac{\delta^2 \bar{\Phi}}{\delta \bar{M}_i(1) \delta \bar{M}_k(3)} \equiv \langle 11|I|3k \rangle. \quad (88)$$

Finally, solution of (87) with $n = 1$ gives,

$$\bar{M}_2(12) = \sum_{i=1}^{\infty} \langle 11|(1 - \bar{M}I)^{-1}|2\rangle \bar{M}_{i+1}(2). \quad (89)$$

A few examples will serve to make the above more concrete:

1. The Effective-field Approximation

This well known approximation (Smart, 1966), the analog of the Hartree approximation in quantum statistics, follows from the choice at Step (a) of (see eqn (55)) $\bar{\Phi}[\bar{M}, v] = \frac{1}{2}$ and corresponds to the sum of all bare free energy graphs ((20) and Rule 1) having no closed loops. Physically the effective-field approximation is expected to be best for systems with long-range interactions, although it is worth emphasising that effective-field *critical behaviour* has no relation to reality for any physical system with finite range interactions.

Continuing to Step (b) we find,

$$\bar{G}_1(1) = \sum_2 v(12) \bar{M}_1(2) \quad \text{and} \quad \bar{G}_n = 0, \quad n > 1. \quad (90)$$

Thus (Step (c)),

$$\begin{aligned} \bar{M}_n(1) &= \left[\exp \bar{G}_1(1) \frac{\delta}{\delta h} \right] M_n^0[1; \tilde{h}]|_{\tilde{h}=h(1)} \\ &= M_n^0 \left(h(1) + \sum_2 v(12) \bar{M}_1(2) \right). \end{aligned} \quad (91)$$

For $n = 1$ and in the translationally invariant situation (21) this reduces for $s = \frac{1}{2}$ to the well-known self-consistency condition,

$$M_1 = \tanh(h + V(0)M_1), \quad (92)$$

where $V(0) \equiv V(\mathbf{k} = 0)$. The free energy in this approximation is

$$\frac{W}{N} = \ln [\exp(h + V(0)\bar{M}_1) + \exp(-h - V(0)\bar{M}_1)] - \frac{1}{2} V(0) \bar{M}_1^2. \quad (93)$$

Finally, from (88)

$$\bar{I}_n(12) = \begin{cases} v(12), & \text{when } l = n = 1 \\ 0, & \text{otherwise} \end{cases} \quad (94)$$

and

$$M_2(k) = \sum_{\mathbf{r}} e^{-ik \cdot \mathbf{r}} M_2(\mathbf{r}) = \frac{\bar{M}_2}{1 - \bar{M}_2 V(\mathbf{k})}. \quad (95)$$

2. Ring graphs:

The first correction to the effective-field approximation in powers of the inverse range of the potential is usually taken to be the inclusion in Φ of graphs with ring topologies.[†] Here, therefore, we choose (Bloch and Langer, 1965; Horwitz and Callen, 1961),

$$\bar{\Phi}[\bar{M}, v] = \frac{1}{2} \text{ (diagram)} + \frac{1}{6} \text{ (diagram)} + \dots + \frac{1}{2n} \left[\text{diagram} \right]_n + \dots \quad (96)$$

with

$$\bar{G}_1(1) = \sum_2 v(12) \bar{M}_1(2),$$

$$\bar{G}_2(1) = \frac{1}{2} \left[\text{diagram} + \text{diagram} + \dots \right], \quad (97)$$

$$\bar{G}_n = 0 \quad \text{for } n > 2.$$

[†] The inverse-range expansion loses its meaning in the critical region and the above language is intended to be suggestive only (Horwitz and Callen, 1961; Englert, 1963).

[‡] The quantum analog of the ring approximation is the self-consistent dielectric, bubble, or K -approximation. Propagator lines become points and correspond to our vertex functions.

In the translationally invariant limit,

$$\bar{G}_1 = V(0)\bar{M}_1, \quad \bar{G}_2 = \frac{1}{2}C(\mathbf{r} = 0),$$

where

$$C(\mathbf{r}) = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{V(\mathbf{k})}{1 - \bar{M}_2 V(\mathbf{k})}, \quad (98)$$

so that the two fundamental semi-invariants are determined (Step (c)) by the non-linear coupled equations:

$$\begin{aligned} \bar{M}_1 &= \left[\exp G_2 \frac{\delta^2}{\delta \tilde{h}^2} \right] M_1^0 (\tilde{h} + V(0)\bar{M}_1) |_{\tilde{h}=h} \\ \bar{M}_2 &= \left[\exp G_2 \frac{\delta^2}{\delta \tilde{h}^2} \right] M_2^0 (\tilde{h} + V(0)\bar{M}_1) |_{\tilde{h}=h}, \end{aligned} \quad (99)$$

which can, for example, be recast along the lines of (50).

The matrix $\bar{I}$ now has (in its valence indices) two non-vanishing components

$$\begin{aligned} \bar{I}_{11}(12) &= v(12) \\ \bar{I}_{22}(12) &= \frac{1}{2}[C(12)]^2. \end{aligned} \quad (100)$$

1. Analysis of the two-point function

Before proceeding to the two-point or *bond* renormalisation, we must analyse the structure of the 2-rooted graphs and their contributions to the correlations $\mathcal{M}_2$ more carefully than heretofore (De Dominicis, 1962; Stell, 1964).

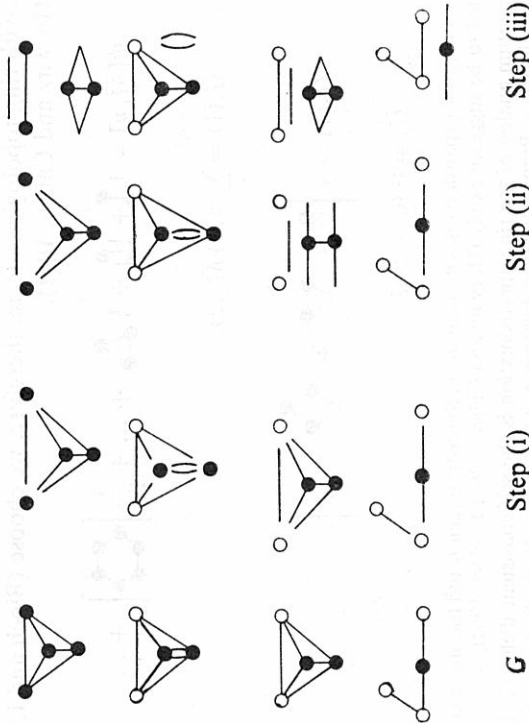


Fig. 10. Disassembly of various vertex pairs.

Disassembly of a vertex pair

A vertex pair V_1 and V_2 of a (1-irreducible) graph G is *disassembled* by the following procedure:

- (i) Clip away all edges from both V_1 and V_2 (thereby dividing G into a number of disjoint parts).
- (ii) Reattach to V_1 and V_2 all clipped edges belonging to disconnected parts containing external vertices.
- (iii) If there are any additional disconnected parts,

- (a) rejoin the clipped ends of each (V_1 ends and V_2 ends separately)
- (b) insert in the rooted part a single edge joining V_1 and V_2 .

This process is pictured for several graphs in Fig. 10. The definitions which follow refer to 1-irreducible graphs.

Graphical definitions III

The *unrooted parts* formed in the process of *disassembly* are called *2-insertions*. A 2-insertion which is not simply a bond is complex.

An articulation pair of a rooted graph is a pair of vertices the *disassembly* of which produces either one complex 2-insertion or any two or more 2-insertions. An articulation pair of an unrooted graph is a pair of vertices the *disassembly* of which produces either two complex 2-insertions or any three or more 2-insertions.

A graph is 2-irreducible if it contains no articulation pairs.

The 2-skeleton of a rooted graph is the graph produced by *disassembling* all pairs of vertices and discarding all 2-insertions thereby produced.

The following definitions classify important types of 2-rooted graphs:

The 2-rooted graph consisting of a single edge joining the external vertices is called the bond graph.

An internal vertex of a 2-rooted graph is said to be a nodal point if all paths between the two external vertices pass through it. A graph is nodal if it contains one or more nodal points; otherwise it is non-nodal.

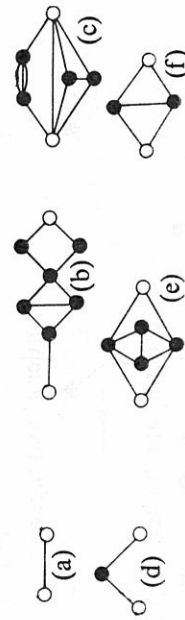


Fig. 11. Various 1-irreducible 2-rooted graphs: (a) the bond graph; (b) a simple nodal graph (two nodal points); (c) a ladder graph; (d) a simple, nodal graph; (e) an elementary graph (contains one internal articulation pair); (f) the lowest order elementary graph.

Again, the essential observation is that the symmetry factor of a nodal graph is just the product of the symmetry factors of its non-nodal components. A matrix notation is useful:

$$\begin{aligned}\langle 1|I|n2\rangle &\equiv I_n(12), \\ \langle 1|M|n2\rangle &\equiv \delta(12)M_{1+n}(1), \\ \langle 1|C|n2\rangle &\equiv C_n(12),\end{aligned}\quad (108)$$

where matrix multiplication involves a sum over valences and lattice points. It is easy formally to write C in terms of I ,

$$C = I(1 - MI)^{-1}, \quad (109)$$

or inverting,

$$I = (1 + CM)^{-1}C, \quad N = C - (1 + CM)^{-1}C, \quad (110)$$

which will be useful below.



$$g = 2 \times 2 \times 2! \times 3! = 48$$

FIG. 12. A ladder graph and its symmetry factor. The symmetry factor is composed multiplicatively of the symmetry factors of the simple or bond insertions ($2 \times 2 \times 1 \times 1 \times 1$) and factorials due to the presence of identical insertions ($2! \times 3!$).

The general 2-rooted graph is either a bond graph, a simple graph (S), or a ladder graph (L),

$$C_{ln} = v_{ln} + S_{ln} + L_{ln}, \quad (111)$$

where

$$v_{ln}(12) = \begin{cases} v(12), & l = n = 1 \\ 0, & \text{otherwise.} \end{cases} \quad (112)$$

Each ladder graph L is composed of two or more simple or bond graphs glued together at the external vertices (see e.g. Fig. 11(b) and (f)). The symmetry factor for the ladder graph is just the product of the symmetry factors for the individual insertions times a product of factorials due to the presence of identical 2-insertions, as illustrated in Fig. 12. Therefore, C can be written entirely in terms of $(v + S)$,

$$C_{ln} = (v + S)_{ln} + \frac{1}{2} \sum_{km} \delta(l - k - s)(v + S)_{km}(v + S)_s \delta(n - m - t) + \dots, \quad (113)$$

or in matrix notation,

$$C = (e^{v+s} - 1), \quad (114)$$

where the δ -functions, which guarantee proper valence at the external vertices, have been left implicit. Equation (114) can, of course, be inverted to give $(v + S)$ in terms of C . Substitution in (111) evaluates,

$$L = C - \ln(1 + C), \quad (115)$$

or, more explicitly expanding the logarithm,

$$\begin{aligned}L_{ln}(12) &= \frac{1}{2} \sum_{k,m} \delta(l - k - s) C_{km}(12) C_s(12) \delta(n - m - t) \\ &\quad - \frac{1}{2} \sum_{s,t} \delta(l - k - s - u) C_{km}(12) C_{st}(12) C_{uv}(12) \\ &\quad \times \delta(n - m - t - v) + \dots\end{aligned} \quad (116)$$

Having succeeded in expressing $L[C]$ and $N[M, C]$, we now make the crucial observations that all non-nodal graphs are bond, ladder, or elementary,

$$I_{ln} = v_{ln} + L_{ln} + E_{ln}, \quad (117)$$

or, alternatively, that all simple graphs are either nodal or elementary,

$$S_{ln} = N_{ln} + E_{ln}. \quad (118)$$

When combined with (104) or (111), respectively, these give

$$C_{ln} = v_{ln} + N_{ln} + L_{ln} + E_{ln}. \quad (119)$$

Now, solve for v and substitute (110) and (115),

$$v = (1 + CM)^{-1}C - C + \ln(1 + C) - E. \quad (120)$$

This constitutes $v[M, C]$ provided we can express $E[M, C]$, which will be done in Section II.J. The lowest order elementary graph is shown as Fig. 11(f).

The approximation $E = 0$ in (120) provides an integral equation for C , which in the context of the classical fluid is called the hypernetted chain equation (Van Leuwen *et al.*, 1959; Meeron, 1960; Stell, 1964). The corresponding equation in the magnetic case was first derived by Englert (1963) and has been studied by Orleans (1963).

J. Renormalisation of the interaction

Bond or interaction renormalisation is the name given to the next (two-point) level of selective resummation. The development is closely parallel to Section II.E. There M^0 was replaced by M as the graphical vertex element with a consequent restriction to 1-irreducible graphs. Here we will replace the potential v by the general two-point correlation factor C introduced in (101).

Consider a general 1-irreducible n -rooted connected graph, $n \geq 3$. Such a graph has a unique 2-irreducible 2-skeleton defined (in Section II.I) by

successive disassembly of all vertex pairs, a process which for each pair of vertices cuts away all 2-insertions and replaces them by a single edge. Conversely, every such graph can be constructed in a unique way by decorating its underlying 2-skeleton, i.e., replacing each edge in turn by an appropriate 1-irreducible 2-rooted graph. This process is illustrated in Figure 13. It produces every 1-irreducible n -rooted graph once and only once.

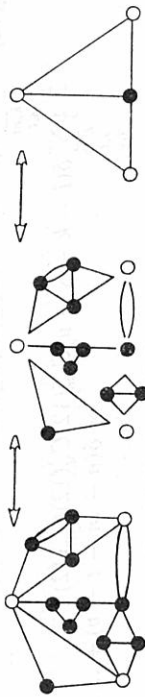


FIG. 13. A 3-rooted graph reduced to (built up from) its 2-skeleton.

Summation of all graphs having the same 2-skeleton is now accomplished simply by reinterpreting skeletal lines as the correlation factor C (instead of v). As in the case of vertex renormalization,[†] the only subtle point is that the symmetry factors work out. In building each 1-irreducible (but not 2-irreducible) graph by decoration of its 2-skeleton, the symmetry factor of each 2-rooted decoration enters multiplicatively. The lowering of skeletal symmetry due to distinct decorations of previously equivalent skeletal edges is taken care of by the cross-terms in the product of C 's for the bond-renormalised graph.

We illustrate by treating the 3-point correlations $\mathcal{M}_3$, eqn (31), explicitly. Note first that the two-point parts $M_{21}(12)$, etc., may, by a trivial extension of the reasoning leading to (103), be written as,

$$M_{21}(12) = \sum_{l,n=1}^{\infty} M_{l+2}(1)C_{ln}(12)M_{n+1}(2). \quad (121)$$

The full expression then is,

$$\begin{aligned} \mathcal{M}_3(123) - \delta(123)M_3(1) - \delta(12)M_{21}(13) - \delta(13)M_{21}(32) - \delta(23)M_{21}(21) \\ = \left\{ \begin{array}{l} \text{sum of all topologically distinct 3-rooted 2-irreducible} \\ \text{connected graphs according to Rule 12.} \end{array} \right\} \quad (122) \end{aligned}$$

Rule 12: Three-point Correlations, Vertex and Bond Renormalised Form

- (a) Assign labels 1, 2, 3 to the external vertices and dummy labels to all internal vertices.
- (b) For each edge joining vertices i and j write a factor $C_{ij}(ij)$.

[†] There is a close parallel here with the discussion surrounding eqn (43), particularly if one chooses (as we shall not) to discuss the decoration in terms of constituent 2-insertions.

To write the proper vertex factors it is convenient to define the *generalised valence* (g -valence) n_i of a vertex i as,

$$n_i = \sum_{\text{incident edges}} l_i,$$

i.e., the g -valence is obtained by summing the valence labels l_i of all correlation factors incident at i .

- (c) For each internal vertex i of g -valence n_i write a factor $M_n(i)$. For each external vertex i of g -valence n_i write a factor $M_{n+1}(i)$.
- (d) Sum each internal vertex label freely over the entire lattice. Sum all valence labels $l_i = 1, 2, \dots$
- (e) Divide by the symmetry factor of the 3-rooted graph.

The first few graphs in the sum (122) are,

$$\begin{aligned} & \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} + \begin{array}{c} 1 \\ | \\ 2 \quad 3 \end{array} + \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} + \begin{array}{c} 1 \\ | \\ 2 \quad 3 \end{array} + \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} + \begin{array}{c} 1 \\ | \\ 2 \quad 3 \end{array} + \dots \\ & + \text{order } C^4 \text{ and higher.} \end{aligned} \quad (123)$$

As an application of Rule 12, the fifth term in (123) is,

$$\begin{aligned} & \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} = \sum_{l_1, \dots, l_6} C_{l_1 l_4}(14) C_{l_2 l_5}(24) C_{l_3 l_6}(34) M_{l_4+l_5+l_6}(4) M_{l_1+1}(1) M_{l_2+1}(2) \\ & \quad M_{l_3+1}(3). \end{aligned} \quad (124)$$

Formulations such as (122) are only useful if correlation factors C are known, so we must return to the problem raised in Section II.1 of finding the relation $v[M, C]$, whose inversion $C[M, v]$ evaluates the correlations. Expression (120) lacks only $E[M, C]$. All elementary 2-rooted graphs may be obtained by decoration of 2-rooted graphs which are elementary and contain no articulation pair except the pair consisting of the two external vertices.[†] Such graphs will be called *elementary 2-rooted 2-skeletons*.

To express E_{ln} , then, one calculates the contributions of all l , n -valent elementary 2-rooted 2-skeletons according to Rule 13.

Rule 13: Elementary Correlation Factors, Vertex and Bond Renormalised Form

- (a) Assign the labels 1 and 2 to the external vertices and dummy labels to all internal vertices.

[†] The two external vertices of a 2-rooted graph are always an articulation pair. Disassembly at the external vertices leaves the bond graph.

- (b) For each edge joining vertices i and j write a factor $C_{ij}(ij)$.
 (c) For each internal vertex i of g -valence n_i (see Rule 12) write a factor $M_n(i)$. For each external vertex write a factor 1.
 (d) Sum each internal vertex label freely over the entire lattice. Sum all valence labels $l_i = 1, 2, \dots$.
 (e) Divide by the symmetry factor of the 2-rooted graph.

The first few terms are:

$$E_{in}(12) = \frac{1}{2} \left[\text{graph 1} + \frac{1}{2} \text{graph 2} + \frac{1}{2} \text{graph 3} + \frac{1}{2} \text{graph 4} + \frac{1}{6} \text{graph 5} + \frac{1}{6} \text{graph 6} \right] + \text{graphs with at least four internal vertices.} \quad (125)$$

There are 56 distinct elementary 2-rooted 2-skeletons with four internal points, not counting as distinct graphs which differ only through interchange of external vertices (such as the third and fourth or seventh and eighth graphs in (125)).

$$\frac{1}{2} \left[\text{graph 1} + \frac{1!2!}{3!} \left[\frac{1}{4} \text{graph 2} + \frac{1}{2} \text{graph 3} \right] \right]$$

Fig. 14. A self-field graph (1-insertion) can be formed by superposing the external vertices of the corresponding 2-rooted simple graphs. See eqn (126).

The self-fields G_n may be expressed in terms of the correlation factors C_{in} . The general relation is,

$$G_{l+n}(1) = \frac{l!n!}{(l+n)!} S_{in}(11). \quad (126)$$

The presence only of simple graphs on the right of (126) guarantees that the result of superposing root points will be a 1-insertion. The factorials compensate for the difference in symmetry factors between the 1-rooted and 2-rooted graphs and the fact that a given 1-insertion may be obtained from several distinct simple graphs, as illustrated in Fig. 14. Equation (126) can be expressed directly in terms of C by using (114) and noting $v(11) = 0$,

$$G_{l+n}(1) = \frac{l!n!}{(l+n)!} \langle l1 | \ln(1+C) | n1 \rangle. \quad (127)$$

The full renormalisation $[M^0, v] \rightarrow [M, C]$ is then expressed by equation (48) and (120),

$$M^0 = e^{-G[C]} M, \quad v = (1 + CM)^{-1} C - C + \ln(1 + C) - E[M, C], \quad (128)$$

which invert the bare perturbation-theoretic relations $M[M^0, v]$ (28) and $C[M^0, v]$. The only graphical part of (128) is $E[M, C]$, eqn (125), which (although it is still an infinite graphical sum) requires *enormously fewer graphs* to any given order in v than analogous bare expansions such as (30). On the other hand, regarded as an integral equation for C in terms of v , eqn (128) is highly non-linear. Even the simplest approximation $E = 0$ (Orlans, 1963) cannot be solved exactly.

Iterative solution in powers of v (Ferrer *et al.*, 1969; Moore *et al.*, 1969) is simple in principle though still time-consuming. To find all terms in M and C up to a given power of v , one needs only a finite number of graphs in E . In zero field, for example, E first enters in order v^7 and the single graph Fig. 11(f) suffices through order v^8 (v^9 with nearest-neighbour interactions only on loose-packed lattices). It is convenient to iterate simultaneously both of eqns (128). Once the M 's and C 's are known from (128) or some approximation thereto, then (103) gives the pair correlations $\mathcal{M}_2(12)$ and (122) (and higher order analogs) give $\mathcal{M}_n, n > 2$.

To complete the story we need an expression for $W[M, C]$, the fully renormalised generalisation of the vertex renormalised (60). This derivation can be done topologically in analogy to De Dominicis' work on the classical fluid (De Dominicis, 1962). The algebraic derivation (Wortis, *et al.*, 1969) simply integrates eqn (7). To carry out this integration efficiently it is necessary to introduce a slight generalisation of our present development. We merely quote the result (Wortis *et al.*, 1969),

$$W = \sum_1 M_0(1) - \sum_1 \sum_{l,n=1}^{\infty} G_l(1) M_l(1) + \frac{1}{2} \sum_{l,n=1}^{\infty} v(12) [\mathcal{M}_2(12) + \mathcal{M}_1(1) \mathcal{M}_1(2)] + \Omega[M, C], \quad (129)$$

where

$$\begin{aligned} \Omega[M, C] = & \frac{1}{2} \sum_{l,n=1}^{\infty} \sum_{l,n=1}^{\infty} M_l(1) \langle l1 | \ln(1+C) - (1+C) \ln(1+C) | 2n \rangle M_n(2) \\ & + \frac{1}{2} \text{Tr}[\ln(1+MC) - MC + \frac{1}{2} MCMC] \\ & + \left\{ \begin{array}{l} \text{sum of all topologically distinct 2-irreducible} \\ \text{unrooted graphs}^\dagger \text{ according to Rule 14.} \end{array} \right\} \end{aligned} \quad (130)$$

[†]The graphs , , and are specifically excluded.

Rule 14: The Functional $\Omega[M, C]$, Vertex and Bond Renormalised Form

- Assign dummy labels to all internal vertices.
- For each edge joining vertices i and j write a factor $C_{ii,j}(ij)$.
- For each internal vertex of g -valence n_i (see Rule 12) write a factor $M_{n_i}(i)$.
- Sum each internal vertex label freely over the entire lattice. Sum all valence labels $l_i = 1, 2, \dots$
- Divide by the symmetry factor of the unrooted graph.

The trace in (130) refers to the matrix representation (108), e.g.

$$\frac{1}{2} \text{Tr} [\ln (1 + MC)] = \frac{1}{2} \sum_i \sum_{n=1}^{\infty} \langle n! |\ln (1 + MC)| n! \rangle. \quad (131)$$

The first four terms in the graphical sum are,

$$\frac{1}{4!} \text{ (triangle)} + \frac{1}{3!} \text{ (square)} + \frac{1}{12} \text{ (pentagon)} + \frac{1}{5!} \text{ (hexagon)} + \text{graphs with six or more vertices.} \quad (132)$$

The evaluation (130) is, of course, not necessary in series calculations, since, once $\mathcal{M}_1$ and $\mathcal{M}_2$ are known explicitly to a given order in v , eqn (7) can be integrated term-by-term.

K. Further renormalisation: comments

Vertex (1-point) and interaction (2-point) renormalisations have now been carried out explicitly. De Dominicis (1962) has given a very brief discussion of 3-point renormalisation for the classical fluid. We shall not develop higher order renormalisations, but the lines which such a development would follow should now be apparent: l -vertex disassembly defines l -insertions and an l -skeleton. All order n correlations $n > l$ can be summed by decorating l -skeletons with combinations of l -insertions. The only point requiring finesse is the derivation of the higher order analogs of eqns (128), expressing the l -order potentials (identically zero for the Ising model) in terms of the l -correlation factors.

III. Further Applications and Extensions of the Linked-Cluster Expansion

A. Orientation

The linked-cluster expansion was developed in detail in Section II for the specific example of the spin s Ising model. This is the simplest and most studied application. The bare graphical expansions of Section II.B were

renormalised with respect to vertices in Sections II.E and II.F and then with respect to edges in Sections II.I and II.J. The renormalisation process depended on two quite general properties, (i) graph topology and (ii) the freedom of the lattice sums in part (d) of all graphical rules.

Linked-cluster expansions closely analogous to the Ising expansion exist for a wide variety of other systems, both quantum and classical. The criterion is simply that the system be *spatially local in the absence of interactions*. Once a bare (unrenormalised) expansion is known to exist which is graphical and contains free lattice sums, its renormalisation is immediate. The presence of extra variables, non-commutativity of operators, etc., present only clerical problems and have no influence on the arguments leading to renormalisation.

In this section we shall briefly sketch some of these extensions and generalisations. Our purpose will be to reduce the analysis to a stage where the Ising development is clearly applicable. Generalisation of results from Section II can then be written down by inspection. The subsections which follow are intended to be independent of one another, although they will draw extensively on the material of Section II.

B. Inclusion of additional local fields

Inclusion in the Hamiltonian (1) or (2) of additional spatially local fields is a useful if trivial generalisation. Consider, for example, the dimensionless Hamiltonian,

$$-\beta\mathcal{H} = \frac{1}{2} \sum_{i,j} v(12) \mu(1) \mu(2) + \sum_i h(1) \mu(1) + \sum_i \Delta(1) X(1), \quad (133)$$

where $X(1)$ is a local operator belonging to the dynamical subsystem at site 1 and Δ is an external field which couples locally to X . For example, a single-ion magnetic anisotropy would be written with $X(1) = \mu^2(1)$ and $\Delta(1)$ the local anisotropy field.[†]

The free energy W is now a functional of Δ in addition to h and v , $W[h, \Delta, v]$. The Ising derivative relations (4)–(8) continue to hold; however, in addition,

$$\frac{\delta W}{\delta \Delta(1)} = \langle X(1) \rangle, \quad \frac{\delta^2 W}{\delta \Delta(1) \delta \Delta(2)} = \langle X(1) X(2) \rangle - \langle X(1) \rangle \langle X(2) \rangle, \text{ etc.} \quad (134)$$

[†] This model, which has been studied for $s = 1$ by Griffiths (1970), Blume *et al.* (1971), and Oitmaa (1970), is interesting in that (at least in the effective-field approximation) it exhibits either a second order transition, a first order transition, or no transition at all depending on the value of Δ .

The initial condition (9) and (10) is modified by Δ ,

$$W_0[h, \Delta] = \sum_1 M_0^0(1) = \sum_1 \left[\ln \sum_{\mu=-1}^1 \exp[h(1)\mu + \Delta(1)X] \right]. \quad (135)$$

For example, in the Griffiths-Blume model referred to above,

$$M_0^0(1) = M_0^0[1; h, \Delta] = \ln(1 + 2e^{\Delta} \cosh h). \quad (136)$$

The bare semi-invariants continue to be defined by (14),

$$M_n^0(h, \Delta) = \frac{\partial^n}{\partial h^n} M_0^0(h, \Delta), \quad (137)$$

but now depend parametrically on Δ at each site.

In the development of unrenormalised perturbation theory, the semi-invariants appeared as derivatives of W_0 , i.e. directly in the form (137). Thus, Rules 1 through 4 (eqns (20), (28), (30), and (31)) are unchanged except that they are now to be carried out in terms of the bare semi-invariants (135), (137), instead of (10), (14). The subsequent renormalisation, which simply replaces the bare semi-invariants M_n^0 by their renormalised analogs, goes through unchanged.

Once having incorporated Δ as a parameter into the semi-invariants, it is extremely easy to generate correlations involving the operators X . The procedure mimics the discussion preceding eqns (27)–(31). For example, to find $\langle X \rangle$ using (134), we note that Δ appears in the unrenormalised expansion for W only through the semi-invariants, so, schematically,

$$\frac{\delta W}{\delta \Delta(1)} = \sum_{n=0}^{\infty} \frac{\delta W[M_n^0, v]}{\delta M_n^0(1)} \frac{\delta M_n^0(1)}{\delta \Delta(1)}. \quad (138)$$

In graphical terms, $\delta W / \delta M_n^0$ roots one l -valent vertex of a previously unrooted graph and $\delta M_n^0(1) / \delta \Delta(1)$ replaces the semi-invariant at that vertex by a new type of mixed semi-invariant generated from $M_0^0(h, \Delta)$ by both h and Δ derivatives. Denote

$$M_{l,n}^0(h, \Delta) \equiv \frac{\partial^l \partial^n}{\partial h^l \partial \Delta^n} M_0^0(h, \Delta). \quad (139)$$

Then (cf. (28)),

$$\langle X(1) \rangle = \left\{ \begin{array}{l} \text{sum of all topologically distinct} \\ \text{1-rooted connected graphs according} \\ \text{to Rule 2'.} \end{array} \right\} \quad (140)$$

Rule 2':

Same as Rule 2 for $n = 1$ except:

(c) For each l -valent internal vertex i write a factor $M_l^0(i)$. For each l -valent external vertex write a factor $M_{l,1}^0(1)$.

Since the operator X is not coupled to the interaction, it appears only at external vertices, and the generalised semi-invariants (139) $n \neq 0$ do not enter self-fields. Thus, vertex renormalisation of the generalised semi-invariants follows (42),

$$M_{l,n}(1) = \left[\exp \sum_{k=1}^{\infty} G_k[1; M, v] \frac{\delta^k}{\delta h^k} \right] M_{l,n}^0[1; \tilde{h}, \Delta]_{\tilde{h}=h(1)}. \quad (141)$$

The analog of (45) is

$$\begin{aligned} \langle X(1)X(2) \rangle - \langle X(1) \rangle \langle X(2) \rangle &= \left\{ \begin{array}{l} \text{sum of all topologically distinct} \\ \text{2-rooted 1-irreducible connected} \\ \text{graphs according to Rule 5'.} \end{array} \right\} \\ &= \delta(12)M_{0,2}(1) + \end{aligned} \quad (142)$$

Rule 5':

Same as Rule 5 except:

(c) For each l -valent internal vertex i write a factor $M_l(i)$. For each l -valent external vertex i write a factor $M_{l,1}(i)$.

Correspondingly, the analogue of (103) is,

$$\langle X(1)X(2) \rangle - \langle X(1) \rangle \langle X(2) \rangle = \delta(12)M_{0,2}(1) + \sum_{l,n=1}^{\infty} M_{l,1}(1)C_{ln}(12)M_{n,1}(2), \quad (143)$$

so that, if the correlation factors C have been found in the calculation of $\mathcal{M}_2(12)$, the extra work required to complete the X - X correlations is minimal.

C. Several commuting variables: the classical Heisenberg model

The first application of the linked-cluster method to the derivation of the high-temperature expansions useful in the study of critical phenomena was in connection with the anisotropic classical Heisenberg model. Joyce (1967) developed the unrenormalised form of the expansion by generalising the work of Horwitz and Callen (1961) and generated the first few terms in perturbation theory. Jasnow and Wortis (1968) and Jasnow (1969) applied the vertex renormalisation to go to higher orders. The Hamiltonian is,

$$\mathcal{H} = - \sum_{\langle ij \rangle} J^{\alpha}(ij) t_{\alpha i} t_{\alpha j} - m \sum_i \mathbf{H}(i) \cdot \mathbf{t}_i, \quad (144)$$

$\alpha = x, y, z$

where the sum is over distinct pairs $i \neq j$, $J(ij)$ is the exchange coupling between sites i and j , m is the magnetic moment, and $\mathbf{H}(i)$ is the magnetic field at site i . $\mathbf{t}_i$ is a classical unit vector associated with site i . Rewriting (144) in dimensionless units,

$$-\beta\mathcal{H} = \frac{1}{2} \sum_{i,j} v^z(12)t_z(1)t_z(2) + \sum_{\alpha=x,y,z} h^\alpha(1)t_\alpha(1), \quad (145)$$

where $v = \beta J$, $h = \beta mH$, and numerical arguments stand for lattice sites. We define as usual,

$$W[\mathbf{h}, v] = \ln \text{Tr} \exp(-\beta\mathcal{H}), \quad (146)$$

where the trace indicates integration of each unit vector $\mathbf{t}$ over its unit sphere. Cumulant correlations are generated via the second term on the right of (145),

$$\frac{\delta W}{\delta h^\alpha(1)} = \langle t_\alpha(1) \rangle = \mathcal{M}_\alpha(1),$$

$$\frac{\delta^2 W}{\delta h^\alpha(1)\delta h^\beta(2)} = \langle t_\alpha(1)t_\beta(2) \rangle - \langle t_\alpha(1) \rangle \langle t_\beta(2) \rangle = \mathcal{M}_{\alpha\beta}(12), \quad (147)$$

and so forth, where the Greek subscripts stand for Cartesian coordinates and are paired with the space indices.

The differential equation which (in analogy to (8)) generates the linked-cluster expansion is

$$\frac{\delta W}{\delta v^z(12)} = \frac{\delta^2 W}{\delta h^z(1)\delta h^z(2)} + \frac{\delta W}{\delta h^z(1)} \frac{\delta W}{\delta h^z(2)}. \quad (148)$$

The non-interacting free energy W_0 is

$$W_0[\mathbf{h}] = \sum_i M_0^0(1),$$

$$M_0^0(1) = M_0^0[1; h(1)] = \ln \int d\Omega \exp[h(1) \cdot \mathbf{t}] = \ln \left[\frac{4\pi \sinh|h(1)|}{|h(1)|} \right]. \quad (149)$$

The bare semi-invariants, which enter the Taylor series coefficients as the cumulant correlations associated with an isolated site, now carry an extra set of Cartesian subscripts,

$$M_n^0(1, \alpha_1 \dots \alpha_n) \equiv \frac{\delta^n M_0^0(1)}{\delta h^{\alpha_1}(1) \dots \delta h^{\alpha_n}(1)}, \quad (150)$$

which is totally symmetrical in the α 's.

In constructing the Taylor development of the free energy W along the lines of (11), there is an extra Cartesian index α which goes with each poten-

tial factor $v^z(12)$ and must be summed[†] $\alpha = x, y, z$. The two $(\delta/\delta h^\alpha)$'s which go with v^z by (148) tie each edge v^z to bare semi-invariants containing α at its two ends. The *topology of the graphs* on the other hand depends only on spatial structure and goes through exactly as for the Ising model. The rule for the unrenormalised free energy, which could also be derived along the lines of Section II.C, is (cf. Rule 1, eqn (20)),

$$W[\mathbf{h}, v] = \left\{ \begin{array}{l} \text{sum of all topologically distinct unrooted} \\ \text{connected graphs according to Rule 15.} \end{array} \right\}$$

Rule 15: Classical Heisenberg Model Free Energy W , Unrenormalised Form

- Assign a dummy label to each internal vertex and a Cartesian index to each edge.
- For each edge indexed γ and joining vertices i and j write a factor $v^\gamma(ij)$.
- For each l -valent vertex i write a factor $M_l^0(i, \gamma_1 \dots \gamma_l)$, where $\gamma_1 \dots \gamma_l$ are the Cartesian indices of the edges incident at i .
- Sum each internal vertex label freely over the entire lattice. Sum the Cartesian index of each edge $\gamma = x, y, z$.
- Divide by the symmetry factor of the unrooted graph.

Since topology is unchanged from the Ising case, the first few contributing graphs are given by (20). For example,

$$\frac{1}{2} \text{graph} = \frac{1}{2} \sum_{1,2,3} v^{\gamma_1}(12) v^{\gamma_2}(23) M_1^0(1, \gamma_1) M_2^0(2, \gamma_1 \gamma_2) M_1^0(3, \gamma_2). \quad (151)$$

Linked-cluster rules for the cumulant correlations are derived via (147). $W[M^0, v]$ depends on the fields $\mathbf{h}$ only through the semi-invariants M^0 . Differentiation fixes a vertex, increases its order by one, and attaches a Cartesian label. The reduction of the unrooted symmetry factor to that appropriate to the rooted graph depends on topology alone and does not differ from the Ising case. As an example, we give the expression for the pair correlations (cf. Rule 3, eqn (30)), which are the easiest route to the susceptibility matrix:

$$\mathcal{M}_{\alpha\beta}(12) - \delta(12)M_2(1, \alpha\beta) = \left\{ \begin{array}{l} \text{sum of all topologically distinct} \\ \text{2-rooted connected graphs according} \\ \text{to Rule 16.} \end{array} \right\} \quad (152)$$

[†] For an interacting system of classical plane rotors (Bowers and Joyce, 1967; Stanley, 1968), the unit vector $\mathbf{t}$ is restricted to a plane. The trace operation is an angular integral from 0 to 2π , thus changing the evaluation (149)–(150) of the semi-invariants. The sum over α is only $\alpha = x, y$. Otherwise, everything is exactly as here sketched.

Rule 16: Classical Heisenberg Model Pair Correlations, Unrenormalised Form

- Assign a dummy label to each internal vertex and a Cartesian index γ to each edge. Assign the pair $(\alpha, 1)$ to one of the two external vertices and $(\beta, 2)$ to the other.
- For each edge indexed γ and joining vertices i and j write a factor $v^\gamma(ij)$.
- For each l -valent internal vertex i write a factor $M_l^0(i, \gamma_1 \dots \gamma_l)$. For each l -valent external vertex $i = 1, 2$ write a factor $M_{l+1}^0(i, \gamma, \gamma_1 \dots \gamma_l)$ with $\gamma = \alpha(\beta)$ for vertex 1(2). The indices $\gamma_1 \dots \gamma_l$ are the Cartesian labels of incident edges.
- Sum each internal vertex label freely over the entire lattice. Sum the Cartesian index of each edge $\gamma = x, y, z$.
- Divide by the symmetry factor of the 2-rooted graph.

Contributing graphs are given by (30).

Renormalisations go through exactly as for the Ising model. The extra complexity introduced by the Cartesian sums is entirely clerical. The renormalisation of the semi-invariants will serve to illustrate this point. The self-field $G_n(1)$ at the point 1 must now carry the n Cartesian indices of the edges incident at the external vertex. In the process of decoration, by which the renormalised semi-invariant is formed, each of these edges must be matched to the bare semi-invariant. Thus, the self-field (cf. Rule 6, eqn (47)),

$$G_n(1, \alpha_1 \dots \alpha_n) = \left\{ \begin{array}{l} \text{sum of all topologically distinct 1-} \\ \text{irreducible 1-insertions which are} \\ \textit{n-valent at the external vertex} \\ \text{according to Rule 17.} \end{array} \right\} \quad (153)$$

Rule 17: Classical Heisenberg Model Self-Field, Vertex Renormalised Form

- Assign the Cartesian indices $\alpha_1, \dots, \alpha_n$ to the n edges incident at the external vertex 1. Assign a Cartesian index γ to each additional edge. Assign a dummy label to each internal vertex.
- For each edge indexed γ and joining vertices i and j write a factor $v^\gamma(ij)$.
- For each l -valent internal vertex i write a factor $M_l(i, \gamma_1 \dots \gamma_l)$ (the renormalised semi-invariant to be defined below). For the external vertex write a factor of 1.
- Sum each internal vertex label freely over the entire lattice. Sum the Cartesian indices of all edges not incident at the external vertex $\gamma = x, y, z$.
- Divide by the symmetry factor of the 1-rooted graph.

Contributing graphs and symmetry factors are as given by eqn (47). One representative term in the evaluation of G_3 is shown in Fig. 15. Unlike the semi-invariants, $G_n(1, \alpha_1 \dots \alpha_n)$ is not in general symmetric under permutations of its Cartesian indices,[†] though it will, of course, reflect any symmetries of the Hamiltonian (144), (145). The renormalised semi-invariant is (cf. (42)),

$$M_n(1, \alpha_1 \dots \alpha_n) = \left[\exp \sum_{l=1}^{\infty} \sum_{\gamma_1 \dots \gamma_l} G_l(1, \gamma_1 \dots \gamma_l) \frac{\delta^l}{\delta \tilde{h}^{\gamma_1} \dots \delta \tilde{h}^{\gamma_l}} \right] M_n^0[1, \alpha_1 \dots \alpha_n; \tilde{h}]|_{\tilde{h}=h(1)}. \quad (154)$$

The vertex renormalised form of (152) sums 1-irreducible 2-rooted graphs by a rule identical to Rule 16 except for the replacement $M^0 \rightarrow M$.

$$\frac{1}{2} \sum_{\gamma=x,y,z} v^{\alpha_1}(12) v^{\alpha_2}(13) v^{\alpha_3}(23) \times M_2(2, \alpha_1 \gamma) M_3(3, \alpha_2 \alpha_3 \gamma)$$

FIG. 15. A contribution to G_3 .

Like G , the correlation factors C must now carry Cartesian indices labeling lines incident at external vertices. Thus,

$$\begin{aligned} \mathcal{M}_{\alpha\beta}(12) - \delta(12)M_2(1, \alpha\beta) \\ = \sum_{l,n=1}^{\infty} \sum_{\alpha_1 \dots \alpha_l} M_{l+1}(1, \alpha\alpha_1 \dots \alpha_l) C_{ln}(1, \alpha_1 \dots \alpha_l; 2, \beta_1 \dots \beta_n) M_{n+1}(2, \beta\beta_1 \dots \beta_n). \end{aligned} \quad (155)$$

D. Non-commuting variables: the quantum Heisenberg model

In treating the quantum many-particle problem (Martin and Schwinger, 1959; Kadanoff and Baym, 1962; Abrikosov *et al.*, 1965), it is customary to focus attention not on the operator $\exp(-\beta\mathcal{H})$, whose trace gives the thermodynamics, but rather on the time-ordered unitary operator,

$$U(t_1, t_2) = \left[\exp \left(-i \int_{t_2}^{t_1} dt \mathcal{H}(t) \right) \right]_+, \quad (156)$$

which generates the time translations of the state vector of the system from time t_2 to time t_1 . Time-dependence is therefore introduced *ab initio* into the

[†] In the renormalisation, G is contracted with the symmetrical semi-invariant, thus picking out its symmetrical part, so it is possible to define G to be symmetrised without changing what follows.

formalism, making all calculation significantly more involved than is the case classically. This time-dependence also seems inescapable when dealing with *magnetic* systems, and it appreciably enhances the difficulty of practical algebraic calculation. However, it is important to emphasise at the outset that, once all definitions have been properly generalised, the topological properties of the theory are identical to those of the Ising model, discussed in Section II.

The Hamiltonian of the spin s anisotropic Heisenberg Model is

$$\mathcal{H} = -\frac{1}{s^2} \sum_{\langle ij \rangle} \sum_{\alpha=x,y,z} J^\alpha(ij) s_{i\alpha} s_{j\alpha} - \frac{m}{s} \sum_i \mathbf{H}(i) \cdot \mathbf{s}_i. \quad (157)$$

It is convenient to introduce the normalised operators $\mathbf{t}(1) = [\mathbf{s}(\mathbf{r}_1, t_1)]/s$, where in dealing with quantum systems the numerical label 1 refers to both the lattice site $\mathbf{r}_1$ and the time t_1 . In units such that $\hbar = 1$,

$$\mathbf{t}(1) \cdot \mathbf{t}(1) = \left(1 + \frac{1}{s}\right), \quad \mathbf{t}(1) \times \mathbf{t}(2)|_{t_1=t_2} = i\delta(\mathbf{r}_1, \mathbf{r}_2) \frac{\mathbf{t}(1)}{s}. \quad (158)$$

In terms of the variables $J \equiv v$, $m\mathbf{H} \equiv \mathbf{h}$, then,

$$-\mathcal{H}(t) = \frac{1}{2} \sum_{\mathbf{r}_1, \mathbf{r}_2} \sum_{\alpha} v^\alpha(\mathbf{r}_1, \mathbf{r}_2) t_\alpha(\mathbf{r}_1, t) t_\alpha(\mathbf{r}_2, t) + \sum_{\mathbf{r}, \alpha} h^\alpha(\mathbf{r}, t) t_\alpha(\mathbf{r}, t). \quad (159)$$

It is important in the development which follows formally to maintain $\mathbf{h}$ and v as functions of time *independently variable* for each point and bond.

Now, in the physical limit, in which $\mathcal{H}$ is time-independent, $U(t_1, t_2) = \exp(-i\mathcal{H}t)$ and $U(-i\beta, 0) = \exp(-\beta\mathcal{H})$. This observation motivates the restriction of all time variables to the pure imaginary interval $[0, -i\beta]$, where for purposes of time-ordering "larger" times are those lying nearer $-i\beta$. Note that for a finite system with a bounded Hamiltonian (like (159)) there is certainly no difficulty in analytically continuing the time variables in (156). We shall not attempt to discuss the more formidable problem of the $N \rightarrow \infty$ limit. The proper definition of the generalised free energy W in the presence of time-variation is

$$W[\mathbf{h}, v] = \ln \text{Tr } U(-i\beta, 0). \quad (160)$$

Functional differentiation† generates time-ordered correlation functions in

† See footnote on p. 145. Functional differentiation with respect to a continuous function $f(t)$ may be defined by approximating f by a sequence of functions which are piecewise constant, $f = f_i$ for t in the interval Δt_i . Then t in Δt_i ,

$$\frac{\delta}{\delta f(t)} \equiv \lim_{\Delta t_i \rightarrow 0} \frac{1}{\Delta t_i} \frac{\delta}{\delta f_i}.$$

which all times are restricted to $[0, -i\beta]$,

$$(-i) \frac{\delta W[\mathbf{h}, v]}{\delta h^\alpha(1)} = \langle t_\alpha(1) \rangle = \mathcal{M}_\alpha(1),$$

$$(-i)^2 \frac{\delta^2 W[\mathbf{h}, v]}{\delta h^\alpha(1) \delta h^\beta(2)} = \langle (t_\alpha(1) t_\beta(2))_+ \rangle - \langle t_\alpha(1) \rangle \langle t_\beta(2) \rangle = \mathcal{M}_{\alpha\beta}(12), \quad (161)$$

which should be compared with (147). In (161) all operators have Heisenberg time-dependence $X(t) = U^{-1}(t, 0) X(0) U(t, 0)$ and the expectation value is defined with respect to $U(-i\beta, 0)$,

$$\langle X \rangle = \frac{\text{Tr } U(-i\beta, 0) X}{\text{Tr } U(-i\beta, 0)}. \quad (162)$$

The notation $()_+$ indicates an ordering of all operators right to left by increasing times. The physical correlation functions (for real times) may be obtained from (161) and higher order analogs by analytic continuation in the time variables. Of course, W requires no analytic continuation and, when $\mathbf{h}$ and v are time-independent, $\mathcal{M}_\alpha(1)$ is, too. The procedure for continuation is discussed at length in the many-particle literature and is not of immediate relevance to our present concerns, which are mainly topological. The analog of the functional differential equation (8) is

$$(-i) \frac{\delta W}{\delta v^\alpha(12)} = \langle t_\alpha(1) t_\alpha(2) \rangle|_{t_1=t_2+}$$

$$= (-i)^2 \left[\frac{\delta^2 W}{\delta h^\alpha(1) \delta h^\beta(2)} + \frac{\delta W}{\delta h^\alpha(1)} \frac{\delta W}{\delta h^\beta(2)} \right] |_{t_1=t_2+}. \quad (163)$$

In the absence of interaction (cf. (9) and (10)),

$$W[\mathbf{h}, v = 0] = W_0[\mathbf{h}] = \sum_i M_0^0(i),$$

where,

$$M_0^0(1) = \ln \text{Tr} \left(\exp i \int_0^{-i\beta} dt \sum_\alpha h^\alpha(1) t_\alpha(1) \right) +. \quad (164)$$

The unrenormalised semi-invariants are now time-dependent,

$$M_n^0(\mathbf{r}, t_1 \dots t_n) = \frac{(-i)^n \delta^n M_0^0(\mathbf{r})}{\delta h^{\alpha_1}(\mathbf{r}, t_1) \dots \delta h^{\alpha_n}(\mathbf{r}, t_n)}. \quad (165)$$

Unlike (10) or (149), (164) cannot in general (i.e. for time-dependent $\mathbf{h}$) be evaluated in closed form, so (165) cannot be used in a practical way to

evaluate the semi-invariants. This need not cause difficulty in practice, since $\mathbf{h}$ is normally time-independent in the physical limit, and each M_n^0 can then be evaluated directly. For example, when $\mathbf{h}(\mathbf{r}) = 0$,

$$M_1^0(\mathbf{r}, t) = \langle t_a(\mathbf{r}, t) \rangle = 0,$$

$$M_2^0(\mathbf{r}, t_1 t_2) = \langle (t_{a_1}(\mathbf{r}, t_1) t_{a_2}(\mathbf{r}, t_2))_+ \rangle - \langle t_{a_1}(\mathbf{r}, t_1) \rangle \langle t_{a_2}(\mathbf{r}, t_2) \rangle$$

$$= \frac{\delta(\alpha_1 \alpha_2)}{3} \left(1 + \frac{1}{s} \right),$$

$$M_3^0(\mathbf{r}, t_1 t_2 t_3) = \begin{cases} 0, & \text{unless } (\alpha_1 \alpha_2 \alpha_3) \text{ are some permutation of } (xyz), \\ \frac{(+i)}{6s} \left(1 + \frac{1}{s} \right), & \text{for time orderings giving } (xyz) \text{ and} \\ \text{cyclic permutations,} \\ \frac{(-i)}{6s} \left(1 + \frac{1}{s} \right), & \text{for time orderings giving } (yxz) \text{ and} \\ \text{cyclic permutations.} \end{cases} \quad (166)$$

When $\mathbf{h} = 0$, M_n^0 is constant for any given time ordering and the evaluation reduces to finding traces of spin-operator products. The presence of a time-independent non-zero field makes things only marginally more complicated. Relevant traces have been evaluated to high order by Dalton and Rimmer (1968).

It is sometimes convenient to have equations of motion for the M_n^0 's. When $v = 0$,

$$i \frac{\partial}{\partial t} t_a = [t_a, \mathcal{H}] = (-i) \sum_{\beta} \varepsilon^{\alpha\beta\gamma} \frac{h^{\beta}}{s} t_{\gamma}, \quad (167)$$

where $\varepsilon^{\alpha\beta\gamma}$ is the completely antisymmetric matrix,

$$\varepsilon^{\alpha\beta\gamma} = \begin{cases} 1, & \text{when } \alpha\beta\gamma \text{ is a cyclic permutation of } xyz, \\ -1, & \text{when } \alpha\beta\gamma \text{ is a cyclic permutation of } yxz, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for example,

$$\sum_{\gamma} (\mathcal{G}_0^{-1}(t_1))^{\alpha\gamma} M_2^0(\mathbf{r}, t_1 t_2) \begin{matrix} \gamma\beta \\ \mathbf{r}, t_1 t_2 \end{matrix} = - \frac{\delta(t_1 t_2)}{s} \sum_{\gamma} \varepsilon^{\alpha\beta\gamma} M_1^0(\mathbf{r}, t_1),$$

where

$$(\mathcal{G}_0^{-1}(t))^{\alpha\gamma} \equiv \delta^{\alpha\gamma} i \frac{\partial}{\partial t} + i \sum_{\beta} \varepsilon^{\alpha\beta\gamma} \frac{h^{\beta}}{s}. \quad (168)$$

The general form is, schematically,

$$\mathcal{G}_0^{-1} M_n^0 = \frac{1}{s} \sum M_{n-1}^0 \delta, \quad (169)$$

which can help to evaluate those parts of the semi-invariants that are not time-independent. In the classical limit $s \rightarrow \infty$, the semi-invariants become entirely time-independent and go over into their classical forms (149), (150).

Unrenormalised perturbation theory for the Heisenberg model has been developed by Stinchcombe *et al.* (1963) and by Vaks *et al.* (1968a, b). To develop W in powers of v , there are as usual two routes. Either we directly expand the time-ordered exponential U (Englert, 1963) (cf. Section II.C) or we use (163) to generate the coefficients in the Taylor expansion (cf. (11) and (12)). It is clear from either point of view that each potential line v carries a factor i and requires a time integral over the interval $[0, -i\beta]$. Otherwise, the topology of the development parallels exactly Sections II.B, II.C, and III.C. The result is

$$W = \left\{ \begin{array}{l} \text{sum of all topologically distinct unrooted} \\ \text{connected graphs according to Rule 18.} \end{array} \right\} \quad (170)$$

Rule 18: Heisenberg Model Free Energy W , Unrenormalised Form

- Assign to each vertex a dummy spatial label $\bar{\mathbf{r}}$. Assign to each edge a Cartesian index γ and a time variable $\bar{t}$.
- For each edge labeled $\gamma, \bar{t}$ and joining vertices $\bar{\mathbf{r}}_i$ and $\bar{\mathbf{r}}_j$ write a factor $iv'(\bar{\mathbf{r}}_i, \bar{\mathbf{r}}_j, \bar{t})$.
- For each l -valent vertex $\mathbf{r}$ with incident edges $\gamma_1 \bar{t}_1, \dots, \gamma_l \bar{t}_l$ write a factor

$$M_l^0 \left(\begin{matrix} \bar{\mathbf{r}}, & \gamma_1 \dots \gamma_l \\ \bar{t}_1 \dots \bar{t}_l \end{matrix} \right).$$

- Sum each internal vertex label freely over the entire lattice. Integrate each time variable $\int_0^{-i\beta} dt$. Sum each Cartesian index $\gamma = x, y, z$.
- Divide by the symmetry factor of the unrooted graph.

The first few contributing graphs are given by (20). Note that in the classical limit each time integral provides a factor $(-i\beta)$ which combines with (iv) to form βv which was called v in III.C. It is also possible (and convenient when v is isotropic) to use a spherical representation, t_+, t_-, t_{-} .

The correlation functions (161) are derived by differentiation. For example,

$$\mathcal{M}_{\alpha\beta}(12) - \delta(\mathbf{r}_1\mathbf{r}_2)M_2(\mathbf{r}_1, \mathbf{r}_2) = \left\{ \begin{array}{l} \text{sum of all topologically distinct} \\ \text{2-rooted connected graphs} \\ \text{according to Rule 19.} \end{array} \right\} \quad (171)$$

Rule 19: Heisenberg Model Pair Correlations, Unrenormalised Form

- (a) Assign to each internal vertex a dummy spatial label $\bar{\mathbf{r}}$. Assign to each edge a Cartesian index γ and a time variable $\bar{t}$. Assign to the external vertices 1 and 2 the labels $(r_1 t_1 \alpha)$ and $(r_2 t_2 \beta)$.
- (b) For each edge labeled $\gamma, \bar{t}$ and joining vertices $\bar{\mathbf{r}}_i$ and $\bar{\mathbf{r}}_j$ write a factor $iv^\gamma(\mathbf{r}_i \bar{\mathbf{r}}_j \bar{t})$.
- (c) For each l -valent internal vertex $\bar{\mathbf{r}}$ with incident edges $\gamma_1 \bar{t}_1, \dots, \gamma_l \bar{t}_l$ write a factor

$$M_l^0\left(\begin{array}{c} \gamma_1 \quad \gamma_l \\ \bar{\mathbf{r}}, \bar{t}_1 \dots \bar{t}_l \end{array}\right).$$

For each l -valent external vertex $i = 1, 2$ write a factor

$$M_{l+1}^0\left(\begin{array}{c} \gamma\gamma_1 \quad \gamma_l \\ \mathbf{r}_i, \bar{t}_1 \dots \bar{t}_l \end{array}\right),$$

where $\gamma = \alpha(\beta)$ if $i = 1(2)$.

- (d) Sum each internal vertex label freely over the entire lattice. Integrate each internal time $[0, -i\beta]$. Sum the Cartesian index of each edge $\gamma = x, y, z$.
- (e) Divide by the symmetry factor of the 2-rooted graph.

Graphs are given by (30). For example,

$$\frac{1}{2} \circlearrowleft_{\alpha,1} \circlearrowright_{\beta,2} = \frac{(i)^2}{2} \sum_{\bar{\mathbf{r}}, \bar{t}} \int_0^{-i\beta} d\bar{t}_1 d\bar{t}_2 v^{\gamma_1}(\mathbf{r}_1 \bar{\mathbf{r}}_1) v^{\gamma_2}(\bar{\mathbf{r}}_2 \bar{\mathbf{r}}_2) M_2^0\left(\begin{array}{c} \alpha \quad \gamma_1 \\ \mathbf{r}_1, \bar{t}_1 \bar{t}_2 \end{array}\right) \times M_2^0\left(\begin{array}{c} \gamma_1 \gamma_2 \\ \mathbf{r}_2, \bar{t}_1 \bar{t}_2 \end{array}\right) M_2^0\left(\begin{array}{c} \gamma_2 \beta \\ \mathbf{r}_2, \bar{t}_2 \bar{t}_2 \end{array}\right). \quad (172)$$

The first few terms of the Heisenberg model correlations have been evaluated by another method by Tahir-Kheli and McFadden (1969) and McFadden and Tahir-Kheli (1970a, b).

Renormalisation is straightforward. In calculating insertions such as the self-fields G the external vertex takes a factor 1 and the γ, t labels of all

edges incident at it remain unsummed. Renormalisation proceeds as previously (cf. (42), (154)),

$$M_n\left(\begin{array}{c} \alpha\beta \\ \mathbf{r}, \bar{t} \end{array}\right) = \left\{ \exp \sum_{i=1}^{\infty} (-i)^i \int_0^{-i\beta} d\bar{t}_1 \dots d\bar{t}_i \sum_{\gamma_1 \dots \gamma_i} G_i\left(\begin{array}{c} \gamma_1 \dots \gamma_i \\ \mathbf{r}, \bar{t}_1 \dots \bar{t}_i \end{array}\right) \times \frac{\delta^i}{\delta h^{\gamma_1}(\mathbf{r}, \bar{t}_1) \dots \delta h^{\gamma_i}(\mathbf{r}, \bar{t}_i)} \right\} M_n^0\left(\begin{array}{c} \alpha\beta \\ \mathbf{r}, \bar{t} \end{array}\right). \quad (173)$$

The analog of the many-particle Dyson equation is made most evident by applying $\mathcal{G}_0^{-1}$ to (173). The result in rather schematic form is (cf. (169)),

$$\mathcal{G}_0^{-1} M_n = \frac{1}{S} \sum M_{n-1} \delta + \frac{1}{S} \sum G_l M_{n+l-1}. \quad (174)$$

Renormalisation has not been carried out systematically in the literature beyond the molecular field approximation, $\Phi = \frac{1}{2} \bullet$. Better approximations based on (174) seem to give high-temperature results similar to Résibois and De Leener (1966, 1969) and De Leener and Résibois (1966, 1969).

E. Fermion variables

The only complications produced by the introduction of fermion variables are sign factors. In the fourth of his papers on what is now known as the Hubbard model, Hubbard (1965) has developed the unrenormalised perturbation theory for the fermion Hamiltonian,

$$\mathcal{H} = \sum_{\mathbf{r}, \mathbf{r}_2} t^{\sigma}(\mathbf{r}_1, \mathbf{r}_2) a_{\sigma}^+(\mathbf{r}_1) a_{\sigma}(\mathbf{r}_2) + I \sum_{\mathbf{r}} n_{\uparrow}(\mathbf{r}) n_{\downarrow}(\mathbf{r}), \quad (175)$$

($n_{\sigma} = a_{\sigma}^+ a_{\sigma}$) where I represents an intratomic coulomb repulsion and $t(\mathbf{r}_1, \mathbf{r}_2)$ is an interatomic hopping integral. To use the methods we have developed, one regards the second term in (175) as the (spatially local) non-interacting Hamiltonian and expands in powers of the kinetic energy. In order to have the freedom of functional methods it is necessary to introduce anticommuting fields (Schwinger, 1951) which couple directly to the fermion creation and destruction operators,

$$\mathcal{H}'(t) = \sum_{\mathbf{r}, \sigma} \xi_{\sigma}(\mathbf{r}, t) a_{\sigma}(\mathbf{r}, t) + \eta_{\sigma}(t) a_{\sigma}^+(\mathbf{r}, t). \quad (176)$$

The reader is referred to Hubbard's work for further details. Renormalisation has not yet been used.

Appendix: The Relation of the Free Multiplicities to the Weak Lattice Constants

Part (d) of all linked-cluster-expansion rules requires for a given graph the summation of a number of terms, each one associated with a specific realisation of that graph on the underlying lattice, i.e. with a specific assignment of each graphical vertex to a given lattice site. When interactions are uniform, isotropic in the lattice, and restricted to nearest neighbors (e.g. (21), (23)), all terms in such a summation are identical. In such a situation the total contribution of the graph is the value of any one term times the number of terms. The number of terms is called the *free multiplicity*.

We confine our main discussion to unrooted graphs for the sake of simplicity. For specificity we shall always take the underlying lattice to be large and toroidally connected. An *embedding of a graph G in a lattice* is an assignment of the graph vertices to the sites of the lattice in such a way that all graph edges lie along nearest-neighbour bonds of the lattice. *Free* embedding allows several distinct graph vertices to be assigned to the same lattice site. *Weak* embedding allows no more than one vertex to be assigned to each site. The *free multiplicity* $m(G)$ of an unrooted graph G on a given lattice is the number of distinct *free* embeddings of the *labelled* graph G (i.e. the vertices are regarded as distinguishable) in the lattice *per site of the lattice*. The *weak lattice constant* $p(G)$ of the unrooted graph G in the lattice is the number of distinct *weak* embeddings of the *unlabelled* graph G in the lattice *per site of the lattice*. We show below that $m(G)$ and $p(G)$ are closely related.†

We assert that for a given lattice:

(i) For a properly chosen set of graphs $G \in S$, $m(G)$ is related to $p(G)$ by a conversion matrix A (Sykes *et al.*, 1966),

$$m(G) = \sum_{G' \in S} A(G|G')p(G'), \quad A(G|G') \geq 0.$$

(ii) A is a non-singular matrix and can therefore be inverted, giving the weak lattice constants p in terms of the m 's.

Note first of all that any graph with multiple edges has the same embeddings as the corresponding *reduced* graph in which all multiple edges have been replaced by single edges. We therefore restrict attention to graphs having only single edges.

Define a *collapse of a graph G* as a graph formed from G by (a) identifying (figuratively, "gluing together") previously distinct vertices of G † and (b)

† The weak lattice constants are closely related to the strong lattice constants and derivable from them, as shown by Sykes *et al.* (1966). See Domb this volume, Chapter 1, Section IV.

‡ No "loops" (i.e. edges with both ends at the same vertex) may be formed in this process.

reducing all multiple edges thus formed. The graph G and all its collapses form the *collapse chain* of G . The members of the collapse chain of G can be ordered (in general, in a variety of ways), $G = G_1, G_2, \dots$, so that no graph G_n collapses into any graph to its left ($l < n$). Such an ordering is a *standard ordering*. The collapse chain of the graph $\bigwedge$ is shown in Fig. 16. The *collapse matrix* $C(G|G') \geq 0$ is the number of distinct ways in which the *labelled* graph G collapses into the *unlabelled* graph G' .

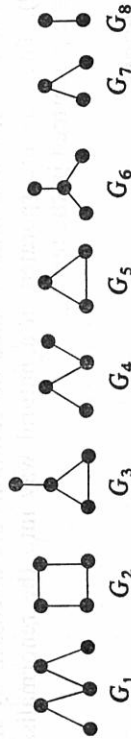


Fig. 16. A collapse chain arranged in standard order.

In the process of counting the free multiplicity of G , all collapses G' are included. Each collapse G' has $g(G')p(G')$ labelled weak embeddings per site of the lattice. Thus,

$$m(G) = \sum_{G'} C(G|G')g(G')p(G'),$$

where the sum is over the graphs in the collapse chain of G . This proves (i). The matrices C and A for the collapse chain shown in Fig. 16 are given below.

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 1 & 4 & 1 \\ 1 & 0 & 0 & 0 & 2 & 1 & & \\ 1 & 0 & 2 & 0 & 0 & 0 & & \\ 1 & 1 & 0 & 2 & 1 & & & \\ 1 & 0 & 0 & 0 & & & & \\ 1 & 3 & 1 & & & & & \\ 1 & 1 & & & & & & \\ 1 & & & & & & & \end{bmatrix} \quad C = \begin{bmatrix} 2 & 8 & 4 & 4 & 18 & 6 & 8 & 2 \\ 8 & 0 & 0 & 0 & 0 & 4 & 2 & \\ 2 & 0 & 12 & 0 & 0 & 0 & 0 & \\ 2 & 6 & 0 & 4 & 2 & & & \\ 6 & 0 & 0 & 0 & & & & \\ 6 & 4 & 2 & & & & & \\ 2 & 2 & & & & & & \\ 2 & & & & & & & \end{bmatrix} \quad (A.2)$$

To prove part (ii) of the original assertion note that, when the collapse chain of G is arranged in standard order, both C and A are triangular. All diagonal elements are positive, so $\det A \neq 0$ and A^{-1} exists.

There is no difficulty in extending the above to sets of graphs formed by taking the union of two or more collapse chains.

The free multiplicity of an n -rooted graph G on a given lattice is the number of free embeddings of the labelled graph G in the lattice *subject to the condition that the external vertices are fixed on the lattice sites* $\mathbf{r}_1, \dots, \mathbf{r}_n$.

The simplifying feature of the free multiplicities (relative, for example, to the weak or strong lattice constants) is that different parts of the graph do not interfere with one another, since they can interpenetrate on the lattice. Thus, the free multiplicities are larger than the weak or strong lattice constants but often far easier to evaluate. For example, the free multiplicity of the unrooted chain of n links is just q^n (q = coordination number). Similarly the free multiplicity of a 1-rooted graph whose external vertex is an articulation point is just the product of the free multiplicities of the associated 1-insertions. These and other related observations are discussed by Jasnow (1969). They are incorporated in a natural way into the renormalisations we have performed in the text.

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4. Asymptotic Analysis of Coefficients

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I. Introduction	181
A. General problem	181
B. Basic properties of power series	184
C. Comment on content	186
II. Ratio Method	187
A. Preliminary estimates	187
B. Neville tables	191
C. Refined estimates	192
D. Loose-packed lattices	195
E. Extrapolation formulae	199
III. Padé Approximants	202
A. Background	202
B. Theory	203
C. Padé method	206
D. Applications	210
IV. Method of <i>N</i> Point Fits	219
V. Transformations of Expansion Variables	224
VI. Applications of Darboux's Theorems	232
A. Introduction	232
B. Close-packed lattices	232
C. Loose-packed lattices	234
D. Contour map method	238
Acknowledgments	241
References	241

I. Introduction

A. General problem

It can be seen in other chapters of this volume how power series expansions may be developed for various lattice models in any dimension. The expansions are exact as far as they go, and clearly represent a good deal of

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